# Isentropic Hydrodynamics in Plane Symmetric Space-Times*

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Use is made of a form of the stress energy tensor of a perfect fluid, previously derived for special relativity, to show that for irrotational isentropic motions a co-moving coordinate system exists in which both sides of the Einstein gravitational field equations may be expressed in terms of the dependent variables of the self-gravitational problem for a perfect fluid. It is shown that for a space-time with plane symmetry the field equations and the assumption of isentropy imply the conservation of mass. General methods for dealing with these field equations are given for the static and spatially independent cases. Approximate solutions are obtained for other specific cases. The general exact solution is obtained for the incompressible case. Properties of the incompressible case are discussed.

## 1. INTRODUCTION

THE problem of determining the motion of a fluid subject only to its own gravitational field and internal forces and further such that various initial and boundary conditions are satisfied is a problem in general relativity. It is the purpose of this paper to restate this problem, using a form of the stress-energy tensor previously given by Eckart[1] and derived by Taub[2] for special relativity by kinetic theory arguments, and to obtain some solutions of the equations involved. We shall restrict ourselves to situations where the space-time and the fluid motion have plane symmetry. That is, the problem is invariant under transformations of the form

$$y^*=y+a,\quad z^*=z+b,$$

and

$$y^*=y\cos\theta+z\sin\theta,\quad z^*=-y\sin\theta+z\cos\theta.$$

However, the methods developed will apply equally well to the spherically symmetric case.

The metric of a space-time admitting plane symmetry may[3] be written as

$$ds^2=e^{2F}dt^2-\frac{1}{c^2}e^{2G}dx^2-\frac{1}{c^2}e^{2H}(dy^2+dz^2),\qquad(1.1)$$

where $F$, $G$, and $H$ are functions of $x$ and $t$ alone.

We take the stress energy tensor of a perfect fluid to be

$$T^{\mu\nu}=\sigma u^\mu u^\nu-g^{\mu\nu}p/c^2,\qquad(1.2)$$

where $u^\mu$ is the four-dimensional velocity vector of the fluid and satisfies

$$g_{\mu\nu}u^\mu u^\nu=u^\mu u_\mu=1,\qquad(1.3)$$

$p$ is the pressure and

$$\sigma=\rho\left(1+\frac{\epsilon}{c^2}+\frac{p}{\rho c^2}\right),\qquad(1.4)$$

where $\rho$ is the density as measured by an observer at rest with respect to the fluid, $c$ is the special relativity velocity of light, and $\epsilon$ is the internal energy of the fluid as measured by an observer at rest with respect to the fluid.

The fluid is characterized by specifying its caloric equation of state, that is, by specifying $\epsilon$ as a function of pressure and density

$$\epsilon=\epsilon(p,\rho).\qquad(1.5)$$

In case

$$\epsilon\equiv 0,\qquad(1.6)$$

the fluid is said to be incompressible. We note that the form of the stress-energy tensor usually used in works on general relativity is that here used for the incompressible fluid.

For gases we shall require [as is the case in special relativity, *cf* reference (2)] that

$$1+\frac{\epsilon}{c^2}\geq\frac{3}{2}\frac{p}{\rho c^2}+\left[1+\frac{9}{4}\left(\frac{p}{\rho c^2}\right)^2\right]^{\frac{1}{2}}.$$

If the equality sign holds in this relation, we shall call the gas in question a "limiting gas." If

$$p/\rho c^2\gg 1,$$

the gas will be said to be "hot." A degenerate gas is said to be one which is limiting and hot, that is, for which

$$1+\frac{\epsilon}{c^2}=\frac{3p}{\rho c^2}.\qquad(1.7)$$

It follows from Eqs. (1.2), (1.3), (1.4), and (1.7) that

$$T=T^{\mu\nu}g_{\mu\nu}=\sigma-\frac{4p}{c^2}=\rho\left(1+\frac{\epsilon}{c^2}\right)-\frac{3p}{c^2}=0.\qquad(1.8)$$

The equations determining the motion of the fluid are five in number, namely,

$$T^{\mu\nu}{}_{;\nu}=0\qquad(1.9)$$

and

$$(\rho u^\mu)_{;\mu}=0,\qquad(1.10)$$

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[1] C. Eckart, Phys. Rev. **58**, 919 (1940).
[2] A. H. Taub, Phys. Rev. **74**, 328 (1948).
[3] A. H. Taub, Ann. Math. **53**, 472 (1951).

where the semicolon denotes the covariant derivative with respect to the space-time with the metric tensor

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{1.11}$$

In problems in which the self-gravitational effects of the fluid may be neglected, the $g_{\mu\nu}$ are taken to be the gravitational potentials of some external gravitational field determined by the Einstein field equations

$$R^{\mu\nu} - \tfrac{1}{2} g^{\mu\nu} R = -kc^2 T^{*\mu\nu}, \tag{1.12}$$

where $R^{\mu\nu}$ is the Ricci tensor of the Riemannian space whose metric is given by (1.11), $R$ is the scalar curvature of this space,

$$k = 8\pi G/c^2, \tag{1.13}$$

with $G$ being Newton's constant of gravitation, and $T^{*\mu\nu}$ is the stress-energy tensor of the matter creating the external gravitational field.

If the only matter creating the gravitational field is the fluid whose motion is to be studied the problem will be said to be a self-attracting one. In this case the tensor $T^{*\mu\nu}$ in Eq. (1.12) is the tensor $T^{\mu\nu}$ given by Eq. (1.2) and is the same tensor which occurs in Eqs. (1.9) which are now a consequence of (1.12) as follows from the Bianchi identities. The self-attracting problems are those in which one attempts to determine the functions $g_{\mu\nu}$, $u^\mu$, $p$, and $\rho$ satisfying Eqs. (1.12), (1.2), (1.3), (1.10), and hence (1.9).

Because of the assumption of plane symmetry equation (1.1) replaces (1.11) and there are only three nonvanishing $g_{\mu\nu}$. Our problem then becomes one of determining eight independent functions, namely, three functions which determine the gravitational potentials and five other functions which determine the velocity field of the fluid and the pressure and density fields. The fundamental relations between these functions are given by Eqs. (1.12) and (1.10), where the right-hand sides of Eqs. (1.12) involve the unknown functions $p$, $\rho$, and $u^\mu$. In order to proceed with the solution of Eqs. (1.12) and (1.10) where we know neither the right-hand sides nor the left-hand sides we show that in the isentropic case there exists a coordinate system, a co-moving one, in which the velocity field has a particularly simple form and then determining the metric tensor and the pressure and density fields.

For nonisentropic flows the system of Eqs. (1.12) and (1.10) may be made more tractable by using the method devised by McVittie.[4] In this method the metric tensor first is reduced to a simple form and then use is made of the fact that the coefficients of the metric tensor must satisfy certain consistency equations in order that the Einstein tensor $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ be of the form (1.2). These consistency equations are differential equations which involve only the coefficients of the metric tensor. We shall restrict this paper to a discussion of isentropic flows.

[4] G. C. McVittie, J. Rational Mech. Anal. 4, 201 (1955).

Before going into the detailed discussion of the program outlined in the previous paragraphs we shall discuss the significance of Eqs. (1.9) and (1.10). The former equations are the generalization of the classical equations of conservation of energy and momentum.

If Eqs. (1.2) are substituted into Eqs. (1.9) we have

$$T^{\mu\nu}{}_{;\nu} = \sigma u^\mu{}_{;\nu} u^\nu + u^\mu (\sigma u^\nu)_{;\nu} - (1/c^2) g^{\mu\nu} p_{;\nu} = 0. \tag{1.14}$$

Multiplying this equation by $u_\mu$ and summing, we obtain

$$(\sigma u^\nu)_{;\nu} = (1/c^2) u^\nu p_{;\nu} \tag{1.15}$$

after using (1.3) and its consequence

$$u_\mu u^\mu{}_{;\nu} = 0. \tag{1.16}$$

When Eq. (1.15) is substituted into (1.14) we obtain

$$\sigma u^\mu{}_{;\nu} u^\nu = (1/c^2)(g^{\mu\nu} - u^\mu u^\nu) p_{;\nu}. \tag{1.17}$$

These four equations contain, among them, three independent ones in view of Eq. (1.16). They are the generalization to general relativity of the classical equations of conservation of momentum. Equations (1.17) and (1.15) are of course equivalent to (1.14) which in turn are a consequence of the Einstein field equations (1.12).

Equation (1.10) is the general relativity form of the conservation of mass equation. With its aid we may interpret Eq. (1.15) which may be written as

$$(\sigma/\rho)(\rho u^\nu)_{;\nu} + \rho \left(1 + \frac{\epsilon}{c^2} + \frac{p}{\rho c^2}\right)_{;\nu} u^\nu = \frac{1}{c^2} u^\nu p_{;\nu}$$

in virtue of Eq. (1.4). This in turn may be written as

$$(\sigma c^2/\rho)(\rho u^\nu)_{;\nu} + \epsilon_{;\nu} u^\nu + p(1/\rho)_{;\nu} u^\nu = 0. \tag{1.18}$$

When Eq. (1.10) holds, Eq. (1.18) is the equation of conservation of energy for it states that along the motion

$$d\epsilon + p dv = 0,$$

where $v$ is the specific volume;

$$v = 1/\rho$$

and

$$df = f_{;\nu} u^\nu.$$

Thus the five Eqs. (1.9) and (1.10) are the generalizations of the five conservation laws on which the classical theory of hydrodynamics is based.

Incidentally, we may note that it follows from Eqs. (1.10) and (1.18) that if $\epsilon \equiv 0$ then

$$\rho_{;\nu} u^\nu = 0, \tag{1.19}$$

and therefore the conservation of mass equation reduces to

$$u^\nu{}_{;\nu} = 0$$

in this case. Equation (1.19) states that the density is a constant along the world lines followed by the elements of the fluid. Thus if the density is initially independent of the coordinates it will always remain so. This is the justification for stating that the caloric equation of state $\epsilon \equiv 0$ is the appropriate one for an incompressible fluid.

## 2. ENTROPY

The specification of the caloric equation of state, that is, the specification of $\epsilon$ as a function of $p$ and $\rho$, determines the rest temperature and the rest specific entropy $S$ of the gas as functions of $p$ and $\rho$. This result has been noted by Eckart.[1] The equation defining these quantities is

$$d\epsilon + p d(1/\rho) = T ds, \tag{2.1}$$

where $T$ is chosen to be an integrating factor so that $dS$ is a perfect differential.

For a degenerate gas [see Eq. (1.7)],

$$d\epsilon = 3(1/\rho)dp + 3pd(1/\rho), \tag{2.2}$$

and Eq. (2.1) becomes

$$TdS = 3(1/\rho)dp + 4pd(1/\rho). \tag{2.3}$$

From this it follows that we may write

$$RT = p(1/\rho), \tag{2.4}$$

where $R$ is a constant which may be identified with the gas constant.

Substituting (2.4) into (2.3), we obtain

$$dS = 3Rd(\log(p\rho^{-4/3})). \tag{2.5}$$

If, as is the case in the classical theory of a perfect gas,

$$\epsilon = \frac{1}{\gamma - 1} p/\rho, \tag{2.6}$$

where $\gamma$ is a constant equal to the ratio of specific heats of the gas, then Eq. (2.4) holds and

$$dS = \frac{R}{(\gamma - 1)} d[\log(p\rho^{-\gamma})]. \tag{2.7}$$

The constant $(\gamma - 1)/R$ is then the specific heat at constant volume. The entropy of a degenerate gas is the same function of pressure and density as is the entropy of a perfect gas in classical theory with ratio of specific heats $\gamma = 4/3$.

In the incompressible case

$$\epsilon = 0,$$

and taking (2.4) as the definition of temperature, we then obtain from (2.1)

$$dS = \frac{R}{(1/\rho)} d(1/\rho). \tag{2.8}$$

Equation (1.18) may now be rewritten as

$$TS_{;\mu} u^\mu = 0. \tag{2.9}$$

That is, the entropy along the world-line of a "particle" of the fluid is independent of the position along the world-line. If on any surface intersecting the world-lines of the particle of the fluid, the entropy does not vary from world-line to world-line, then we have

$$S = \text{constant} \tag{2.10}$$

throughout the fluid.

Motions in which Eqs. (2.10) holds are called isentropic motions. For a degenerate and a classically perfect gas these are characterized by

$$p = p_0(\rho/\rho_0)^\gamma, \tag{2.11}$$

where $p_0$ and $\rho_0$ are constants and where $\gamma = 4/3$ in the former case. For the incompressible case, isentropic motions are characterized by

$$\rho = \rho_0 = \text{constant}. \tag{2.12}$$

In all of these cases, it follows that for isentropic motions the equation

$$-\phi = \int_{p_0}^{p} \frac{dp}{\sigma c^2}, \tag{2.13}$$

where the lower limit is the same constant that enters into Eq. (2.11) in case that equation is used to define $\rho$ as a function of $p$ in the expression for $\sigma$, and $p_0$ is arbitrary if Eq. (2.12 obtains, defines a function $\phi$ such that

$$-\phi_{,\mu} = p_{,\mu}/\sigma c^2, \tag{2.14}$$

where the comma denotes the ordinary derivative.

Equation (2.13) may be used to express $p$, $\rho$, and $\sigma$ as functions of $\phi$. This will be done for certain specific caloric equations of state below. We first obtain some general expressions in terms of $\phi$. Equation (2.14) implies that

$$p' = dp/d\phi = -\sigma c^2. \tag{2.15}$$

where all quantities are functions of $\phi$ alone as follows from the assumption of isentropy and its consequence, Eq. (2.13).

It is a consequence of Eq. (2.10) and the definition of entropy, Eq. (2.1), that

$$\epsilon_{,\mu} - (p/\rho^2)\rho_{,\mu} = 0.$$

Hence it follows from Eqs. (1.4) and (2.14) that

$$\left(\frac{\sigma c^2}{\rho}\right)_{,\mu} = (\epsilon + p/\rho)_{,\mu} = \frac{1}{\rho} p_{,\mu} = -\frac{\sigma c^2}{\rho} \phi_{,\mu}. \tag{2.16}$$

Integrating this equation, we obtain

$$\rho_0 \sigma c^2/\rho = K e^{-\phi} = -\rho_0 p'/\rho,$$

where $K$ is a constant.

It has been shown[2] that the special relativity theory expression for $a$, the velocity of sound, is given by

$$\frac{a^2}{c^2}=\frac{\rho^2}{\sigma}\frac{d}{d\rho}\left(\frac{\sigma}{\rho}\right),$$

where $\sigma$ is considered as a function of $\rho$ alone in virtue of the isentropy assumption. We may also write this as

$$\frac{a^2}{c^2}=\frac{\rho^2}{\sigma c^2}\frac{d}{d\phi}\left(\frac{\sigma c^2}{\rho}\right)\Big/\frac{d\rho}{d\phi}=-\frac{\rho}{\rho'}, \qquad (2.17)$$

where again the prime denotes the derivative with respect to $\phi$.

We now turn our attention to expressing various quantities as functions of $\phi$. That is, we evaluate the integral occurring in Eq. (2.13) for a specific $\epsilon(p,\rho)$, using the relation between $p$ and $\rho$ given by the requirement that the entropy is a constant. It may be verified that if $\epsilon$ is given by Eq. (2.6) and hence Eq. (2.11) holds, then

$$p=p_0\psi^n, \qquad (2.18)$$

where

$$\psi=(1+\delta)e^{-\phi}-\delta, \qquad (2.19)$$

$$\delta=\rho_0c^2/np_0, \qquad (2.20)$$

$$n=\gamma/(\gamma-1). \qquad (2.21)$$

In the incompressible case, Eqs. (2.18) through (2.20) hold and Eq. (2.21) is replaced by

$$n=1. \qquad (2.22)$$

For a degenerate gas, Eqs. (1.4), (1.7), and (2.13) define $\phi$ for arbitrary motions, that is, isentropic and nonisentropic ones. For such a gas, Eqs. (2.18), (2.19), and (2.21) hold with $\gamma=4/3$ (that is, $n=4$) in the latter and Eq. (2.20) becomes

$$\delta=0. \qquad (2.23)$$

It now follows from Eq. (2.11) that

$$\rho=\rho_0\psi^{n-1}, \qquad (2.24)$$

if $\epsilon$ is given by Eq. (2.6) and the motion is isentropic. In this case it follows from Eq. (2.17) that

$$\frac{a^2}{c^2}=-\frac{\rho}{\rho'}=\frac{(\gamma-1)}{1+\delta}\psi e^{\varphi}.$$

If we evaluate this expression when $p=p_0$, $(\rho=\rho_0)$, that is when $\psi=1$, and hence $\phi=0$, we obtain for the square of the ratio of the velocity of sound to the velocity of light

$$a^2/c^2=(\gamma-1)/1+\delta).$$

Hence the case $\delta\ll1$ is that in which the state $p_0$, $\rho_0$ is such that velocity of sound is as large as it can be. The case $\delta\gg1$ corresponds to the state $p_0$, $\rho_0$ being one usually contemplated in classical hydrodynamics where the velocity of sound is small compared to the velocity of light. The latter limiting case will therefore be called the classical one and the former the relativistic one. The case of the degenerate gas is seen to be an extreme relativistic case.

In the classical case, we may write

$$\psi=\delta(e^{-\phi}-1), \qquad (2.25)$$

and in the relativistic one,

$$\psi=e^{-\phi}. \qquad (2.26)$$

It is a consequence of Eqs. (2.15) and (2.18) that

$$\sigma c^2=-p'=n(1+\delta)p_0e^{-\phi}\psi^{n-1}. \qquad (2.27)$$

In the incompressible case, since $n=1$,

$$p+p'=-\delta p_0. \qquad (2.28)$$

If $n\neq1$ we have

$$p+p'=p\left(1-n\frac{e^{\phi_0-\phi}}{e^{\phi_0-\phi}-1}\right), \qquad (2.29)$$

where

$$e^{\phi_0}=1+(1/\delta). \qquad (2.30)$$

Hence when

$$\phi\ll\phi_0=\log[1+(1/\delta)],$$

we may write

$$p+p'=(1-n)p/p_0 \qquad (2.31)$$

and

$$p=p_0(1+\delta)^ne^{-n\phi}. \qquad (2.32)$$

The quantity $\psi$ is positive for $\phi$ in the range

$$-\infty<\phi<\phi_0 \qquad (2.33)$$

and is a monatonically decreasing function of $\phi$. It approaches $-\delta$ as $\phi$ approaches infinity. Since we require both $p$ and $\rho$ to be non-negative, the range of the variable $\phi$ must be restricted to that given by the expressions (2.33).

### 3. VELOCITY FIELD IN ISENTROPIC MOTION

In this section, we prove that for isentropic irrotational motion of a fluid, there exists a co-moving coordinate system in which the metric tensor has a particularly simple form. The theorem is then applied to space-times with plane symmetry. It may equally well be applied to space-times with spherical symmetry.

Consider the vector field

$$V^\mu=e^{-\phi}u^\mu, \qquad (3.1)$$

where $\phi$ is the function defined by Eq. (2.13) and $u^\mu$ is the velocity field of the fluid. Then

$$\Omega_{\mu\nu}=V_{\mu;\nu}-V_{\nu;\mu}=e^{-\phi}[\omega_{\mu\nu}-(\phi,_\nu u_\mu-\phi_\mu u_\nu)], \qquad (3.2)$$

where

$$\omega_{\mu\nu}=u_{\mu;\nu}-u_{\nu;\mu}. \qquad (3.3)$$

Hence

$$V_\lambda\Omega_{\mu\nu}+V_\mu\Omega_{\nu\lambda}+V_\nu\Omega_{\lambda\mu}=e^{-2\phi}(u_\lambda\omega_{\mu\nu}+u_\mu\omega_{\nu\lambda}+u_\nu\omega_{\lambda\mu}), \qquad (3.4)$$

and

$$V^\mu\Omega_{\mu\nu}=e^{-2\phi}[u^\mu\omega_{\mu\nu}-\phi_{,\mu}(\delta_\nu{}^\mu-u^\mu u_\nu)]. \tag{3.5}$$

It follows from Eqs. (1.3) and (1.17) that

$$u^\mu\omega_{\mu\nu}=-u_{\nu;\mu}u^\mu=-\frac{p_{,\mu}}{c^2\sigma}(\delta_\nu{}^\mu-u^\mu u_\nu).$$

Therefore Eq. (3.5) becomes

$$V^\mu\Omega_{\mu\nu}=-e^{-2\phi}\left(\frac{p_{,\mu}}{\sigma c^2}+\phi_{,\mu}\right)(\delta_\nu{}^\mu-u^\mu u_\nu)=0. \tag{3.6}$$

An irrotational motion[5] is one for which

$$u_\lambda\omega_{\mu\nu}+u_\mu\omega_{\nu\lambda}+u_\nu\omega_{\lambda\mu}=0. \tag{3.7}$$

Thus for an isentropic irrotational motion, Eq. (3.6) holds and the right-hand side of Eq. (3.4) vanishes. The latter fact implies that in a coordinate system in which $V^i=0$ ($i=1,2,3$), $V^4\neq 0$.

$$\Omega_{ij}=0.$$

In this coordinate system we have, as a consequence of Eq. (3.6),

$$\Omega_{4i}=0.$$

Hence, in this coordinate system,

$$\Omega_{\mu\nu}=0. \tag{3.8}$$

Since Eq. (3.7) is a tensor equation, it holds in all coordinate systems.

Therefore there exists a function $T(x)$ such that

$$V_\mu=\partial T/\partial x^\mu. \tag{3.9}$$

If we introduce a coordinate system in space-time

$$x^{4*}=T(x),$$

and the curves of parameter $x^{i*}$ ($i=1,2,3$) are the geodesics orthogonal to the surface $T=$constant, the line element becomes

$$ds^2=e^{2\phi}dT^2-g_{ij}{}^*(x^\mu)dx^{*i}dx^{*j}, \tag{3.10}$$

where $g_{ij}{}^*(x^\mu)$ are functions of $T$ and $x^{i*}$, since

$$g^{44*}=g^{\mu\nu}\frac{\partial T}{\partial x^\mu}\frac{\partial T}{\partial x^\nu}=g^{\mu\nu}V_\mu V_\nu=e^{-2\phi}g^{\mu\nu}u_\mu u_\nu=e^{-2\phi}.$$

In the coordinate system in which (3.10) holds, we have

$$V^{\mu*}=e^{-2\phi}\delta_4{}^\mu.$$

Hence

$$u^{\mu*}=e^\phi V^{\mu*}=e^{-\phi}\delta_4{}^\mu. \tag{3.11}$$

Thus for isentropic, irrotational flow there exists a coordinate system, a co-moving one, in which the line element is given by Eq. (3.10) and the velocity field by Eq. (3.11), with $\phi$ as yet an undetermined function of the coordinates.

[5] K. Godel, Revs. Modern Phys. **21**, 447 (1949).

In a coordinate system in a space-time with plane symmetry in which the line element is given by Eq. (1.1) we have $u_2=u_3=0$ and $u_1$ and $u_4$ functions of $x$ and $t$. It may readily be verified that Eq. (3.7) is satisfied. Hence for isentropic motion we may introduce a coordinate system such that

$$ds^2=e^{2\phi}dt^2-(e^{2G}/c^2)dx^2-(e^{2H}/c^2)(dy^2+dz^2), \tag{3.12}$$

and in this coordinate system

$$u^\mu=e^{-\phi}\delta_4{}^\mu. \tag{3.13}$$

We note that for a degenerate gas, Eqs. (3.12) and (3.13) hold irrespective of whether the motion is isentropic or not, since Eq. (2.13) defines the function $\phi$ for such a gas for an arbitrary motion.

## 4. FIELD EQUATIONS IN ISENTROPIC FLOW

The field equations (1.12) may be written as

$$R_\nu{}^\mu-\tfrac{1}{2}\delta_\nu{}^\mu R=-kc^2\left(\sigma u^\mu u_\nu-\frac{p}{c^2}\delta_\nu{}^\mu\right). \tag{4.1}$$

In view of Eqs. (3.13), these reduce to

$$\begin{aligned}R_4{}^4-\tfrac{1}{2}R&=-kc^2[\sigma-(p/c^2)],\\ R_1{}^1-\tfrac{1}{2}R&=kp,\\ R_2{}^2-\tfrac{1}{2}R&=kp,\\ R_1{}^4&=0,\end{aligned} \tag{4.2}$$

where

$$R=R_4{}^4+R_1{}^1+2R_2{}^2. \tag{4.3}$$

since $R_3{}^3=R_2{}^2$.

These may be rewritten as

$$\begin{aligned}R_4{}^4-R_1{}^1&=-k\sigma c^2,\\ \tfrac{1}{2}(R_1{}^1+R_4{}^4)&=-kp,\\ R_1{}^4&=0,\\ R_1{}^1&=R_2{}^2.\end{aligned} \tag{4.4}$$

The right-hand sides of the first two of these equations are functions of $\phi$ as determined in Sec. 2. Moreover $\phi$ and its first two derivatives with respect to $x^1$ and $x^4$ enter into the left-hand sides of these same two equations since $g_{44}=e^{2\phi}$. Hence the field equations (4.4) now involve only the functions $\phi$, $G$, and $H$ of the line element (3.12).

The four equations (4.4) are not independent, since we have the relation

$$\partial p/\partial x^\mu=-c^2\sigma\partial\phi/\partial x^\mu,$$

as follows from the definition of $\phi$ given in Sec. 2. If the left-hand sides of Eq. (4.14) are substituted for $p$ and $\sigma$ in this equation, we obtain equations involving the derivatives of the Ricci tensor. It may be verified that these are satisfied as a consequence of the Bianchi identities, which will be discussed in the next section.

### 5. FIELD EQUATIONS IN THE PLANE SYMMETRIC CASE

When the line element of space-time is given by Eqs. (3.12), the nonvanishing components of the Ricci tensor are

$$\begin{aligned}R_4{}^4&=e^{-2\phi}[G_{tt}+2H_{tt}+2H_t{}^2+G_t{}^2-(G_t+2H_t)\phi_t]\\&\quad-c^2e^{-2G}[\phi_{xx}+\phi_x(\phi_x-G_x)+2\phi_xH_x],\\R_1{}^1&=e^{-2\phi}[G_{tt}+G_t(G_t-\phi_t)+2G_tH_t]\\&\quad-c^2e^{-2G}[\phi_{xx}+\phi_x(\phi_x-G_x)\\&\quad+2H_{xx}+2H_x(H_x-G_x)],\\R_3{}^3&=R_2{}^2=e^{-2\phi}[H_{tt}+2H_t{}^2+H_t(G_t-\phi_t)]\\&\quad-c^2e^{-2G}[H_{xx}+2H_x{}^2+H_x(\phi_x-G_x)],\\R_1{}^4&=2e^{-2\phi}[H_{xt}-H_t\phi_x-H_xG_x+H_tH_x].\end{aligned}\tag{5.1}$$

The subscripts denote partial differentiation with respect to the variables indicated.

The field equations (4.4) then become

$$\begin{aligned}R_4{}^4-R_1{}^1&=2e^{-2\phi}[H_{tt}+H_t{}^2-H_t(G_t+\phi_t)]\\&\quad+2e^{-2G}c^2[H_{xx}+H_x{}^2-H_x(G_x+\phi_x)]\\&=-k\sigma c^2,\\\tfrac{1}{2}(R_4{}^4+R_1{}^1)&=e^{-2\phi}[G_{tt}+(G_t-\phi_t)(G_t+H_t)+H_{tt}+H_t{}^2]\\&\quad-c^2e^{-2G}[\phi_{xx}+(\phi_x+H_x)(\phi_x-G_x)\\&\quad+H_{xx}+H_x{}^2]=-kp,\\R_1{}^1-R_2{}^2&=e^{-2\phi}[G_{tt}-H_{tt}+(G_t-H_t)(G_t-\phi_t)\\&\quad+2H_t(G_t-H_t)]\\&\quad-c^2e^{-2G}[\phi_{xx}+H_{xx}+(\phi_x-G_x)(\phi_x-H_x)\\&\quad-2H_xG_x]=0,\\R_1{}^4&=2e^{-2\phi}[H_{xt}-H_xG_t-H_t\phi_x+H_tH_x]=0.\end{aligned}\tag{5.2}$$

We introduce the dimensionless variables

$$\xi=x/x_0,\quad \tau=ct/x_0,\tag{5.3}$$

where

$$x_0=(kp_0/c^2)^{-\frac{1}{2}}$$

and $p_0$ is a constant with the dimensions of a pressure, which may be taken to be the same constant used in Sec. 2. Equations (5.2), after being multiplied by $x_0{}^2/c^2$, become

$$\begin{aligned}&2e^{-2\phi}[H_{\tau\tau}+H_\tau{}^2-H_\tau(G_\tau+\phi_\tau)]\\&\quad+2e^{-2G}[H_{\xi\xi}+H_\xi{}^2-H_\xi(G_\xi+\phi_\xi)]=-\sigma c^2/p_0,\\&e^{-2\phi}[G_{\tau\tau}+(G_\tau+H_\tau)(G_\tau-\phi_\tau)+H_{\tau\tau}+H_\tau{}^2]\\&\quad-e^{-2G}[\phi_{\xi\xi}+H_{\xi\xi}+H_\xi{}^2+(\phi_\xi+H_\xi)(\phi_\xi-G_\xi)]\\&\quad=-p/p_0,\\&B=e^{-2\phi}[G_{\tau\tau}-H_{\tau\tau}+(G_\tau-H_\tau)(G_\tau-\phi_\tau+2H_\tau)]\\&\quad-e^{-2G}[\phi_{\xi\xi}+H_{\xi\xi}+(\phi_\xi-G_\xi)(\phi_\xi-H_\xi)-2H_\xi G_\xi]=0,\\&2e^{-2\phi}A=2e^{-2\phi}[H_{\xi\tau}-H_\tau\phi_\xi-H_\xi G_\tau+H_\tau H_\xi]=0.\end{aligned}\tag{5.4}$$

In these equations, $\sigma$ and $p$ are functions of $\phi$, since the motion is assumed to be isentropic and irrotational. For the various specific fluids treated in Sec. 2, explicit forms of these functions have been obtained. We wish to discuss the solution of these equations subject to the additional condition that the density $\rho$, which is also a function of $\phi$, satisfy the conservation-of-mass equation

$$(\rho u^\nu)_{;\nu}=\frac{1}{\sqrt{(-g)}}\frac{\partial(\sqrt{(-g)}\rho u^\nu)}{\partial x^\nu}=\frac{1}{\sqrt{(-g)}}(e^{G+2H}\rho)_t=0,\tag{5.5}$$

where $\rho$ is a function of $\phi$ alone.

The five differential equations in the system of Eqs. (5.4) and (5.5) are not independent. Because of the Bianchi identities there are two differential relations between them. We shall now examine this system in detail.

On subtracting the third of Eqs. (5.4) from the second, we obtain

$$e^{-2\phi}[2H_{\tau\tau}+3H_\tau{}^2-2H_\tau\phi_\tau]-e^{-2G}H_\xi[H_\xi+2\phi_\xi]=-p/p_0.\tag{5.6}$$

We subtract this from the first of Eqs. (5.4) and get

$$e^{-2\phi}H_\tau[H_\tau+2G_\tau]-e^{-2G}[2H_{\xi\xi}+3H_\xi{}^2-2H_\xi G_\xi]=(\sigma c^2/p_0)-(p/p_0).\tag{5.7}$$

The first of Eqs. (5.4) may be obtained by subtracting (5.7) from (5.6).

If we differentiate Eq. (5.6) with respect to $\xi$, we find after some manipulation that the result may be written as

$$2e^{-2\phi}[A_\tau+(G_\tau-\phi_\tau+2H_\tau)A]+2H_\xi B=0,\tag{5.8}$$

where the functions $A$ and $B$ are defined in terms of $G$, $H$, and $\phi$ and their derivatives by the third and fourth of Eqs. (5.4). In deriving (5.8) the first of Eqs. (5.4) is used to replace the term $\sigma c^2$ which enters in the evaluation of $p_\xi=-\sigma c^2\phi_\xi$.

If we differentiate Eq. (5.7) with respect to $\tau$ and use the first of Eqs. (5.4) we may write the result after some manipulation, using Eqs. (2.16) as

$$-\left(G_\tau+2H_\tau+\frac{\rho_\tau}{\rho_0}\right)\frac{\sigma c^2}{p_0}+2H_\tau B-2e^{-2G}[A_\xi+(\phi_\xi+2H_\xi-G_\xi)A]=0.\tag{5.9}$$

Equations (5.8) and (5.9) may be written as

$$e^{-(G+\phi+H)}(Ae^{G-\phi+2H})_\tau+H_\xi B=0,\tag{5.10}$$

and

$$e^{-(G+\phi+2H)}(Ae^{G-\phi+2H})_\xi-H_\tau B=-\frac{\sigma c^2}{2p_0}\left(G_\tau+2H_\tau+\frac{\rho_\tau}{\rho}\right).\tag{5.11}$$

Thus, the field equations, Eqs. (5.4), imply the conservation of mass, Eq. (5.5). Conversely, Eqs. (5.5), (5.6), and (5.7) together with the fourth of Eqs. (5.4), namely, the equation $A=0$, imply $B=0$. The reason for this is that we cannot have $H_\xi\equiv 0$ and $H_\tau\equiv 0$ where $p\neq 0$ and $\sigma\neq 0$, that is, where there is matter.

We shall therefore take as the system of equations to be solved, the Eqs. (5.5), (5.6), (5.7), and the fourth of Eqs. (5.4). Equation (5.5) may be written as

$$G_\tau+2H_\tau+(\rho_\tau/\rho)=0 \tag{5.12}$$

or

$$G=-2H-\log(\rho/\rho_0)-\log\beta(\xi). \tag{5.13}$$

Equations (5.6), (5.7), and the fourth of Eqs. (5.4) may then be written as

$$2e^{-\phi}(H_\tau e^{-\phi})_\tau+3(H_\tau e^{-\phi})^2-\left(e^{2H}\frac{\rho}{\rho_0}H_\eta\right)\times\left(e^{2H}\frac{\rho}{\rho_0}H_\eta+2e^{2H}\phi_\eta\frac{\rho}{\rho_0}\right)=-\frac{p}{p_0}, \tag{5.14}$$

$$-e^{-\phi}H_\tau\left[e^{-\phi}H_\tau+2e^{-\phi}\frac{\rho_\tau}{\rho}\right]-2\left(e^{2H}\frac{\rho}{\rho_0}H_\eta\right)_\eta\frac{e^{2H}\rho}{\rho_0}-3\left(e^{2H}\frac{\rho}{\rho_0}H_\eta\right)^2=\frac{\sigma c^2}{p_0}-\frac{p}{p_0}, \tag{5.15}$$

and

$$\frac{e^\phi}{\rho}[(e^{3H})_\tau\rho e^{-\phi}]_\eta=(e^{3H})_\tau\frac{\rho_\eta}{\rho}-(e^{3H})_\eta\frac{\rho_\tau}{\rho}, \tag{5.16}$$

respectively, where the spatial variable $\eta$ is such that for any function

$$f_\eta=\beta(\xi)f_\xi.$$

## 6. BOUNDARY CONDITIONS

The equations (5.14) to (5.16) must be supplemented by additional information in order that we may determine the functions $\phi$, and $H$. This information is in the form of initial and boundary conditions. That is, values of $\phi$ and $H$ and their first derivatives on certain loci in space-time must be prescribed in order that we may determine $\phi$ and $H$ over a region of space-time.

We now describe two types of physical problems that have been treated both in classical hydrodynamics and in special relativistic dynamics and deduce for these problems certain boundary conditions. In subsequent sections, we show that these boundary conditions are indeed sufficient to determine a unique solution to the problem.

We first discuss the paths followed by particles in the fluid. That is, the curves

$$dx^\mu/ds=u^\mu(x) \tag{6.1}$$

subject to the initial conditions at $s=0$ given by

$$x^\mu=x_0^\mu. \tag{6.2}$$

In the co-moving coordinate system used in the preceding sections, Eqs. (6.1) become

$$dx^4/ds=\exp[-\phi(x^1,x^4)],$$
$$dx^i/ds=0,\quad i=1,2,3,$$

and have as their solution

$$x^i=x_0^i,\quad i=1,2,3,$$
$$s-s_0=\int\exp\phi(x_0^1,x^4)dx^4. \tag{6.3}$$

That is, the world-line of any particle of the fluid has constant $x^1$, $x^2$, $x^3$ throughout the fluid's motion. Thus the co-moving coordinates are the analogs of Lagrange coordinates in classical hydrodynamics. The function $\phi(x^1,x^4)$ evaluated at $x^1=x_0^1$, describes the motion of the "particle located at $x_0^1$" and as this function changes, the motion of this particle changes.

One class of plane-symmetric problems in classical physics may be stated as follows. The gas is initially (at $x^4=0$) at rest and at constant entropy with pressure distribution $p(x^1)$, and the motion of one of the particles is prescribed. Determine the subsequent motion of the gas from the laws of conservation of energy, mass, and momentum. In classical theory and in special relativity, the prescribed particle motion is such to increase the volume occupied by the gas, the motion is isentropic, and the flow variables are continuous and are determined by the boundary data. In case the volume occupied by the gas decreases, shocks occur. One would therefore expect that the corresponding data would determine a unique solution of the field equations of the preceding paragraph for the cases where the volume occupied by the gas increases, and that physically unacceptable mathematical solutions of the field equations will arise in the other case where, for example, $u^\mu(x,t)$ [that is $\phi(x,t)$] will not be single-valued functions of $x$ and $t$ as they must.

Since $p$ is a function of $\phi$ and since the particle motion is also determined by $\phi$, the corresponding problem in general relativity is as follows: Given $\phi(x,0)$ and $\phi(x_0,t)$, determine a solution of the field equations. This problem will be discussed after the treatment of a second type of problem which will now be formulated.

Consider a body of gas confined to a region of space-time subject only to the self-gravitational force and the hydrodynamic forces. The space-time outside of this region is presumed to be empty, that is, the metric tensor satisfies

$$R_{\mu\nu}=0 \tag{6.4}$$

in this region.

The boundary between the region where Eqs. (5.14) to (5.16) hold and the region where (6.4) holds is given by the "outermost" particle's world-line, say by the world-line of the particle at $x_0$. On this world-line we assume that $p=p(\phi)=0$. There are therefore two equivalent characterizations of the boundary: one being

$$x=x_0$$

and the other being

$$\phi(x,t)=\text{constant}.$$

These will be the same if and only if

$$\phi_t=0. \tag{6.5}$$

That is, $\phi$ is independent of $t$ and hence $p$ and $\sigma$ are independent of $t$.

We shall assume, as is customarily done in general relativity, that the metric tensor $g_{\mu\nu}$ and its first derivatives are continuous across the boundary $p=0$ (that is, at $x=x_0$). Thus (6.5) obtains both in the region occupied by the gas and on the boundary of the region where (6.4) holds.

The justification for the assumptions: $p=0$ on the boundary, the coincidence of this locus and the world-lines of the outermost particles, and the continuity of the metric tensor and its derivatives across the boundary, is to be found in the general relativity formulation of the Rankine-Hugoniot relations. That is, the relations that must replace the differential form of the conservation laws across surfaces where these cannot apply, surfaces where discontinuities in the dependent hydrodynamical or gravitational variables may occur. We shall discuss these generalized Rankine-Hugoniot equations in a subsequent paper.

If Eqs. (6.5) hold, it follows from Eq. (5.12) and the fact that $\rho=\rho(\phi)$ that

$$[(e^{3H})_\tau \rho e^{-\phi}]_\eta=[\rho e^{-\phi}(e^{3H})_\tau]\frac{\rho_\eta}{\rho}.$$

Hence,

$$e^{3H}=K(\tau)e^{\phi}+L(\xi), \tag{6.6}$$

where $K$ and $L$ are arbitrary functions of their arguments. It is a consequence of previous results[3] that the only solution of Eqs. (6.4) in a coordinate system in which (3.12) holds satisfying (6.5) is given by the static solution

$$ds^2=(a+bx)^{-\frac{1}{2}}\left(dt^2-\frac{1}{c^2}dx^2\right) - (a+bx)\frac{1}{c^2}(dy^2+dz^2), \tag{6.7}$$

where $a$ and $b$ are constant. In this coordinate system then

$$-2G=-2\phi=H=\tfrac{1}{2}\log(a+bx). \tag{6.8}$$

It then follows that on the boundary ($\phi=\phi_0$), $K(\tau)=0$. This implies that $H$ is independent of $\tau$ inside the region occupied by the gas. Thus it is a consequence of the assumption that $p=0$ coincide with the world-line of one of the particles that in the coordinate system used the metric is static outside the matter and this implies in turn that it is static inside the matter, that is, the functions $H$ and $\phi$ and hence the function $G$ are independent of $\tau$.

## 7. STATIC SOLUTIONS

If in Eqs. (5.5) to (5.7) and the fourth of Eqs. (5.4) the dependent variables are independent of $\tau$, then the first and last of these are automatically satisfied and the system of equations becomes

$$e^{-2G}H_\xi(H_\xi+2\phi_\xi)=p/p_0 \tag{7.1}$$

and

$$e^{-2G}(2H_{\xi\xi}+3H_\xi^2-2H_\xi G_\xi)=p/p_0-\sigma c^2/p_0. \tag{7.2}$$

These may be written as

$$H_\eta(H_\eta+2\phi_\eta)=p/p_0 \tag{7.3}$$

and

$$2H_{\eta\eta}+3H_\eta^2=p/p_0-\sigma c^2/p_0, \tag{7.4}$$

where the spatial variable $\eta$ is such that

$$e^G d\xi=d\eta. \tag{7.5}$$

We may consider $H_\eta$ as a function of $\phi$ and write

$$H_\eta=Y(\phi). \tag{7.6}$$

Equation (7.3) may then be written as

$$\phi_\eta=[p(\phi)-Y^2]/2Y, \tag{7.7}$$

from which we may determine $\phi$ as a function of $\eta$ by a quadrature

$$\eta-\eta_0=\int_\phi^\phi \frac{2Yd\phi}{p(\phi)-Y^2}. \tag{7.8}$$

Furthermore we may also determine $H$ as a function of $\phi$ (or $\eta$) by a quadrature

$$H-H_0=\int_{\eta_0}^\eta H_\eta d_\eta=\int_{\phi_0}^\phi \frac{H_\eta}{\phi_\eta}d\phi=\int_{\phi_0}^\phi \frac{2Y^2}{p/p_0-Y^2}d\phi. \tag{7.9}$$

The constants $H_0$ and $\phi_0$ are the values of $H$ and $\phi$, respectively, at $\eta=\eta_0$.

The equation determining $Y(\phi)$ is obtained from substituting

$$H_{\eta\eta}=Y'(\phi)\phi_\eta=Y'(p/p_0-Y^2)/2Y$$

into Eq. (7.4), where the prime denotes differentiation with respect to $\phi$. Thus

$$Y'[(p-Y^2)/p_0]+3Y^3=[(p+p')/p_0]Y. \tag{7.10}$$

We define $\phi_0$ by the equation

$$p(\phi_0)=0. \tag{7.11}$$

Then $\eta_0$, and $\phi_0$ and $H_0$ determine the values of the independent variable and the metric tensor on the boundary between the regions occupied by matter and empty space-time. If the metric is to have continuous derivatives at $\eta=\eta_0$ (or $\phi=\phi_0$), the initial value of $Y$

for the differential equation (7.10) must be the value of $Y$ taken on by the solution of the equation

$$Y'Y^2-3Y^3=0 \tag{7.12}$$

at $\phi=\phi_0$. Equation (7.12) is obtained from

$$H_\eta(H_\eta+2\phi_\eta)=0 \tag{7.13}$$

and

$$2H_{\eta\eta}+3H_{\eta\xi}^2=0, \tag{7.14}$$

the field equations

$$R_{\mu\nu}=0,$$

in the same way as (7.10) is obtained from (7.3) and (7.4).

The solution of (7.12) is

$$Y=2Ce^{3\phi}, \tag{7.15}$$

where $C$ is a constant. Hence the initial value for the function defined by the Eq. (7.10) is

$$Y_0=Y(\phi_0)=2Ce^{3\phi_0}. \tag{7.16}$$

In the region occupied by matter the line element is given by

$$ds^2=\frac{x_0^2}{c^2}e^{2\phi}d\tau^2-\frac{x_0^2}{c^2}\left(\frac{2Y}{p/p_0-Y^2}\right)^2d\phi^2 -\frac{e^{2H}}{c^2}(dy^2+dz^2), \tag{7.17}$$

where $Y(\phi)$ is determined by (7.10) and (7.16), and $H(\phi)$ is given by (7.9). In empty space-time, in the same coordinate system, the metric tensor is given by

$$ds^2=\frac{x_0^2}{c^2}e^{2\phi}d\tau^2-\frac{x_0^2}{c^2C^2}e^{-6\phi}d\phi^2-L^2\frac{e^{-4\phi}}{c^2}(dy^2+dz^2), \tag{7.18}$$

where $L$ is a constant which may be taken to be one by changing the units of length in the $y$ and $z$ directions.

Thus we see that specification of the function $\epsilon(p,\rho)$, that is, a statement of the gas that is present, determines the function $p(\phi)$ (see Sec. 2) which in turn determines the coefficients of the first-order differential equation (7.10). The line elements (7.17) is then determined. It will of course be a function of the constant $C$. This constant may be related to another constant $M$, the mass per unit area present, by the relation

$$\int_{\phi_0}^{\phi_1}\rho u^4\surd(-g)d\phi=\int_{\phi_0}^{\phi_1}\rho e^{-\phi}\surd(-g)d\phi=M, \tag{7.19}$$

where the integration is carried out over the region occupied by the gas. From (7.17) and (7.19), we have

$$\int_{\phi_0}^{\phi_1}\rho\frac{2Ye^{2H}}{p/p_0-Y^2}d\phi=M. \tag{7.20}$$

Since $Y$ and $H$ depend on the constant $C$ in virtue of Eq. (7.16), it is clear that the value of $C$ determines the value of $M$ and conversely.

In the next section, we shall give a parametric representation of the function $Y(\phi)$ defined by Eq. (7.10). This representation is particularly suitable when the function $p(\phi)$ is one of those discussed in Sec. 2.

## 8. EXAMPLES OF STATIC SOLUTIONS

For the various functions $\epsilon(p,\rho)$ discussed in Sec. 2, the function $p(\phi)$ is given by Eq. (2.18). In particular, the value of $\phi$ determined by the equation

$$p(\phi)=0$$

is given by

$$\phi=\phi_0=\log[1+(1/\delta)].$$

In case $n\neq1$, we shall use the approximation represented by Eqs. (2.31) and (2.32) in discussing Eq. (7.10). It is convenient to discuss the function $Y(\phi)$ defined by the latter equation in terms of the parameter $H$. It is a consequence of Eq. (7.9) that

$$\frac{d\phi}{dH}=\frac{p/p_0-Y^2}{2Y^2}. \tag{8.1}$$

Then we have from (7.10) and (8.1)

$$\frac{dY}{dH}=\frac{[(p+p')/p_0]-3Y^2}{2Y}. \tag{8.2}$$

We define the variable $u$ by the equation

$$Y^2=(p/p_0)u. \tag{8.3}$$

Equations (8.1) and (8.2) then become

$$d\phi/dH=(1-u)/2u \tag{8.4}$$

and

$$\frac{du}{dH}=\frac{p+p'}{p}-3u-\tfrac{1}{2}(p'/p)(1-u). \tag{8.5}$$

In the region where Eqs. (2.31) and (2.32) hold, Eq. (8.5) becomes

$$\frac{du}{dH}=1-\frac{n}{2}-\left(2+\frac{n}{2}\right)u. \tag{8.6}$$

Therefore

$$u=\frac{n-2}{6+n}(Ae^{-\frac{1}{2}(6+n)H}-1), \tag{8.7}$$

where $A$ is a constant of integration. Substituting this value of $u$ into (8.4) and integrating, we obtain

$$\phi=-\frac{n+2}{n-2}H-\frac{1}{n-2}\log(Ae^{-\frac{1}{2}(6+n)H}-1)+B, \tag{8.8}$$

where $B$ is another constant of integration.

Equation (2.32) is

$$p=p_0(1+\delta)^n e^{-n\phi}. \tag{8.9}$$

Hence

$$Y^2=p_0(1+\delta)^n e^{-n\phi}u. \tag{8.10}$$

The line element (7.17) may then be written as

$$ds^2=\frac{x_0^2}{c^2}e^{2\phi}d\tau^2-\frac{x_0^2}{c^2}\frac{dH^2}{Y^2}-\frac{e^{2H}}{c^2}(dy^2+dz^2), \tag{8.11}$$

where $\phi$ and $Y^2$ are given as functions of $H$ by Eqs. (8.8), (8.10), and (8.7). The constants of integration, $A$ and $B$, occurring in these formulas may be evaluated when the line element is known for values of $\phi$ which do not satisfy the condition $\phi_0\gg\phi$. We shall illustrate the procedure for doing this by making the *inconsistent* assumption that the line element determined above extends to the boundary given by $\phi_0$. This assumption is inconsistent in that Eq. (8.9) is not the correct expression for the pressure at the boundary and in particular this $p$ does not vanish at the boundary. This will have the effect of causing a violation of the boundary condition which requires that the components of the metric tensor have continuous derivatives at the boundary. Nevertheless, for illustrative purposes we determine the constants of integration $A$ and $B$ by requiring that the components of the metric tensor given by the line element (8.11) take on the values of the metric tensor given by Eq. (7.18) at the boundary. The latter equation may be written as

$$ds^2=\frac{x_0^2}{c^2}e^{-H}d\tau^2-\frac{x_0^2}{c^2}\frac{1}{C^2}e^{3H}dH^2-\frac{e^{2H}}{c^2}(dy^2+dz^2). \tag{8.12}$$

If we normalize the variable $H$ so that $H=H_0$ is the value of $H$ at the boundary then at the boundary we must have

$$2\phi(H_0)=-H_0=2\phi_0,\quad Y(H_0)=Ce^{-\frac{3}{2}H_0}. \tag{8.13}$$

We may of course choose $H_0=0$. Then

$$\phi(0)=\phi_0,\quad Y^2(0)=C^2. \tag{8.14}$$

It is a consequence of (8.8) and the first of (8.14) that we must have

$$\phi=-\frac{1}{(n-2)}\left[(n+2)H+\log\left(\frac{Ae^{-\frac{1}{2}(6+n)H}-1}{A-1}\right)\right]. \tag{8.15}$$

It then follows from Eq. (8.10) that

$$\begin{aligned}Y^2(0)&=p_0(1+\delta)^n u(0)e^{-n\phi_0}\\ &=p_0\delta^n\left(\frac{n-2}{n+6}\right)(A-1)=C^2.\end{aligned} \tag{8.16}$$

## 9. INCOMPRESSIBLE CASE—STATIC SOLUTIONS

In this case, Eq. (2.28) holds and as a result Eq. (8.2) may be written as

$$dY/dH=-(\delta+3Y^2)/2Y.$$

The solution of this equation is given by

$$Y^2=\tfrac{1}{3}\delta(A^2e^{-3H}-1), \tag{9.1}$$

where $A$ is a constant of integration whose evaluation will be discussed later.

Equation (8.11) may be written as

$$\frac{d\phi}{dH}=\frac{\delta(e^{\phi_0-\phi}-1)}{2Y^2}-\tfrac{1}{2}, \tag{9.2}$$

where $Y^2$ is given as a function of $H$ by Eq. (9.1). Multiplying this equation by $e^{\phi}$ and setting

$$\Phi=e^{\phi}-e^{\phi_0}, \tag{9.3}$$

we have

$$\frac{d\Phi}{dH}=-\tfrac{1}{2}\left(\frac{\delta}{Y^2}+1\right)-\tfrac{1}{2}e^{\phi_0}.$$

The solution to this equation satisfying

$$\Phi(H_0)=\Phi(-2\phi_0)=0 \tag{9.4}$$

is

$$\Phi(H)=-\tfrac{1}{2}e^{\phi_0}e^{-\frac{1}{2}H}(A^2-e^{3H})^{\frac{1}{2}}\int_{-2\phi_0}^{H}\frac{e^{u/2}du}{(A^2-e^{3u})^{\frac{1}{2}}}. \tag{9.5}$$

Equations (9.4) are consequences of Eq. (9.3) and the first of Eqs. (8.13) which define $H_0$ as that value of $H$ so that $\phi=\phi_0$ and further state that $H_0=-2\phi_0$.

The region occupied by the incompressible fluid is represented by the range of variation of the variable $H$. This range must be such that given by Eq. (9.5) is negative and $Y^2$ given by (9.1) is positive. Both of these conditions will be satisfied if we choose the constant of integration such that

$$A^2\geq 1 \tag{9.6}$$

and restrict $H$ to the range

$$-2\phi_0\leq H\leq\tfrac{2}{3}\log A. \tag{9.7}$$

The total mass present will be given by

$$\int\rho_0\surd(-g)u^4dx^1dx^2dx^3.$$

This quantity will be proportional to the number

$$\begin{aligned}M&=\int_{-2\phi_0}^{\frac{2}{3}\log A}\frac{e^{2H}}{Y}dH=\int_{-2\phi_0}^{\frac{2}{3}\log A}\frac{e^{2H}dH}{(A^2e^{-3H}-1)^{\frac{1}{2}}}\\ &=\tfrac{2}{3}\int_{-2\phi_0}^{\frac{2}{3}\log A}\frac{e^{2H}d(e^{2H/3})}{(A^2-e^{3H})^{\frac{1}{2}}},\end{aligned}$$

or

$$M=\tfrac{2}{3}A^{4/3}\int_{\omega_0}^{\pi/2}\sin^{4/3}\omega d\omega, \tag{9.8}$$

where

$$e^{\frac{3}{2}H}=A\sin\omega \tag{9.9}$$

and

$$\sin\omega_0 = \frac{e^{-3\phi_0}}{A} = 1 \Big/ \left(1+\frac{1}{\delta}\right)^3 A. \qquad (9.10)$$

Equation (9.8) serves to determine $A$ as a function of $M$.

If $H$ is in the range given by (9.7), then it follows from Eq. (9.5) that $\Phi(H)$ is negative and hence $p(H)$ is positive as must be the case. We may write Eq. (9.5) as

$$-e^{\phi_0}F(\omega) = \Phi = -\frac{e^{\phi_0}}{2}\frac{\cos\omega}{\sin^{\frac{1}{3}}\omega}\int_{\omega_0}^{\omega}\frac{d\omega'}{\sin^{\frac{2}{3}}\omega'};$$

$$0 \le \omega_0 \le \omega \le \frac{\pi}{2}. \qquad (9.11)$$

The pressure is then given by

$$p = \rho_0 c^2\left(\frac{1}{1-F} - 1\right). \qquad (9.12)$$

We have thus determined a solution of the field equations for an incompressible fluid which have the property that the pressure vanishes for the beginning of the range given by the inequalities (9.7) and is positive throughout this range. However, this solution cannot be joined to the solution for empty space-time at the point $H = -2\phi_0$. The reason for this is evident from the differential equations satisfied by $Y$ on both sides of the point $H = -2\phi_0$. For the empty region, we have

$$dY/dH = -\tfrac{3}{2}Y,$$

whereas for the region occupied by the incompressible fluid we have the equation before (9.1). Since $\delta \neq 0$, $dY/dH$ cannot be continuous across the boundary and hence we cannot satisfy the boundary condition that the derivatives of the metric tensor be continuous across the boundary. We then have the result:

An incompressible fluid cannot bound a vacuum in a space-time with plane symmetry unless the boundary condition of the continuity of the derivatives of the metric tensor is violated. As is well known, a similar result holds in the spherically symmetric case.

Equation (9.1) is the general solution of the field equations for $Y(H)$. The general solution of the field equations for $\Phi(H)$ is obtained by adding a constant to the right hand side of Eq. (9.5). The constant of integration occurring in Eqs. (9.1) and the modified (9.5) may be evaluated by fitting these solutions to other solutions of the field equations, say to solutions corresponding to another gas.

If the same boundary condition is not to be violated we must have that the equation

$$p_1(\phi_1) = p_0\delta(e^{\phi_0-\phi_1} - 1)$$

must imply the equation

$$(p_1 + p_1')_{\phi=\phi_1} = -\delta p_0,$$

where $p_1(\phi)$ is the pressure as a function of $\phi$ for the second gas and $\phi = \phi_1$ corresponds to the boundary between the two gases.

## 10. SPATIALLY INDEPENDENT SOLUTIONS

In this case the functions $\phi$ and $H$ are independent of the variable $\xi$ and the field equations (5.6), (5.7), and (5.15) reduce to

$$e^{-2\phi}(2H_{\tau\tau} + 3H_\tau^2 - 2H_\tau\phi_\tau) = -p/p_0 \qquad (10.1)$$

and

$$e^{-2\phi}H_\tau[3H_\tau + 2(\rho'/\rho)\phi_\tau] = (p+p')/p_0. \qquad (10.2)$$

We define

$$Y(\phi) = e^{-\phi}H_\tau. \qquad (10.3)$$

Then

$$e^{-\phi}(H_{\tau\tau} - \phi_\tau H_\tau) = Y'\phi_\tau,$$

where the prime denotes the derivative with respect to $\phi$. Equations (10.1) and (10.2) may then be written as

$$2e^{-\phi}Y'\phi_\tau + 3Y^2 = -p/p_0 \qquad (10.4)$$

and

$$3Y^2 + 2Y(\rho'/\rho)e^{-\phi}\phi_\tau = (p+p'/p_0. \qquad (10.5)$$

Eliminating $e^{-\phi}\phi_\tau$ from these two equations, we obtain

$$Y' = -Y\frac{\rho'}{\rho}\left(\frac{p}{p_0} + 3Y^2\right) \Big/ \left(\frac{p+p'}{p_0} - 3Y^2\right) \qquad (10.6)$$

and

$$2e^{-\phi}\phi_\tau = \frac{\rho}{\rho' Y}\left(\frac{p+p'}{p_0} - 3Y^2\right). \qquad (10.7)$$

Equation (10.6) is an Abel differential equation for the variable $Y^2$. When it is solved for $Y = Y(\phi)$ we may substitute its solution into Eq. (10.7) and determine $\phi(\tau)$. We shall illustrate the method of dealing with these equations for the special case of the degenerate gas where

$$p/p_0 = e^{-4\phi}$$

and

$$\rho/\rho_0 = e^{-3\phi}.$$

Equation (10.6) then becomes

$$Y' = -Y(e^{-4\phi} + 3Y^2)/(e^{-4\phi} + Y^2).$$

If we define

$$T = (e^{2\phi}Y)^2,$$

the differential equation becomes

$$T' = 4T + 2e^{4\phi}YY' = 4T - 2T(1+3T)/(1+T)$$

that is,

$$T' = 2T(1-T)/(1+T).$$

Hence $T$ is defined as a function of $\phi$ by the equation

$$e^{2\phi} = AT/(1-T)^2, \qquad (10.8)$$

where $A$ is a constant of integration.

Equation (10.7) may be written as

$$2e^{-\phi}\frac{d\phi}{d\tau}=\frac{1}{Y}(e^{-4\phi}+Y^2)$$

or as

$$e^{\phi}d\tau=\frac{2T^{\frac{1}{2}}}{1+T}e^{2\phi}d\phi=\frac{AT^{\frac{1}{2}}}{(1-T)^3}dT. \quad (10.9)$$

The last equation follows from Eq. (10.8).

The function $H$ may be determined as a function of $T$ from Eq. (10.3) which may be written as

$$H=\int e^{\phi}Yd\tau=\int\frac{AT^{\frac{1}{2}}}{(1-T)^3}YdT$$
$$=\int\frac{dT}{1-T}=-\log\frac{B}{1-T}, \quad (10.10)$$

where $B$ is another constant of integration.

The line element is given by

$$ds^2=\frac{x_0^2}{c^2}e^{2\phi}d\tau^2-\frac{x_0^2}{c^2}e^{-4H}\left(\frac{\rho_0}{\rho}\right)^2d\xi^2-\frac{1}{c^2}e^{2H}(dy^2+dz^2).$$

In view of Eqs. (10.9) and (10.8), this may be written as

$$ds^2=\frac{x_0^2A^2T}{c^2(1-T)^6}dT^2-\frac{x_0^2}{c^2}\frac{A^3T^3d\xi^2}{B^4(1-T)^2}$$
$$-\frac{1}{c^2}\frac{B^2}{(1-T)^2}(dy^2+dz^2). \quad (10.11)$$

In deriving Eq. (10.11), we have used Eqs. (10.8), (10.9), (10.10), and the fact that

$$(\rho_0/\rho)^2=e^{6\phi}=A^3T^3/(1-T)^6$$

for the degenerate gas. The pressure distribution is given by

$$p/p_0=e^{-4\phi}=(1-T)^4/A^2T^2. \quad (10.12)$$

## 11. INCOMPRESSIBLE CASE—TIME-DEPENDENT SOLUTIONS

We may now turn our attention to the second type of boundary conditions discussed in Sec. 6. The problems in which these conditions occur involve time dependent solutions of the system of equations consisting of Eqs. (5.5) to (5.7) and the fourth of Eq. (5.4). We shall confine our discussion of these equations to the incompressible case. Then by introducing the variable $\eta$ used at the end of Sec. 5, we may write Eq. (5.13) as

$$G=-2H. \quad (11.1)$$

Equations (5.6), (5.7) and the fourth of Eqs. (5.4) become

$$e^{-2\phi}(2H_{\tau\tau}+3H_\tau^2-2H_\tau\phi_\tau)-e^{4H}H_\eta(H_\eta+2\phi_\eta)$$
$$=-p/p_0=-\delta(e^{\phi_0-\phi}-1), \quad (11.2)$$

$$-3e^{-2\phi}H_\tau^2-e^{4H}(2H_{\eta\eta}+7H_\eta^2)=-(p+p')/p_0=\delta \quad (11.3)$$

$$2e^{-2\phi}(H_{\eta\tau}-H_\tau\phi_\eta+3H_\tau H_\eta)=0, \quad (11.4)$$

respectively, where we have used Eqs. (2.18) to (2.22) and Eq. (2.28).

Equation (11.4) may be written as

$$H_{\eta\tau}+3H_\eta H_\tau=H_\tau\phi_\eta.$$

Hence

$$(e^{3H})_\tau=\sqrt{3}\alpha(\tau)e^{\phi}, \quad (11.5)$$

where $\alpha(\tau)$ is an arbitrary function of its argument.

Equation (11.5) may be integrated further to give

$$e^{3H}=\int\sqrt{3}\alpha(\tau)e^{\phi}d\tau+\beta(\eta), \quad (11.6)$$

where $\beta(\eta)$ is an arbitrary function of its argument.

Substituting from Eq. (11.5) into (11.3), we have

$$\alpha^2e^{-6H}+e^{4H}(2H_{\eta\eta}+7H_\eta^2)=-\delta. \quad (11.7)$$

This equation may be considered as an ordinary differential equation for $H$ provided the constants of integration are functions of $\tau$.

A first integral of equation (11.7) is given by

$$H_\eta^2=-\tfrac{1}{3}\delta e^{-4H}+\tfrac{1}{3}\alpha^2e^{-10H}+\tfrac{1}{3}\epsilon(\tau)e^{-7H}, \quad (11.8)$$

where $\epsilon(\tau)$ is an arbitrary function of its argument, as may be verified by differentiation. If we set

$$h=e^{3H},$$

Equation (11.8) may be written as

$$h_\eta^2=3h^{-4/3}\left[\left(\alpha^2+\frac{1}{4}\frac{\epsilon^2}{\delta}\right)-\delta\left(h-\frac{1}{2\delta}\epsilon\right)^2\right] \quad (11.9)$$

or as

$$F(h,\tau)\equiv\int_{h_0}^{h}\frac{h^{2/3}dh}{\left\{3\left[\left(\alpha^2+\frac{1}{4}\frac{\epsilon^2}{\delta}\right)-\delta\left(h-\frac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]\right\}^{\frac{1}{2}}}$$
$$=\eta-\eta_0, \quad (11.10)$$

where $\tau$ is kept constant in the integration and $h_0(\tau)$ is the value of $h$ at $\eta=\eta_0$.

Equation (11.5) may be written as

$$h_\tau=\sqrt{3}\alpha(\tau)e^{\phi}. \quad (11.11)$$

We must now examine the remaining field equation (11.2). Subtracting Eq. (11.3) from (11.2) one obtains

$$\tfrac{2}{3}e^{-2\phi-3H}[(e^{3H})_{\tau\tau}-(e^{3H})_\tau\phi_\tau]+\tfrac{2}{3}e^H[(e^{3H})_{\eta\eta}-(e^{3H})_\eta\phi_\eta]$$
$$=-\delta e^{\phi_0-\phi}.$$

Multiplying by $\frac{3}{2}e^{\phi}$ and substituting $h$ for $e^{3H}$, we may write the resulting equation as

$$\frac{1}{h}(e^{-\phi}h_\tau)_\tau+h^{\frac{1}{3}}e^{\phi}(h_{\eta\eta}-h_\eta\phi_\eta)=-\tfrac{3}{2}\delta e^{\phi_0}.$$

This in turn may be written as

$$\frac{1}{h}(\sqrt{3}\alpha_\tau)-h^{\frac{1}{3}}h_\eta^{2}\left(\frac{e^{\phi}}{h_\eta}\right)_\eta=-\tfrac{3}{2}\delta e^{\phi_0}.$$

Substituting from (11.11), we have

$$\frac{1}{h}(\sqrt{3}\alpha_\tau)-\frac{h^{\frac{1}{3}}h_\eta^{2}}{\sqrt{3}\alpha}\left(\frac{h_\tau}{h_\eta}\right)_\eta=-\tfrac{3}{2}\delta e^{\phi_0}. \qquad (11.12)$$

We may evaluate $h_\tau$ from the first of Eqs. (11.10) as

$$h_\tau=-F_\tau/F_h,$$

where the subscript denotes the partial derivative and hence

$$F_h=h^{\frac{2}{3}}\Big/\left\{3\left[\left(\alpha^2+\tfrac{1}{4}\frac{\epsilon^2}{\delta}\right)-\delta\left(h-\tfrac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]\right\}^{\frac{1}{2}}.$$

Thus

$$F_h h_\eta=1$$

and

$$\frac{h_\tau}{h_\eta}=-F_\tau=\frac{h_0^{\frac{2}{3}}h_{0\tau}}{\left\{3\left[\alpha^2+\tfrac{1}{4}\frac{\epsilon^2}{\delta}-\delta\left(h_0-\tfrac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]\right\}^{\frac{1}{2}}}+\int_{h_0}^{h}\frac{h^{\frac{2}{3}}dh(3\alpha\alpha_\tau+\tfrac{3}{2}\epsilon_\tau h)}{\left\{3\left[\alpha^2+\tfrac{1}{4}\frac{\epsilon^2}{\delta}-\delta\left(h-\tfrac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]\right\}^{\frac{3}{2}}}. \qquad (11.13)$$

Hence

$$\left(\frac{h_\tau}{h_\eta}\right)_\eta=\tfrac{3}{2}\frac{h^{\frac{2}{3}}(3\alpha\alpha_\tau+\tfrac{3}{2}\epsilon_\tau h)h^{-\frac{2}{3}}\left\{3\left[\alpha^2+\tfrac{1}{4}\frac{\epsilon^2}{\delta}-\delta\left(h-\tfrac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]\right\}^{\frac{1}{2}}}{\left\{3\left[\alpha^2+\tfrac{1}{4}\frac{\epsilon^2}{\delta}-\delta\left(h-\tfrac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]\right\}^{\frac{3}{2}}}$$

$$=\frac{3\alpha\alpha_\tau+\tfrac{3}{2}\epsilon_\tau h}{3\left[\alpha^2+\tfrac{1}{4}\frac{\epsilon^2}{\delta}-\delta\left(h-\tfrac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]},$$

$$\frac{1}{\sqrt{3}\alpha}h^{\frac{1}{3}}h_\eta^{2}\left(\frac{h_\tau}{h_\eta}\right)_\eta=\frac{1}{h\sqrt{3}\alpha}(3\alpha\alpha_\tau+\tfrac{3}{2}\epsilon_\tau h),$$

$$\frac{1}{h}(\sqrt{3}\alpha_\tau)-\frac{h^{\frac{1}{3}}h_\eta^{2}}{\sqrt{3}\alpha}\left(\frac{h_\tau}{h_\eta}\right)_\eta=-\tfrac{3}{2}\frac{\epsilon_\tau}{\sqrt{3}\alpha}=-\tfrac{3}{2}\delta e^{\phi_0}.$$

Hence, if we choose $\epsilon$ such that

$$\epsilon_\tau=\sqrt{3}\alpha\delta e^{\phi_0}=\sqrt{3}\alpha(1+\delta), \qquad (11.14)$$

Eqs. (11.10) and (11.11) give the general solution to the field equations (11.2) to (11.4) subject to the continuity of mass equation (11.1).

The line element may be written in terms of the function $h(\eta,\tau)$ defined by Eqs. (11.10) as

$$ds^2=\frac{x_0^2h_\tau^2}{c^2 3\alpha^2}d\tau^2-\frac{x_0^2}{c^2}h^{-4/3}d\eta^2-\frac{h^{\frac{2}{3}}}{c^2}(dy^2+dz^2). \qquad (11.15)$$

We may replace the variable $\tau$ by a new variable $\tau'$ such that

$$d\tau'=d\tau/\sqrt{3}\alpha.$$

This is equivalent to choosing $3\alpha(\tau)=1$. Then the line element (11.15) becomes

$$ds^2=\frac{x_0^2}{c^2}h_\tau^2d\tau^2-\frac{x_0^2}{c^2}h^{-4/3}d\eta^2-\frac{h^{\frac{2}{3}}}{c^2}(dy^2+dz^2), \qquad (11.16)$$

where $h(\eta,\tau)$ is given by

$$\int_{h_0}^{h}\frac{h^{\frac{2}{3}}dh}{\left\{3\left[\left(\tfrac{1}{3}+\tfrac{1}{4}\frac{\epsilon^2}{\delta}\right)-\delta\left(h-\tfrac{1}{2}\frac{\epsilon}{\delta}\right)^2\right]\right\}^{\frac{1}{2}}}=\eta-\eta_0 \qquad (11.17)$$

and

$$\epsilon=(1+\delta)(\tau-\tau_0). \qquad (11.18)$$

If in the above discussion we had chosen $\alpha(\tau)=0$ and $\epsilon(\tau)=\epsilon_0$ a constant, we could have obtained a static solution for the incompressible case. Equations (11.11) would no longer determine the function $\phi$. However, this function could be determined from the equation preceding (11.12), namely,

$$h^{\frac{1}{3}}h_\eta^{2}[e^{\phi}/h_\eta]_\eta=\tfrac{3}{2}(1+\delta). \qquad (11.19)$$

Thus

$$\frac{e^{\phi}}{h_\eta}=\tfrac{3}{2}(1+\delta)\int_{\eta_0}^{\eta}\frac{d\eta}{h^{\frac{1}{3}}h_\eta^{2}}+\text{constant},$$

$$\frac{e^{\phi}}{h_\eta}=\tfrac{3}{2}(1+\delta)\int_{h_0'}^{h}\frac{h^{5/3}dh}{\left\{3\left[\left(1+\tfrac{1}{4}\frac{\epsilon_0^2}{\delta}\right)-\delta\left(h-\tfrac{1}{2}\frac{\epsilon_0}{\delta}\right)^2\right]\right\}^{\frac{3}{2}}}+\text{constant}. \qquad (11.20)$$

The right-hand side of Eq. (11.20) is identical with the right-hand side of Eq. (11.13) if $h_0(\tau)$ entering in that equation is such that at $\tau=\tau_0$ the value of the first term is equal to the constant in Eq. (11.20) and if

$$h_0'=h_0(\tau_0). \qquad (11.21)$$

The line element (11.16) which represents the general time-dependent solution for the metric due to the motion of an incompressible fluid depends on one arbitrary function $h_0(\tau)$. If this function is such that the above conditions are satisfied, we may interpret the solution as follows: At time $\tau=\tau_0$ the gas is at rest for $\eta\neq\eta_0$ and in gravitational equilibrium in the sense that the pressure forces balance the gravitational ones. This is the physical interpretation of the fact that at $\tau=\tau_0$ the metric is given by a static solution of the field equations. At $\eta=\eta_0$, $h=h_0(\tau)$. This determines $\phi(\eta_0,\tau)$ via Eqs. (11.11) and (11.13) and $\phi(\eta_0,\tau)$ determines $u^\mu(\eta_0,\tau)$ and $p(\eta_0,\tau)$. These equations reduce to

$$\phi(\eta_0,\tau)=h_{0\tau}(\tau).$$

Since at $\eta=\eta_0$, $h=h_0(\tau)$ and

$$h_\eta^2(\eta_0,\tau)=3h_0^{-\frac{2}{3}}\left[\left(\tfrac{1}{3}+\tfrac{1}{4}\frac{\epsilon^2}{\delta}\right)-\delta\left(h_0-\frac{\epsilon}{2\delta}\right)^2\right].$$

Thus if $\phi(\eta,\tau_0)$ is chosen so that at time $\tau=\tau_0$ we have a solution of the static field equations, then in the incompressible case the specifications of $\phi(\eta_0,\tau)$ determines, aside from a constant of integration, a general time-dependent solution of the field equation. Thus for this case the second type of boundary value problem discussed in Sec. 6 corresponds to the general time-dependent solution of the field equations.

## 12. CONCLUDING REMARKS

It should be pointed out that for the incompressible case discussed in the preceding section, the stress energy tensor used here reduces to that usually treated for an arbitrary perfect fluid. The restriction to isentropic motions implies that the density is constant throughout. The results obtained in Sec. 9 state that the plane-symmetric analog of the Schwarzschild interior solution has the same difficulty with the boundary conditions as that solution.

The results obtained in Sec. 11 give a complete solution of the field equations and the equations of motion of an incompressible fluid in terms of an arbitrary function of time. This function may in turn be determined from conditions obtaining at a plane of particles located at the plane $\eta=\eta_0$. The actual determination of the function $\phi(\eta_0,\tau)$ depends on further specification of the problem. Thus if, as may be the case, the space-time is created by an incompressible fluid bounded by a compressible one and this in turn is bounded by a vacuum, these gases being initially at rest, then the determination of the function $\phi(\eta_0,\tau)$ is accomplished by knowing the space-time on the compressible side of the compressible-incompressible boundary. This knowledge is in turn dependent on obtaining the solution of the field equations for the compressible case. An approximate method for dealing with these equations can be given and will be discussed in a later paper.

We finally remark that although there seem to be some conceptual difficulties in the notion of an incompressible fluid in special relativity, these are not encountered in general relativity if the definitions given above are used. The results of Sec. 11 show that such a fluid may be started from rest and no difficulties comparable with those occurring in the special theory arise