Approximation Method for High-Energy Potential Scattering*

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An approximation method for high-energy potential scattering is developed that expresses the scattered amplitude in terms of a quadrature, similar to the Born approximation but superior to it in accuracy. It is valid when the potential is slowly varying compared to a wavelength, $|V|/E'$ is small compared to unity, θ is either small or large compared to $(kR)^{-\frac{1}{2}}$, and $|V|R/\hbar v$ is unrestricted in magnitude, where E', θ, k, and v are the kinetic energy, scattering angle, wave number, and speed of the scattered particle, and V and R are rough measures of the strength and range of the scattering potential, which may be complex. For comparison, the Born approximation requires that $|V|R/\hbar v$ be small compared to unity. The procedure consists in summing the infinite Born series after approximating each term by the method of stationary phase. Both the Schrödinger and Dirac equations are treated, and it is expected that the method can be extended to the scattering theory of other wave equations. The relation of the present work to previous work of others is discussed, and the limitations of WKB or eikonal-type approximations are explored. The method is expected to be especially useful for calculating the scattering of fast electrons, neutrons, and protons from nonspherical nuclei.

I. INTRODUCTION

ACCURATE calculations of potential scattering at high energies have been based on the method of partial waves.[1] This method can only be used when the potential is spherically symmetric, and becomes very laborious when the reduced wavelength of the scattered particle is somewhat smaller than the range of the potential. It would be desirable to find an expression for the scattered amplitude in terms of a quadrature, similar to the Born approximation but superior to it in accuracy. One would hope that such a closed form would be better adapted to numerical computation than the partial wave series, but in any event it could be applied to nonspherical scattering potentials. The present investigation was directly stimulated by this last point in connection with the experiments of Hofstadter and collaborators[2] on the scattering of high-energy electrons from heavy nuclei, since it is known that such nuclei often depart significantly from spherical symmetry, and that electron scattering from them cannot be treated reliably by the Born approximation.[3]

As is well known, the Born approximation is expected to be reliable when the true wave function in the scattering region does not differ significantly from the incident plane wave. We shall be concerned with potentials that do not change by an appreciable fraction of their value in a reduced wavelength, that is, with high incident energies and finite, continuous potentials. A rough approximation to the true wave function in the scattering region can then be obtained from a kind of WKB approach, in the following way. The classical trajectories are nearly straight lines in the direction of the positive z axis, so that the situation is approximately one-dimensional. Then the phase of the wave function differs from that of the incident plane wave by

$$\int_{-\infty}^{z}[\kappa(x,y,z')-k]dz', \tag{1}$$

where

$$\hbar^2c^2k^2+m^2c^4=E^2,\quad \hbar^2c^2\kappa^2+m^2c^4=(E-V)^2.$$

Here E and m are the total energy and rest mass of the incident particle, and V is the scattering potential energy. Then if $|\kappa-k|\ll k$, which is the same as the condition $|V|\ll(E-mc^2)$, Eq. (1) is equivalent to

$$-(\hbar v)^{-1}\int_{-\infty}^{z}V(x,y,z')dz',$$

where v is the speed of the particle. The incident plane wave $\exp(ikz)$ must then be replaced by

$$\exp i\left[kz-(\hbar v)^{-1}\int_{-\infty}^{z}V(x,y,z')dz'\right]. \tag{2}$$

An amplitude change of order $(\kappa-k)/k$ has been neglected here.[4]

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[1] For electron scattering, see Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954); for proton scattering, see Melkanoff, Nodvik, and Saxon, Phys. Rev. **100**, 1805 (1955); for neutron scattering, see Culler, Fernbach, and Sherman, Phys. Rev. **101**, 1047 (1956).

[2] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 1131 (1956).

[3] A useful approximation based on a modified plane wave has been described by Yennie, Ravenhall, and Downs, Phys. Rev. **98**, 277 (1955).

[4] Equation (2) or its equivalent appears in a number of earlier papers: G. Moliere, Z. Naturforsch. **2**, 133 (1947); G. Parzen, Phys. Rev. **81**, 808 (1951); E. W. Montroll and J. M. Greenberg, Proceedings of the Symposium on Applied Mathematics **5**, 103 (1954); B. J. Malenka, Phys. Rev. **95**, 522 (1954); I. I. Gol'dman and A. B. Migdal, J. Exptl. Theoret. Phys. U.S.S.R. **28**, 394 (1954) or Soviet Physics—JETP **1**, 304 (1955). Montroll and Greenberg attribute it to Moliere and to R. J. Glauber, Phys. Rev. **91**, 459 (1953); Malenka attributes it to Glauber; Gol'dman and Migdal attribute it to L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Part I, p. 184 (1948).

Thus we expect the Born approximation to be useful at high energies when

$$(\hbar v)^{-1}\left|\int_{-\infty}^{z} V(x,y,z')dz'\right| \ll 1, \tag{3}$$

throughout the scattering region.[5] In the nonrelativistic case, this condition is always satisfied at sufficiently high energy since v can be made as large as desired. On the other hand, v cannot exceed c in the relativistic case, so that for potentials that are large enough in magnitude or range, the Born approximation is never valid. As an example, we expect that high-energy electron scattering from nuclei can be treated by Born approximation if $(Ze^2/\hbar c)\ln\beta \ll 1$, where β is the ratio of the atomic screening radius to the nuclear radius. However, this criterion may be too severe, for we need only require that the *change* of the phase difference between Eq. (2) and the plane wave throughout the principal scattering region be small in order for the Born approximation to be valid, since the constant phase difference is of no significance in potential scattering. This leads to the validity criterion $(Ze^2/\hbar c) \ll 1$.

We shall derive an approximate expression for the scattered amplitude that is valid when $ka \gg 1$, $|V| \ll (E-mc^2)$, and the quantity which appears on the left side of Eq. (3) is unrestricted in magnitude. Here, a is the distance over which the potential changes by an appreciable fraction of itself. Our procedure will consist in summing the infinite Born series after approximating each term by the method of stationary phase. This will be justified if the angle of scattering θ is either somewhat larger or somewhat smaller than $(kR)^{-\frac{1}{2}}$; the form of the calculation is different in the two cases.[6] Here, R is the range of that part of the potential which contributes appreciably to the scattering; in the Coulomb case, it is probably more nearly the nuclear radius than the atomic screening radius. We shall generally assume that R and a are of the same order of magnitude.

This derivation for the Schrödinger case is given in Sec. II, and for the Dirac case in Sec. III. Sections IV and V are devoted to a more detailed study of the Schrödinger case, in order to understand the limitations of the present procedure and to clarify its relationship with earlier work. The wave function is found in Sec. IV, and the WKB or eikonal approximation is examined in Sec. V.

[5] A condition equivalent to Eq. (3) was derived for the partial wave analysis by G. Parzen, Phys. Rev. **80**, 261 (1950).

[6] A somewhat similar procedure has been employed by Glauber and Malenka (reference 4), and by I. I. Shapiro, thesis, Harvard University, May, 1955 (unpublished). Rather than use the stationary phase approximation explicitly on an integral like Eq. (7) below, they approximate the Green's function by itself in a way that is only justified if it is later to be multiplied by a plane wave and integrated over as in Eq. (7). The final results obtained by them are only correct for small angles of scattering (see Sec. II B below).

II. SCHRÖDINGER SCATTERED AMPLITUDE

We wish to find a solution of the Schrödinger wave equation

$$(\nabla^2+k^2-U)\psi=0, \tag{4}$$

where $\hbar^2k^2/2m$ is the kinetic energy of the incident particle and $\hbar^2U/2m=V$ is the scattering potential energy, with the asymptotic form

$$\psi \to \exp(i\mathbf{k}_0\cdot\mathbf{r})+r^{-1}\exp(ikr)f(\mathbf{k}_f,\mathbf{k}_0).$$

Here, $\mathbf{k}_0$ is a vector of magnitude k along the direction of incidence, $\mathbf{k}_f$ is a vector of magnitude k along the direction of observation, and f is the scattered amplitude. As is well known, f can be expressed as the infinite Born series

$$\begin{aligned} f(\mathbf{k}_f,\mathbf{k}_0)=\sum_{n=1}^{\infty}\int\cdots\int \exp(-i\mathbf{k}_f\cdot\mathbf{r}_n)U(\mathbf{r}_n)G(\mathbf{r}_n-\mathbf{r}_{n-1})\\ \times U(\mathbf{r}_{n-1})G(\mathbf{r}_{n-1}-\mathbf{r}_{n-2})U(\mathbf{r}_{n-2})\cdots U(\mathbf{r}_2)\\ \times G(\mathbf{r}_2-\mathbf{r}_1)U(\mathbf{r}_1)\exp(i\mathbf{k}_0\cdot\mathbf{r}_1)d\tau_1\cdots d\tau_n, \end{aligned} \tag{5}$$

where G is the outgoing wave Green's function for the operator (∇^2+k^2)

$$G(\boldsymbol{\varrho})=(-4\pi\rho)^{-1}\exp(ik\rho). \tag{6}$$

We wish to take advantage of the assumed slow variation of U over distances of order k^{-1}. This can be done by rearranging the exponents of the initial and final plane waves and all the Green's functions so that the regions of the integrand which are not oscillating rapidly are explicitly exhibited. In this connection consider the quantity $\exp[i(k\rho-\mathbf{k}\cdot\boldsymbol{\varrho})]$ as a function of $\boldsymbol{\varrho}$. For a particular value of the magnitude ρ of $\boldsymbol{\varrho}$, when it is assumed to be somewhat larger than k^{-1}, this exponential oscillates rapidly as the direction of $\boldsymbol{\varrho}$ is changed, except when $\boldsymbol{\varrho}$ is nearly parallel to $\mathbf{k}$. The phase of the exponential is stationary when $\boldsymbol{\varrho}$ is parallel to $\mathbf{k}$, and the approximation procedure based on this property is called the method of stationary phase.[7] It follows that most of the contribution to the integral

$$I=\int \rho^{-1}g(\boldsymbol{\varrho})\exp[i(k\rho-\mathbf{k}\cdot\boldsymbol{\varrho})]d\tau \tag{7}$$

comes either from a region shaped like a paraboloid of revolution about the direction of $\mathbf{k}$ with vertex at the origin and radius of order $(\rho/k)^{\frac{1}{2}}$ perpendicular to its axis, or else from a spherical region with radius of order k^{-1} about the origin. It is shown in Appendix A that for large k

$$I=(2\pi i/k)\int_0^{\infty} g(\hat{k}\rho)d\rho+O(k^{-2}), \tag{8}$$

where $\hat{k}$ is a unit vector parallel to $\mathbf{k}$.

[7] The method of stationary phase is discussed by C. Eckart, Revs. Modern Phys. **20**, 399 (1948), who gives references to earlier work.

A. Large Scattering Angle

We define vectors $\boldsymbol{\varrho}_1=\mathbf{r}_2-\mathbf{r}_1$, $\boldsymbol{\varrho}_2=\mathbf{r}_3-\mathbf{r}_2$, $\cdots\boldsymbol{\varrho}_{n-1}=\mathbf{r}_n-\mathbf{r}_{n-1}$. In terms of these we can write

$$-\mathbf{k}_f\cdot\mathbf{r}_n+\mathbf{k}_0\cdot\mathbf{r}_1=-\mathbf{k}_f\cdot\boldsymbol{\varrho}_{n-1}-\mathbf{k}_f\cdot\boldsymbol{\varrho}_{n-2}-\cdots-\mathbf{k}_f\cdot\boldsymbol{\varrho}_m+\mathbf{q}\cdot\mathbf{r}_m-\mathbf{k}_0\cdot\boldsymbol{\varrho}_{m-1}-\cdots-\mathbf{k}_0\cdot\boldsymbol{\varrho}_2-\mathbf{k}_0\cdot\boldsymbol{\varrho}_1, \quad (9)$$

where $\mathbf{q}=\mathbf{k}_0-\mathbf{k}_f$ and m can be any of the integers $1, 2, \cdots n$. The Jacobian of the transformation from $\mathbf{r}_1, \mathbf{r}_2, \cdots\mathbf{r}_n$ to $\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2, \cdots\boldsymbol{\varrho}_{n-1}, \mathbf{r}_m$ is unity, and it is easily seen that the infinite limits on the first set of variables transform into infinite limits on the second set. Substitution of Eqs. (6) and (9) into (5) yields a sequence of integrals over $\boldsymbol{\varrho}_1, \cdots\boldsymbol{\varrho}_{n-1}$, each of which is of the form (7), where the functions g involve products of U's. Application of the leading term of Eq. (8) to these $n-1$ integrations yields

$$f(\mathbf{k}_f,\mathbf{k}_0)=(-4\pi)^{-1}\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-i/2k)^{n-1}\times\int d\tau_m\int_0^\infty d\rho_1\cdots\int_0^\infty d\rho_{n-1}\exp(i\mathbf{q}\cdot\mathbf{r}_m)$$
$$\times U[\mathbf{r}_m+\hat{k}_f(\rho_{n-1}+\cdots+\rho_m)]\cdots U(\mathbf{r}_m+\hat{k}_f\rho_m)U(\mathbf{r}_m)$$
$$\times U(\mathbf{r}_m-\hat{k}_0\rho_{m-1})\cdots U[\mathbf{r}_m-\hat{k}_0(\rho_{m-1}+\cdots+\rho_1)]. \quad (10)$$

To simplify the integrations over $\rho_1, \cdots\rho_{m-1}$ we introduce new variables

$$s_1=\rho_1+\cdots+\rho_{m-1}=\rho_1+s_2,\quad s_2=\rho_2+\cdots+\rho_{m-1}=\rho_2+s_3,\quad\cdots,\quad s_{m-1}=\rho_{m-1}.$$

Then since the Jacobian is unity,

$$\int_0^\infty d\rho_1\cdots\int_0^\infty d\rho_{m-1}\times U(\mathbf{r}_m-\hat{k}_0\rho_{m-1})\cdots U[\mathbf{r}_m-\hat{k}_0(\rho_{m-1}+\cdots+\rho_1)]$$
$$=\int_0^\infty ds_{m-1}\int_{s_{m-1}}^\infty ds_{m-2}\cdots\int_{s_2}^\infty ds_1\times U(\mathbf{r}_m-\hat{k}_0 s_{m-1})\cdots U(\mathbf{r}_m-\hat{k}_0 s_1).$$

We now define another set of variables

$$W_l=\int_{s_l}^\infty U(\mathbf{r}_m-\hat{k}_0 s)ds,\quad l=1, 2, \cdots m-1,$$

and rewrite the last multiple integral as

$$\int_0^{W_0}dW_{m-1}\int_0^{W_{m-1}}dW_{m-2}\cdots\int_0^{W_3}dW_2\int_0^{W_2}dW_1,$$

where

$$W_0=\int_0^\infty U(\mathbf{r}_m-\hat{k}_0 s)ds.$$

This is easily evaluated to be

$$W_0^{m-1}/(m-1)!=[(m-1)!]^{-1}\left[\int_0^\infty U(\mathbf{r}_m-\hat{k}_0 s)ds\right]^{m-1}. \quad (11)$$

In similar fashion we introduce new variables

$$s_m=\rho_m,\quad s_{m+1}=\rho_m+\rho_{m+1}=s_m+\rho_{m+1},\quad\cdots s_{n-1}=\rho_m+\cdots+\rho_{n-1}=s_{n-2}+\rho_{n-1},$$

and find that

$$\int_0^\infty d\rho_m\cdots\int_0^\infty d\rho_{n-1}\times U[\mathbf{r}_m+\hat{k}_f(\rho_{n-1}+\cdots+\rho_m)]\cdots U(\mathbf{r}_m+\hat{k}_f\rho_m)$$
$$=\int_0^\infty ds_m\int_{s_m}^\infty ds_{m+1}\cdots\int_{s_{n-2}}^\infty ds_{n-1}\times U(\mathbf{r}_m+\hat{k}_f s_{n-1})\cdots U(\mathbf{r}_m+\hat{k}_f s_m)$$
$$=[(n-m)!]^{-1}\left[\int_0^\infty U(\mathbf{r}_m+\hat{k}_f s)ds\right]^{n-m}. \quad (12)$$

When (11) and (12) are substituted into Eq. (10), the summations over n and m become trivial, and we obtain

$$f(\mathbf{k}_f,\mathbf{k}_0)=(-4\pi)^{-1}\int\exp(i\mathbf{q}\cdot\mathbf{r})U(\mathbf{r})\times\exp\left\{(-i/2k)\left[\int_0^\infty U(\mathbf{r}-\hat{k}_0 s)ds+\int_0^\infty U(\mathbf{r}+\hat{k}_f s)ds\right]\right\}d\tau. \quad (13)$$

Equation (13) is our approximation for the Schrödinger scattered amplitude. It is valid so long as (a) the higher-order terms in Eq. (8) can be neglected, and (b) the stationary phase regions that correspond to the n different values of m are actually distinct from each other, since these n contributions were added together to obtain the nth term of Eq. (10). In regard to (a), we see from Eq. (A.2) of Appendix A that the ratio of the second to the leading term of I is of order R/ka^2, where a is the distance over which the potential changes by an appreciable fraction of itself and R is the range of that part of the potential which contributes appreciably to the scattering. In most cases a and R are of the same order of magnitude, and even in the Coulomb case R is not expected to be much larger than a. Thus (a) requires in the first place that kR be substantially larger than unity. Further, the leading term of I gives rise to the curly bracket term in the exponent of Eq. (13), which is of order $UR/k=2VR/\hbar v$. We therefore expect inclusion of the second term of I to change the exponent by an additive amount of order $(2VR/\hbar v)\times(R/ka^2)=VR^2/E'a^2$, where E' is the kinetic energy of the incident particle, and thus change f by a multiplying factor of this order. Thus in the second place, (a) is roughly equivalent to the requirement that the scattering potential energy be small in comparison with the kinetic energy of the incident particle.

In regard to (b), we note from the discussion of Eq. (7) and Appendix A that the angle by which $\boldsymbol{\varrho}$ can deviate from $\mathbf{k}$ and still lie within the stationary phase region is of order $(k\rho)^{-\frac{1}{2}}$, which we can replace in order of magnitude by $(kR)^{-\frac{1}{2}}$. Now the n stationary phase regions that are added together to obtain the nth term of Eq. (10) differ from each other in that each of the vectors $\boldsymbol{\varrho}_1, \cdots \boldsymbol{\varrho}_{n-1}$ is nearly parallel to either $\mathbf{k}_0$ or $\mathbf{k}_f$. Unless the angle θ between $\mathbf{k}_0$ and $\mathbf{k}_f$ is substantially larger than $(kR)^{-\frac{1}{2}}$, these n regions are not distinct. Thus (b) is equivalent to the requirement that the scattering angle θ be large in comparison with $(kR)^{-\frac{1}{2}}$.

It is interesting to note that Eq. (13) satisfies two general symmetry requirements.[8] The first of these is $f(\mathbf{k}_f,\mathbf{k}_0)=f(-\mathbf{k}_0,-\mathbf{k}_f)$, which expresses the reversibility of the scattered amplitude between any pair of directions; it is easily verified when it is remembered that $\mathbf{q}=\mathbf{k}_0-\mathbf{k}_f$. The second of these is $f(\mathbf{k}_f,\mathbf{k}_0)=f(\mathbf{k}_0,\mathbf{k}_f)$, and is valid only when the scattering potential is symmetric with respect to inversion: $V(\mathbf{r})=V(-\mathbf{r})$. It can be verified by interchanging $\mathbf{k}_0$ and $\mathbf{k}_f$ in Eq. (13), replacing $\mathbf{r}$ by $-\mathbf{r}$ as the variable of integration, and making use of the assumed inversion symmetry.

B. Small Scattering Angle

When θ is substantially smaller than $(kR)^{-\frac{1}{2}}$, the different stationary phase regions coalesce into a single one for each value of n. We choose the z axis in the direction of $\mathbf{k}_0$; then $q_z\approx\frac{1}{2}k\theta^2$ is small in comparison with $1/2R$, so we replace $\exp(iq_zz_m)$ by unity. The integrals that appear in Eqs. (11) and (12) may be written

$$\int_0^\infty U(\mathbf{r}_m-\hat{k}_0s)ds=\int_{-\infty}^{z_m}U(x_m,y_m,z)dz,$$

$$\int_0^\infty U(\mathbf{r}_m+\hat{k}_fs)ds=\int_{z_m}^{\infty}U(x_m,y_m,z)dz,$$

since $\mathbf{k}_f$ is very nearly parallel to the z axis. Equation (10) then becomes

$$f(\mathbf{k}_f,\mathbf{k}_0)=(-4\pi)^{-1}\sum_{n=1}^{\infty}(-i/2k)^{n-1}$$
$$\times\int_{-\infty}^{\infty}dx_m\int_{-\infty}^{\infty}dy_m\exp[i(q_xx_m+q_yy_m)]$$
$$\times\int_{-\infty}^{\infty}dz_mU(x_m,y_m,z_m)[(m-1)!(n-m)!]^{-1}$$
$$\times\left[\int_{-\infty}^{z_m}U(x_m,y_m,z)dz\right]^{m-1}\left[\int_{z_m}^{\infty}U(x_m,y_m,z)dz\right]^{n-m}.$$

The factor $\exp[i(q_xx_m+q_yy_m)]$ cannot be replaced by unity since the transverse components of $\mathbf{q}$ are of order $k\theta$ and hence can be considerably larger than $1/R$. If now we substitute

$$w=\int_{-\infty}^{z_m}U(x_m,y_m,z)dz,\quad \alpha=\int_{-\infty}^{\infty}U(x_m,y_m,z)dz$$

into the integration over z_m, it becomes

$$\int_0^\alpha[(m-1)!(n-m)!]^{-1}w^{m-1}(\alpha-w)^{n-m}dw=\alpha^n/n!,$$

and so is independent of m, as of course it must be. The summation over n is now trivial, and the scattered amplitude becomes

$$f(\mathbf{k}_f,\mathbf{k}_0)=(ik/2\pi)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp i(q_xx+q_yy)$$
$$\times\left\{1-\exp\left[(-i/2k)\int_{-\infty}^{\infty}U(x,y,z)dz\right]\right\}dxdy.\quad(14)$$

Equation (14) is our approximation for the Schrödinger scattered amplitude when θ is substantially smaller than $(kR)^{-\frac{1}{2}}$, and also $1/kR$ and V/E' are small compared to unity. It appears in some of the earlier work[6] as a small-angle approximation, although no discussion of its validity appears to have been given previously. Equation (14) has the form of an integral over impact parameters, denoted by their coordinates x and y on a plane perpendicular to the direction of incidence. If the scattering potential is axially symmetric with axis parallel to the direction of incidence, the variables x, y can be replaced by b, ϕ, where $b=(x^2+y^2)^{\frac{1}{2}}$ is the magnitude of the impact parameter. The integration over ϕ can then be performed at once to give

$$f(\mathbf{k}_f,\mathbf{k}_0)=ik\int_0^\infty J_0(qb)$$
$$\times\left\{1-\exp\left[(-i/2k)\int_{-\infty}^{\infty}U(b,z)dz\right]\right\}bdb.\quad(15)$$

[8] R. Glauber and V. Schomaker, Phys. Rev. **89**, 667 (1953).

We have been unable to obtain a result analogous to Eqs. (13) and (14) that is valid when $\theta \approx (kR)^{-\frac{1}{2}}$. It should be noted, however, that when $VR/\hbar v$ is very small, (13) and (14) reduce to the Born approximation amplitude

$$f_B(\mathbf{k}_f,\mathbf{k}_0) = (-4\pi)^{-1} \int \exp(i\mathbf{q}\cdot\mathbf{r}) U(\mathbf{r}) d\tau, \quad (16)$$

and hence are valid regardless of the magnitudes of θ and kR.

C. Total Cross Section

As is well known, the total cross section is related to the imaginary part of the forward scattered amplitude

$$\sigma = (4\pi/k)\ \mathrm{Im} f(\mathbf{k}_0,\mathbf{k}_0)$$
$$= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{1-\cos\left[(2k)^{-1}\int_{-\infty}^{\infty} U(x,y,z)dz\right]\right\} dxdy. \quad (17)$$

Here, use has been made of Eq. (14).

It is interesting to compare Eq. (17) with the integral over angles of the differential cross section obtained from Eq. (16), when k is very large and V is spherically symmetric. In this case Eq. (17) gives

$$\sigma \underset{k\to\infty}{\longrightarrow} (2\pi/k^2)\int_0^\infty \left[\int_b^\infty (r^2-b^2)^{-\frac{1}{2}} U(r) r dr\right]^2 b db. \quad (18)$$

Equation (16) can be written

$$f_B = -q^{-1}\int_0^\infty \sin qr\, U(r) r dr, \quad q = 2k \sin\tfrac{1}{2}\theta,$$

so that the total Born approximation cross section is

$$\sigma_B \underset{k\to\infty}{\longrightarrow} (2\pi/k^2)\int_0^\infty \left[\int_0^\infty \sin qr\, U(r) r dr\right]^2 (dq/q). \quad (19)$$

Equation (18) has the form of an integral over impact parameters, and Eq. (19) the form of an integral over momentum transfers. If the q integration is performed first in (19), it becomes

$$(\pi/k^2)\int_0^\infty\int_0^\infty U(r)U(r') \ln[(r+r')/|r-r'|] rr' dr dr'.$$

Equation (18) can also be put in this form by doing the b integration first, although the calculation is slightly more complicated in this case. Thus the two expressions agree, as expected.

D. Complex Scattering Potential

In order for a complex scattering potential to represent an absorbing medium, its imaginary part must be negative. Then, since the integrals involving U in the exponents of Eqs. (13) and (14) are multiplied by $-i$, the integrands decrease exponentially rather than increase as the imaginary part of V or the range of V increases, as of course they must. There is nothing in the analysis that prevents the results obtained in this paper from being applied to complex potentials.

III. DIRAC SCATTERED AMPLITUDE

Since the infinite Born series for the Dirac case is perhaps less well known than the analogous Eq. (5) for the Schrödinger case, we start with the Dirac equation

$$(-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + mc^2\beta + E)\psi = V\psi,$$

where $\boldsymbol{\alpha}$ and β are the Dirac matrices. We operate on this from the left with

$$i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - mc^2\beta + E$$

to obtain

$$(\nabla^2 + k^2)\psi = (\epsilon + i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \mu\beta) v\psi, \quad (20)$$

where $k^2 = \epsilon^2 - \mu^2$, $\epsilon = E/\hbar c$, $\mu = mc/\hbar$, $v = V/\hbar c$. Equation (20) can be converted into an integral equation by making use of the Green's function (6) for the operator on the left side to obtain[9]

$$\psi(\mathbf{r}) = a_0 \exp(i\mathbf{k}_0\cdot\mathbf{r}) - (4\pi)^{-1}\int |\mathbf{r}-\mathbf{r}'|^{-1}$$
$$\times \exp(ik|\mathbf{r}-\mathbf{r}'|)(\epsilon + i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}' - \mu\beta) v(\mathbf{r}')\psi(\mathbf{r}') d\tau',$$

where a_0 is one of the two unit spinors associated with the incident plane wave of momentum $\hbar\mathbf{k}_0$ and positive energy, and the prime on the gradient operator denotes differentiation with respect to $\mathbf{r}'$. Integration by parts, followed by transfer of the gradient from $\mathbf{r}'$ to $\mathbf{r}$ in the Green's function, leads to

$$\psi(\mathbf{r}) = a_0 \exp(i\mathbf{k}_0\cdot\mathbf{r}) - (4\pi)^{-1}(\epsilon + i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} - \mu\beta)$$
$$\times \int |\mathbf{r}-\mathbf{r}'|^{-1} \exp(ik|\mathbf{r}-\mathbf{r}'|) v(\mathbf{r}')\psi(\mathbf{r}') d\tau'. \quad (21)$$

The integral equation (21) is exact, and can be iterated to obtain the infinite Born series, which has the asymptotic form

$$\psi \to a_0 \exp(i\mathbf{k}_0\cdot\mathbf{r})$$
$$+ \sum_{n=1}^{\infty} (-4\pi r)^{-1} \exp(ikr)(\epsilon - \boldsymbol{\alpha}\cdot\mathbf{k}_f - \mu\beta)$$
$$\times \int\cdots\int \exp(-i\mathbf{k}_f\cdot\mathbf{r}_n) v(\mathbf{r}_n)(\epsilon + i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_n - \mu\beta)$$
$$\times G(\boldsymbol{\varrho}_{n-1}) v(\mathbf{r}_{n-1})\cdots(\epsilon + i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}_2 - \mu\beta) G(\boldsymbol{\varrho}_1) v(\mathbf{r}_1) a_0$$
$$\times \exp(i\mathbf{k}_0\cdot\mathbf{r}_1) d\tau_1\cdots d\tau_n. \quad (22)$$

[9] G. Parzen, Phys. Rev. 80, 261 (1950).

Here, G is given by Eq. (6), the $\boldsymbol{\varrho}$'s are defined just above Eq. (9), and the subscript on each of the gradient operators denotes the particular $\mathbf{r}$ on which it operates. We wish now to proceed as in Sec. II A, and associate a factor $\exp(-i\mathbf{k}_0\cdot\boldsymbol{\varrho})$ or $\exp(-i\mathbf{k}_f\cdot\boldsymbol{\varrho})$ with each G, in order that the stationary phase approximation may be applied. The difficulty is that these exponentials do not commute with the gradient operators which appear in the integrand of Eq. (22). We note, however, that for any constant vector $\mathbf{k}$ and function $g(\mathbf{r})$

$$\boldsymbol{\nabla} g=\exp(i\mathbf{k}\cdot\mathbf{r})(\boldsymbol{\nabla}+i\mathbf{k})[\exp(-i\mathbf{k}\cdot\mathbf{r})g],$$

so that the substitution (9) can be used to put the exponentials in the right places if each gradient is replaced by $\boldsymbol{\nabla}+i\mathbf{k}_0$ or $\boldsymbol{\nabla}+i\mathbf{k}_f$, as appropriate. The Dirac scattered amplitude in Eq. (22) then becomes

$$\begin{aligned}f_D(\mathbf{k}_f,\mathbf{k}_0)&=(-4\pi)^{-1}(\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}_f-\mu\beta)\\&\times\sum_{n=1}^{\infty}\int\cdots\int v(\mathbf{r}_n)[\epsilon+\boldsymbol{\alpha}\cdot(i\boldsymbol{\nabla}_n-\mathbf{k}_f)-\mu\beta]\\&\times G(\boldsymbol{\varrho}_{n-1})\exp(-i\mathbf{k}_f\cdot\boldsymbol{\varrho}_{n-1})v(\mathbf{r}_{n-1})\cdots\\&\times[\epsilon+\boldsymbol{\alpha}\cdot(i\boldsymbol{\nabla}_{m+1}-\mathbf{k}_f)-\mu\beta]\\&\times G(\boldsymbol{\varrho}_m)\exp(-i\mathbf{k}_f\cdot\boldsymbol{\varrho}_m)v(\mathbf{r}_m)\\&\times\exp(i\mathbf{q}\cdot\mathbf{r}_m)[\epsilon+\boldsymbol{\alpha}\cdot(i\boldsymbol{\nabla}_m-\mathbf{k}_0)-\mu\beta]\\&\times G(\boldsymbol{\varrho}_{m-1})\exp(-i\mathbf{k}_0\cdot\boldsymbol{\varrho}_{m-1})v(\mathbf{r}_{m-1})\cdots\\&\times[\epsilon+\boldsymbol{\alpha}\cdot(i\boldsymbol{\nabla}_2-\mathbf{k}_0)-\mu\beta]\\&\qquad\times G(\boldsymbol{\varrho}_1)\exp(-i\mathbf{k}_0\cdot\boldsymbol{\varrho}_1)v(\mathbf{r}_1)a_0d\tau_1\cdots d\tau_n,\end{aligned}$$

where m can be any of the integers 1, 2, $\cdots n$.

Before applying the stationary phase approximation (8) to each of the factors $G(\boldsymbol{\varrho}_l)\exp(-i\mathbf{k}\cdot\boldsymbol{\varrho}_l)$, we note that

$$\begin{aligned}(i\boldsymbol{\nabla}_{l+1}-\mathbf{k})&[G(\boldsymbol{\varrho}_l)\exp(-i\mathbf{k}\cdot\boldsymbol{\varrho}_l)]\\&=-\rho_l^{-2}\boldsymbol{\varrho}_l(k\rho_l+i)G(\boldsymbol{\varrho}_l)\exp(-i\mathbf{k}\cdot\boldsymbol{\varrho}_l).\end{aligned}$$

It is plausible, and can be verified by direct calculation, that i can be neglected in comparison with $k\rho_l$ in the parenthesis when only the leading term of (8) is retained. We thus obtain, to the same approximation as Eq. (10),

$$\begin{aligned}f_D(\mathbf{k}_f,\mathbf{k}_0)&=(-4\pi)^{-1}(\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}_f-\mu\beta)\\&\times\sum_{n=1}^{\infty}\sum_{m=1}^{n}(-i/2k)^{n-1}\int d\tau_m\int_0^\infty d\rho_1\cdots\int_0^\infty d\rho_{n-1}\\&\times\exp(i\mathbf{q}\cdot\mathbf{r}_m)v[\mathbf{r}_m+\hat{k}_f(\rho_{n-1}+\cdots+\rho_m)]\\&\times\cdots v(\mathbf{r}_m+\hat{k}_f\rho_m)v(\mathbf{r}_m)v(\mathbf{r}_m-\hat{k}_0\rho_{m-1})\\&\times\cdots v[\mathbf{r}_m-\hat{k}_0(\rho_{m-1}+\cdots+\rho_1)]\\&\qquad\times(\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}_f-\mu\beta)^{n-m}(\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}_0-\mu\beta)^{m-1}a_0,\end{aligned}\quad(23)$$

where again we have assumed that the n stationary phase regions are distinct. We are actually interested not so much in f_D itself as in the scalar product of f_D with a_f, which is one of the two unit spinors associated with the final plane wave of momentum $\hbar\mathbf{k}_f$ and positive energy. This quantity $\bar{a}_f f_D$ can be simplified by noting that a_0 and $\bar{a}_f$ satisfy the spinor equations

$$(\epsilon+\boldsymbol{\alpha}\cdot\mathbf{k}_0+\mu\beta)a_0=0,$$
$$\bar{a}_f(\epsilon+\boldsymbol{\alpha}\cdot\mathbf{k}_f+\mu\beta)=0,$$

so that

$$\bar{a}_f(\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}_f-\mu\beta)^{n-m+1}(\epsilon-\boldsymbol{\alpha}\cdot\mathbf{k}_0-\mu\beta)^{m-1}a_0=(2\epsilon)^n\bar{a}_fa_0.$$

On substitution into Eq. (23), it can be treated in exactly the same way that Eq. (10) was treated in Sec. II to yield Eqs. (13) and (14) in the large- and small-angle cases. The result is

$$\bar{a}_f f_D(\mathbf{k}_f,\mathbf{k}_0)=(\bar{a}_fa_0)(E/mc^2)f(\mathbf{k}_f,\mathbf{k}_0),\quad(24)$$

where f is given by (13) or (14); it must be remembered that $U/2k$ and $\epsilon v/k$ are both equal to $V/\hbar v$.

The differential cross section in the Dirac case (ignoring polarization effects) is obtained by averaging the absolute square of Eq. (24) over the two initial spinors a_0 and summing over the two final spinors a_f. The result is

$$\sigma_D(\mathbf{k}_f,\mathbf{k}_0)=(E/mc^2)^2[1-(v^2/c^2)\sin^2(\tfrac{1}{2}\theta)]\sigma(\mathbf{k}_f,\mathbf{k}_0),\quad(25)$$

where

$$\sigma(\mathbf{k}_f,\mathbf{k}_0)=|f(\mathbf{k}_f,\mathbf{k}_0)|^2\quad(26)$$

is the differential cross section in the Schrödinger case. Equations (25), (26), (13), and (14) give the Schrödinger and Dirac cross sections when θ is large and small in comparison with $(kR)^{-\frac{1}{2}}$. The relation (25) between the Dirac and Schrödinger cross sections in our approximation is the same as that derived by Parzen[9] for the Born approximation.

It is to be expected that this approximation can be extended to the scattering theory of other wave equations (electromagnetic, acoustic, other spin values, etc.), although this has not as yet been done. In general, the validity conditions are $kR\gg1$, θ large or small in comparison with $(kR)^{-\frac{1}{2}}$, and Δk (the change in wave number within the scattering region) slowly varying and small in comparison with k, with $R\Delta k$ unrestricted.

IV. WAVE FUNCTION

In this section and the next we examine the Schrödinger case in more detail than the preceding sections, in order to understand the limitations of the present procedure and to clarify its relationship with earlier work. We start by calculating the wave function to the same accuracy as the scattered amplitude (13) and (14), and by the same method.

The infinite Born series for ψ is

$$\psi(\mathbf{r})=\exp(i\mathbf{k}_0\cdot\mathbf{r})+\sum_{n=1}^{\infty}\psi_n(\mathbf{r}),$$

$$\psi_n(\mathbf{r})=\int\cdots\int G(\mathbf{r}-\mathbf{r}_n)U(\mathbf{r}_n)G(\mathbf{r}_n-\mathbf{r}_{n-1})U(\mathbf{r}_{n-1})\cdots$$
$$\times U(\mathbf{r}_2)G(\mathbf{r}_2-\mathbf{r}_1)U(\mathbf{r}_1)\exp(i\mathbf{k}_0\cdot\mathbf{r}_1)d\tau_1\cdots d\tau_n. \quad (27)$$

In analogy with the development between Eqs. (5) and (10), we put

$$\mathbf{k}_0\cdot\mathbf{r}_1=\mathbf{k}_0\cdot\mathbf{r}_m-\mathbf{k}_0\cdot(\mathbf{r}_m-\mathbf{r}_{m-1})-\cdots-\mathbf{k}_0\cdot(\mathbf{r}_2-\mathbf{r}_1).$$

There are now two classes of terms which give rise to stationary phase regions: (a) those for which $\mathbf{r}_m=\mathbf{r}$, $\mathbf{r}_{m-1}=\mathbf{r}_n$, etc., and (b) those for which m is any of the integers $1, 2, \cdots n$. The (a) term for each n is immediately approximated by the methods of Sec. II to give the following contribution to ψ_n:

$$(-i/2k)^n\exp(i\mathbf{k}_0\cdot\mathbf{r})\int_0^\infty d\rho_1\cdots\int_0^\infty d\rho_n U(\mathbf{r}-\hat{k}_0\rho_n)\cdots$$
$$\times U[\mathbf{r}-\hat{k}_0(\rho_n+\cdots+\rho_1)],$$

where $\boldsymbol{\rho}_n=\mathbf{r}-\mathbf{r}_n$ and the other $\boldsymbol{\rho}$'s are defined just above Eq. (9). This can be simplified as in Sec. II A, and the sum over n performed. Inclusion of the incident plane wave of Eq. (27) then leads to the following contribution to ψ:

$$\exp\left\{i(\mathbf{k}_0\cdot\mathbf{r})-(i/2k)\int_0^\infty U(\mathbf{r}-\hat{k}_0s)ds\right\}. \quad (28)$$

We now consider the (b) term that arises from a particular choice of n and m. The integrations over $\mathbf{r}_1, \cdots\mathbf{r}_{m-1}$ can be converted into integrations over $\boldsymbol{\rho}_1, \cdots\boldsymbol{\rho}_{m-1}$ and evaluated as

$$(-i/2k)^{m-1}\int_0^\infty d\rho_1\cdots\int_0^\infty d\rho_{m-1}$$
$$\times U(\mathbf{r}_m-\hat{k}_0\rho_{m-1})\cdots U[\mathbf{r}_m-\hat{k}_0(\rho_{m-1}+\cdots+\rho_1)]$$
$$=[(m-1)!]^{-1}\left[(-i/2k)\int_0^\infty U(\mathbf{r}_m-\hat{k}_0s)ds\right]^{m-1}. \quad (29)$$

To evaluate the integrations over $\mathbf{r}_{m+1}, \cdots\mathbf{r}_n$, we note that the rapidly varying part of the integrand is $\exp[ik(\rho_n+\rho_{n-1}+\cdots+\rho_m)]$. It is apparent that the phase of this term will be stationary when the $\boldsymbol{\rho}$'s are lined up parallel to the vector $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}_m$, in which case the exponential will be equal to $\exp(ik\rho)$. It is shown in Appendix B that the integrations over $\mathbf{r}_{m+1}, \cdots\mathbf{r}_n$ yield

$$(-4\pi\rho)^{-1}\exp(ik\rho)(-i/2k)^{n-m}$$
$$\times\int_0^\rho d\rho_m\int_0^{\rho-\rho_m}d\rho_{m+1}\cdots\int_0^{\rho-\rho_m-\cdots-\rho_{n-2}}d\rho_{n-1}$$
$$\times U[\mathbf{r}_m+\hat{\rho}(\rho_{n-1}+\cdots+\rho_m)]\cdots U(\mathbf{r}_m+\hat{\rho}\rho_m),$$

where $\hat{\rho}$ is a unit vector parallel to $\boldsymbol{\rho}$. This may be simplified in the same way that Eq. (11) was obtained, and leads to

$$(-4\pi\rho)^{-1}\exp(ik\rho)[(n-m)!]^{-1}$$
$$\times\left[(-i/2k)\int_0^\rho U(\mathbf{r}_m+\hat{\rho}s)ds\right]^{n-m}. \quad (30)$$

Substitution of Eqs. (29) and (30) into (27), and summation over m and n, gives for the contribution of the (b) terms to ψ

$$\int(-4\pi\rho)^{-1}\exp(ik\rho)U(\mathbf{r}')\exp(i\mathbf{k}_0\cdot\mathbf{r}')$$
$$\times\exp\left\{(-i/2k)\left[\int_0^\infty U(\mathbf{r}'-\hat{k}_0s)ds\right.\right.$$
$$\left.\left.+\int_0^\rho U(\mathbf{r}'+\hat{\rho}s)ds\right]\right\}d\tau', \quad (31)$$

where now $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}'$.

Before adding together Eqs. (28) and (31) to obtain our approximation for ψ, it is essential to realize that we have assumed that the various stationary phase regions are distinct. In the (a) terms that led to Eq. (28), the n scatterings that give rise to the nth term in the Born series are all nearly forward, so that the particle may be thought of as propagating from $z=-\infty$ to $\mathbf{r}$ along a nearly straight line parallel to its initial direction $\mathbf{k}_0$. In the (b) terms that led to Eq. (31), the particle may be thought of as propagating from $z=-\infty$ to $\mathbf{r}'$ along a nearly straight line parallel to $\mathbf{k}_0$, then being scattered at $\mathbf{r}'$ and propagating along a nearly straight line to the point $\mathbf{r}$. The $(n+1)$ stationary phase regions that contribute to ψ_n [one from the (a) term and n from the (b) terms] are thus distinct only if the point $\mathbf{r}'$ is not too close to the line parallel to $\mathbf{k}_0$ that extends from $z=-\infty$ to $\mathbf{r}$; more precisely, the angle between $\boldsymbol{\rho}$ and $\mathbf{k}_0$ must be somewhat greater than $(k\rho)^{-\frac{1}{2}}$. Thus either we must exclude this range of $\mathbf{r}'$ from the integration in Eq. (31), or we must replace Eq. (28) by $\exp(i\mathbf{k}_0\cdot\mathbf{r})$ and recalculate the (b) terms that led to Eq. (31), taking into account the coalescence of the stationary phase regions for this range of $\mathbf{r}'$. The two procedures are of course equivalent; we use the first, and express our approximation for the Schrödinger wave function as the sum of Eqs. (28) and (31), where we introduce a prime on the $\mathbf{r}'$ integration to denote the exclusion just mentioned:

$$\psi(\mathbf{r})=\exp\left\{i(\mathbf{k}_0\cdot\mathbf{r})-(i/2k)\int_0^\infty U(\mathbf{r}-\hat{k}_0 s)ds\right\}$$
$$+\int'(-4\pi\rho)^{-1}\exp(ik\rho)U(\mathbf{r}')\exp(i\mathbf{k}_0\cdot\mathbf{r}')$$
$$\times\exp\left\{(-i/2k)\left[\int_0^\infty U(\mathbf{r}'-\hat{k}_0 s)ds+\int_0^\rho U(\mathbf{r}'+\hat{\rho}s)ds\right]\right\}d\tau'. \quad (32)$$

Equation (32) can be used as it stands in the vicinity of the scattering potential. In the asymptotic region, for $\theta\gg(kR)^{-\frac{1}{2}}$, the first term becomes $\exp(i\mathbf{k}_0\cdot\mathbf{r})$ and the second term becomes $r^{-1}\exp(ikr)f(\mathbf{k}_f,\mathbf{k}_0)$, where $\mathbf{k}_f$ is parallel to $\mathbf{r}$ and f is given by Eq. (13). However, for $\theta\ll(kR)^{-\frac{1}{2}}$ in the asymptotic region, it must be remembered that the first term of (32) cannot be taken literally as implying that the integral in the exponent vanishes if $r\gg R/\theta$, so that the line from $z=-\infty$ to $\mathbf{r}$ along $\mathbf{k}_0$ misses the scattering potential. The reason is that the $\boldsymbol{\varrho}$'s from which this integral arises can have directions that deviate from that of $\mathbf{k}_0$ by angles of order $(kR)^{-\frac{1}{2}}$, so that the integral always fails to vanish in the small-angle case. It is simpler then to follow the second procedure outlined in the preceding paragraph, in which case the calculation is exactly that which led to Eq. (14) for the scattered amplitude in the small-angle case.

It is interesting to substitute Eq. (32) into the well-known formula

$$f(\mathbf{k}_f,\mathbf{k}_0)=(-4\pi)^{-1}\int\exp(-i\mathbf{k}_f\cdot\mathbf{r})U(\mathbf{r})\psi(\mathbf{r})d\tau \quad (33)$$

for the scattered amplitude, which is exact if ψ is exact. The first term of ψ yields

$$(-4\pi)^{-1}\int\exp(i\mathbf{q}\cdot\mathbf{r})U(\mathbf{r})\times\exp\left[(-i/2k)\int_0^\infty U(\mathbf{r}-\hat{k}_0 s)ds\right]d\tau, \quad (34)$$

while the second term leads to

$$(4\pi)^{-2}\int d\tau\int' d\tau' U(\mathbf{r})U(\mathbf{r}')\rho^{-1}$$
$$\times\exp[i(\mathbf{k}_0\cdot\mathbf{r}'-\mathbf{k}_f\cdot\mathbf{r}+k\rho)]$$
$$\times\exp\left\{(-i/2k)\left[\int_0^\infty U(\mathbf{r}'-\hat{k}_0 s)ds+\int_0^\rho U(\mathbf{r}'+\hat{\rho}s)ds\right]\right\}. \quad (35)$$

Now the first term of ψ given by Eq. (32) is of order unity, while the second term can be shown to be of order V/E' since the stationary phase region that would normally arise from that part of the integrand for which $\boldsymbol{\varrho}$ is nearly parallel to $\mathbf{k}_0$ is excluded. Thus since the derivation of the first term neglected corrections of order V/E' [see the discussion of Eq. (13)] one might think that the second term of ψ should be dropped. However, we shall show in the next paragraph that in the large-angle case, there is a different stationary phase region in the integrand of (35), that in which $\boldsymbol{\varrho}$ is nearly parallel to $\mathbf{k}_f$, which brings (35) up to the same order of magnitude as (34). On the other hand, in the small-angle case where $\mathbf{k}_f$ and $\mathbf{k}_0$ are nearly parallel to each other, this second stationary phase region is not distinct from the first one, which is excluded; thus (35) has to be neglected for consistency in this case. We thus conclude that Eq. (34) by itself is the scattered amplitude in the small-angle case; but for small angles, as pointed out at the beginning of Sec. II B, $q_z z$ is small compared to unity and may be neglected. The z integration is then easily performed, and leads immediately to Eq. (14).

In the large-angle case, where $\mathbf{k}_f$ and $\mathbf{k}_0$ are not close to parallelism, we can put $\mathbf{k}_0\cdot\mathbf{r}'-\mathbf{k}_f\cdot\mathbf{r}+k\rho=\mathbf{q}\cdot\mathbf{r}'+k\rho-\mathbf{k}_f\cdot\boldsymbol{\varrho}$ in the exponent of the integrand of Eq. (35). Thus there is a stationary phase region when $\boldsymbol{\varrho}$ is nearly parallel to $\mathbf{k}_f$, which is distinct from that excluded by the prime of the $\mathbf{r}'$ integration. Then, using Eq. (8) to evaluate the leading term of Eq. (35), we obtain

$$(4\pi)^{-2}(2\pi i/k)\int d\tau'\int_0^\infty d\rho\exp(i\mathbf{q}\cdot\mathbf{r}')U(\mathbf{r}')U(\mathbf{r}'+\hat{k}_f\rho)$$
$$\times\exp\left\{(-i/2k)\left[\int_0^\infty U(\mathbf{r}'-\hat{k}_0 s)ds+\int_0^\rho U(\mathbf{r}'+\hat{k}_f s)ds\right]\right\},$$

where we have replaced $\hat{\rho}$ by $\hat{k}_f$. The ρ integration is now easily performed, and leads to

$$(-4\pi)^{-1}\int\exp(i\mathbf{q}\cdot\mathbf{r})U(\mathbf{r})\times\exp\left[(-i/2k)\int_0^\infty U(\mathbf{r}-\hat{k}_0 s)ds\right]$$
$$\times\left\{\exp\left[(-i/2k)\int_0^\infty U(\mathbf{r}+\hat{k}_f s)ds\right]-1\right\}d\tau, \quad (36)$$

where we have replaced $\mathbf{r}'$ by $\mathbf{r}$ as the variable of integration. It is now apparent that Eqs. (34) and (36) are of the same order of magnitude; indeed, when they are added together, the second term in the curly bracket of (36) just cancels (34), yielding the scattered amplitude (13) obtained earlier.

We see then that Eq. (32) represents the wave function in the vicinity of the scattering potential to an accuracy that is consistent with the calculation of the scattered amplitude by means of Eq. (33), and that both terms of (32) must be retained for this purpose.

V. EIKONAL-TYPE APPROXIMATIONS

As pointed out in Sec. I, Eq. (2), which is the same as Eq. (28), has appeared in several of the earlier papers,[4] usually being derived by a WKB or eikonal-type argument as was done in Sec. I. We have seen in Sec. IV that if this form for ψ is used in conjunction with Eq. (33) to calculate the scattered amplitude, it gives useful results for small angles but not for large angles.

The reason for this failure may be put in the following way. Equation (2) or (28) is incorrect beyond zero order in V/E' for two different reasons. The first of these appears in the neglect of the amplitude change mentioned just below Eq. (2), and also appears in the neglect of the next to the leading terms of the stationary phase approximation that took the (a) terms of Eq. (27) into Eq. (28). The second reason why Eq. (28) is incorrect beyond zero order in V/E' is that the second term of Eq. (32), which arises from the (b) terms of Eq. (27), is not included in it. While both the second term of (32) and the neglected corrections to (28) are of the same order of magnitude, we have seen in Sec. IV that the former leads to a large-angle scattered amplitude that is of the same order as that obtained from the uncorrected Eq. (28), so that the V/E' corrections to (2) or (28) can be neglected consistently. Now as pointed out by Gol'dman and Migdal,[4] continued improvement of the WKB approximation will improve the wave function along the classical trajectories, but will never help in the classically inaccessible regions in which the second term of (32) is significant. Thus the ψ of Eq. (32), which is essential for a correct calculation of the scattering, is beyond the reach of the WKB or eikonal approximation.

In an interesting but unsuccessful attempt to circumvent this limitation, Gol'dman and Migdal have adopted a somewhat different procedure from that represented by Eq. (33). Equation (33) can be derived in the following way. We write the Schrödinger wave equation (4) as

$$(\nabla^2+k^2)\psi=U\psi,$$

and put

$$\psi(\mathbf{r})=\exp(i\mathbf{k}_0\cdot\mathbf{r})+u(\mathbf{r}), \tag{37}$$

so that u satisfies the equation

$$(\nabla^2+k^2)u=U\psi.$$

With the help of the Green's function G of Eq. (6) we obtain

$$u(\mathbf{r})=\int G(\mathbf{r}-\mathbf{r}')U(\mathbf{r}')\psi(\mathbf{r}')d\tau',$$

and the asymptotic form of this immediately yields Eq. (33).

As an alternative, Gol'dman and Migdal rewrite Eq. (4), with the help of Eq. (37), in the form

$$(\nabla^2+k^2-U)u=U\exp(i\mathbf{k}_0\cdot\mathbf{r}), \tag{38}$$

and solve this by using the WKB approximation to the Green's function for the operator on the left side,

$$G(\mathbf{r},\mathbf{r}')=(-4\pi\rho)^{-1}\times\exp\left\{ik\rho-(i/2k)\int_0^\rho U(\mathbf{r}'+\hat{\rho}s)ds\right\}, \tag{39}$$

where $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}'$. The resulting asymptotic form of u gives a scattered amplitude that resembles Eq. (13), but contains only the second of the two integrals in the exponent. This approximation fails for the same reason that the substitution of Eq. (2) or (28) into Eq. (33) fails; the error involved in the use of (28) without the second term of (32) has its counterpart in the neglect of a similarly important term in the approximate Green's function (39). A further possibility, not discussed by Gol'dman and Migdal, is to substitute the ψ obtained from Eqs. (37), (38), and (39) into Eq. (33). While the scattered amplitude obtained in this way is closer to Eq. (13) than either of the WKB results discussed above, it is still not correct.

In conclusion, the author wishes to express his appreciation to Professor D. R. Yennie for several helpful discussions, and in particular for pointing out an error during the early stages of this work. He also wishes to thank Professor H. Levine for an interesting conversation on these matters.

APPENDIX A

We wish to evaluate the integral I of Eq. (7) when it is assumed that g varies slowly over distances of order k^{-1}. We choose spherical coordinates ρ,θ,ϕ with the polar axis along $\mathbf{k}$, and put $\mu=\cos\theta$:

$$I=\int_0^\infty\int_0^{2\pi}\int_{-1}^1 g(\rho,\mu,\phi)\exp[ik\rho(1-\mu)]\rho d\rho d\phi d\mu. \tag{A.1}$$

Now if $k\rho$ is somewhat larger than unity, the exponential will oscillate rapidly except near $\mu=1$ or $\theta=0$; most of the integral will come from the angular region $1-\mu \lesssim (k\rho)^{-1}$ or $\theta\lesssim(k\rho)^{-\frac{1}{2}}$. This region is shaped like a paraboloid of revolution about the polar axis, with radius perpendicular to the axis of order $\rho\theta\sim(p/k)^{\frac{1}{2}}$. Thus if in I we put $d\phi d\mu\sim 2\pi/k\rho$, we find the order of magnitude of I to be $(2\pi/k)\int_0^\infty g(\rho,1,0)d\rho$. All of this assumes that $k\rho$ is somewhat larger than unity. For

$\rho \lesssim k^{-1}$, the exponential in I is of order unity, and this part of I is of order $(2\pi/k^2)g(0,1,0)$. Thus if R is the range of g, the first term is larger than the second by a factor of order kR, which is assumed to be large compared to unity. We can therefore obtain the leading term of I by assuming that ρ is always somewhat larger than k^{-1}.

Our formal procedure consists of integrating Eq. (A.1) by parts with respect to μ[7]:

$$I=\int_0^\infty\int_0^{2\pi}\left\{(i/k\rho)g(\rho,\mu,\phi)\exp[ik\rho(1-\mu)]\Big|_{-1}^{1}-\int_{-1}^{1}(i/k\rho)g'(\rho,\mu,\phi)\exp[ik\rho(1-\mu)]d\mu\right\}\rho d\rho d\phi,$$

where the prime on g denotes differentiation with respect to μ. The second term is of order $(k\rho)^{-1}$ with respect to the first term, since another partial integration will introduce an additional power of $k\rho$ in the denominator. Further, the contribution from the lower limit of the first term will be small compared to that from the upper limit, since the former will contain a rapidly oscillating factor $\exp(2ik\rho)$. The leading term of I is then just the upper limit of the first term

$$I=(2\pi i/k)\int_0^\infty g(\rho,1,0)d\rho+O(k^{-2})$$

as quoted in Eq. (8).

It is possible to go farther in this way, and obtain the next term. To do this it is necessary to break up the ρ integration into two parts, according as ρ is less or greater than ρ_0; ρ_0 is assumed to be large compared to k^{-1}, and small enough so that a few terms in a power series of g suffice for $p\leqq\rho_0$. When the power series expansion is used for $\rho\leqq\rho_0$, and the partial integration procedure for $\rho\geqq\rho_0$, it is possible to keep terms consistently through order k^{-2} and find that the precise choice for ρ_0 drops out of the end result. We obtain in this way

$$I=(2\pi i/k)\int_0^\infty g(\rho,1,0)d\rho-(\pi/k^2)\int_0^\infty \rho h(\rho,1,0)d\rho+O(k^{-3}),\quad \text{(A.2)}$$

$$h=\nabla^2 g.$$

Equation (A.2) can also be derived by a Fourier transform procedure which is equivalent to that used by Malenka and by Shapiro.[6] We replace $\rho^{-1}\exp(ik\rho)$ in Eq. (7) by

$$(2\pi^2)^{-1}\lim_{\epsilon\to 0}\int(\kappa^2-k^2-i\epsilon)^{-1}\exp(i\boldsymbol{\kappa}\cdot\boldsymbol{\varrho})d\tau_\kappa.$$

With $\mathbf{p}=\boldsymbol{\kappa}-\mathbf{k}$, I can be written

$$I=(2\pi^2)^{-1}\lim_{\epsilon\to 0}\int\int(p^2+2\mathbf{p}\cdot\mathbf{k}-i\epsilon)^{-1}g(\boldsymbol{\varrho})\times\exp(i\mathbf{p}\cdot\boldsymbol{\varrho})d\tau d\tau_p.$$

The assumption that g is slowly varying is equivalent to the assumption that the integrand is only large where $p\ll k$. We therefore expand the factor in the denominator in powers of p^2, to obtain for the first two terms

$$I\cong(2\pi^2)^{-1}\lim_{\epsilon\to 0}\int\int(2\mathbf{p}\cdot\mathbf{k}-i\epsilon)^{-1}g(\boldsymbol{\varrho})\exp(i\mathbf{p}\cdot\boldsymbol{\varrho})d\tau d\tau_p-(2\pi^2)^{-1}\lim_{\epsilon\to 0}\int\int(2\mathbf{p}\cdot\mathbf{k}-i\epsilon)^{-2}g(\boldsymbol{\varrho})p^2\times\exp(i\mathbf{p}\cdot\boldsymbol{\varrho})d\tau d\tau_p.\quad \text{(A.3)}$$

We put $g(\boldsymbol{\varrho})=g(x,y,z)$, and let the z axis be parallel to $\mathbf{k}$. Then the integration of $\exp[i(p_x x+p_y y)]$ over p_x and p_y gives $(2\pi)^2\delta(x)\delta(y)$. Also the p_z integral

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(2kp_z-i\epsilon)^{-1}\exp(ip_z z)dp_z$$

is zero for negative z and equal to $\pi i/k$ for positive z. Thus the first term of (A.3) becomes $(2\pi i/k)\times\int_0^\infty g(0,0,z)dz$, in agreement with the first term of (A.2).

For the second term of (A.3), we first replace $p^2\exp(i\mathbf{p}\cdot\boldsymbol{\varrho})$ by $-\nabla^2\exp(i\mathbf{p}\cdot\boldsymbol{\varrho})$, and transfer the Laplacian operator to g by two partial integrations. The p_x and p_y integrals are evaluated as before, and the p_z integral

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}(2kp_z-i\epsilon)^{-2}\exp(ip_z z)dp_z$$

is zero for negative z and equal to $-\pi z/2k^2$ for positive z. Thus the second term of (A.3) becomes $-(\pi/k^2)\times\int_0^\infty zh(0,0,z)dz$, where $h=\nabla^2 g$, in agreement with the second term of (A.2).

APPENDIX B

It seems to be more convenient to treat the factor $\exp[ik(\rho_n+\cdots+\rho_m)]$ that appears in the (b) terms of Sec. IV by an extension of the first method of Appendix A rather than by the Fourier transform procedure. We imagine a straight line running from $\mathbf{r}_m$ to $\mathbf{r}$; z_s denotes the component of $\boldsymbol{\varrho}_s$ along the line, and x_s, y_s denote the two mutually perpendicular components of the transverse displacement of $\mathbf{r}_s$ from this line. We obtain the leading term of the multiple integral for large k if we assume that the x's and y's are small in comparison with the z's. Then we may expand

$$\rho_n+\cdots+\rho_m\cong z_n+\cdots+z_m$$
$$+\tfrac{1}{2}\{z_n^{-1}(x_n^2+y_n^2)+z_{n-1}^{-1}[(x_n-x_{n-1})^2+(y_n-y_{n-1})^2]$$
$$+\cdots+z_{m+1}^{-1}[(x_{m+2}-x_{m+1})^2+(y_{m+2}-y_{m+1})^2]$$
$$+z_m^{-1}(x_{m+1}^2+y_{m+1}^2)\}.$$

We consider first the part of this expression that depends on x_n:

$$z_n^{-1}x_n^2+z_{n-1}^{-1}(x_n-x_{n-1})^2=(z_n+z_{n-1})^{-1}x_{n-1}^2$$
$$+(z_n+z_{n-1})(z_nz_{n-1})^{-1}[x_n-x_{n-1}z_n(z_n+z_{n-1})^{-1}]^2.$$

Integration of $\exp[ik(\rho_n+\cdots+\rho_m)]$ over x_n may be extended from $-\infty$ to $+\infty$, since for large k most of the integral comes from small values of x_n. This yields a factor $[(2\pi i/k)z_nz_{n-1}(z_n+z_{n-1})^{-1}]^{\frac{1}{2}}$, and an identical factor comes from the integration over y_n. The x_{n-1}, y_{n-1} integrations can then be performed, and together yield a factor $(2\pi i/k)(z_n+z_{n-1})z_{n-2}(z_n+z_{n-1}+z_{n-2})^{-1}$. This can be continued until the last integrations, over x_{m+1}, y_{m+1}, which together yield a factor $(2\pi i/k)\times(z_n+\cdots+z_{m+1})z_m(z_n+\cdots+z_m)^{-1}$. When all these factors are multiplied together, we obtain

$$(2\pi i/k)^{n-m}z_n\cdots z_m(z_n+\cdots+z_m)^{-1}. \qquad \text{(B.1)}$$

If now we apply this result to the evaluation of the leading term of the integral

$$(-1/4\pi)^{n-m+1}\int\cdots\int(\rho_n\cdots\rho_m)^{-1}\exp ik(\rho_n+\cdots+\rho_m)$$
$$\times U(\mathbf{r}_n)\cdots U(\mathbf{r}_{m+1})d\tau_{m+1}\cdots d\tau_n, \qquad \text{(B.2)}$$

and remember that the U's are slowly varying, we find that the product of z's in the numerator of Eq. (B.1) cancels the product of ρ's in the denominator of Eq. (B.2), since the ρ's are now to be taken as oriented along the line from $\mathbf{r}_m$ to $\mathbf{r}$. Also, the sum of z's that remains in the exponent and appears in the denominator of Eq. (B.1) can be replaced by $\rho=|\mathbf{r}-\mathbf{r}_m|$, provided that all of the points $\mathbf{r}_{m+1},\cdots\mathbf{r}_n$ lie in this order between $\mathbf{r}_m$ and $\mathbf{r}$. If they are out of order, the exponential factor $\exp[ik(z_n+\cdots+z_m)]$ is rapidly oscillating, and we are not in a stationary phase region. We thus obtain as the stationary phase approximation to Eq. (B.2):

$$(-4\pi\rho)^{-1}\exp(ik\rho)(-i/2k)^{n-m}$$
$$\times\int_0^{\rho}d\rho_m\cdots\int_0^{\rho-\rho_m-\cdots-\rho_{n-2}}d\rho_{n-1}$$
$$\times U[\mathbf{r}_m+\hat{\rho}(\rho_{n-1}+\cdots+\rho_m)]\cdots U(\mathbf{r}_m+\hat{\rho}\rho_m).$$