order to reproduce the level structure of the real nucleus; in our opinion, nuclear models should thus be described in terms of the wave function.

A concrete calculation has not been made in this paper, since our primary concern was how to define and construct the model operators in various nuclear models. Such a calculation will be done at another opportunity.

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Relativistic Theory of Polarization Phenomena

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A covariant form of the polarization formalism of Wolfenstein and Ashkin is developed. After the hole-theory boundary conditions are incorporated, the theory may be transformed into a form in which the positive- and negative-energy components are separated. This form involves two-by-two matrices of the Pauli type and its similarity to the form of the nonrelativistic equations allows the relativistic contributions to be extracted. It is concluded that, with suitable interpretations, the nonrelativistic formalism may be used if an additional rotation of the polarization vector is added at each scattering. The relativistic forms of the Wolfenstein equations for various polarization parameters are then derived.

INTRODUCTION

IN recent experiments at Berkeley[1] and elsewhere,[2] the spin dependence of the nucleon-nucleon interaction has been investigated by scattering beams of nucleons in which the spin directions have been partially aligned by means of previous scattering processes. The analysis of these polarization experiments has been discussed by several authors,[3–5] and the more recent of these discussions are based on the polarization formalism introduced by Wolfenstein and Ashkin. Except for a brief note by Michel and Wightman[6] and the early work of Mott[7] on the double scattering of electrons by fixed source centers, the treatments have been based upon the nonrelativistic Pauli approximation. However, since the present cyclotron energies are within the relativistic range and because of the increased energies now becoming available, it is of importance to extend the polarization formalism into the relativistic region. Such an extension is the object of this paper, and a completely covariant formalism for the description of polarization phenomena in the collisions of relativistic particles is developed.

In the first two sections, the covariant forms of the S matrix and the density matrix for the collision of a Dirac particle with a finite-mass spin-zero particle are obtained and the restrictions upon these forms implied by the hole-theory boundary conditions are imposed. In the third section, these forms are used in the manner developed by Wolfenstein and Ashkin for the nonrelativistic treatment of polarization phenomena and a covariant polarization formalism is obtained. This formalism may be transformed into a relativistic but noncovariant form which separates the scattering of positive and negative energy particles into two distinct parts, each of which is expressed in terms of a form which involves two-by-two matrices and which is quite similar to that obtained in the nonrelativistic treatment. The relativistic effects may be extracted by comparing the theory in this form to the nonrelativistic theory. It is shown that the relativistic effects modify the nonrelativistic formulas by the effects of an additional rotation of the polarization vector, a rotation whose magnitude depends upon the scattering angle in a manner which is explicitly exhibited. The effects are of order $(\gamma-1)$, where γ is the relativistic contraction factor for the Dirac particle as seen in the center-of-mass frame.

In the fourth section the method is extended to the

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collision of two Dirac particles and similar results are obtained. Special attention is paid to the triple-scattering and correlation experiments and the relativistic generalizations of the nonrelativistic formulas are derived.

I. COVARIANT S-MATRIX

In this first section, the covariant form of the S matrix for the collision of a Dirac particle with a spin-zero particle is developed. Relativistic invariance requires that the element of the S matrix which transforms the spinor in the initial state into the spinor in the final state be of the form[8]

$$\mathcal{S}_p(k',t,k)=A+B_\mu\gamma_\mu+\tfrac{1}{2}C_{\mu\nu}\sigma_{\mu\nu}+D_\mu(i\gamma_5\gamma_\mu)+E\gamma_5, \quad (1)$$

where A, B_μ, $C_{\mu\nu}$, D_μ, and E are respectively scalar, vector, antisymmetric tensor, pseudovector, and pseudoscalar functions of the three independent four-momenta k, k', and t. The k and k' denote the relative four-momenta in the initial and final states respectively, while t is the total four-momentum of the system, the sum of the initial or the final four-momenta of the two particles. The general matrix of this form is not consistent, however, with the requirements of hole theory. This interpretation of the Dirac equation requires that a Dirac particle which is in a plane wave state at both $T=+\infty$ and $T=-\infty$ must have the sign of its energy the same at these two times. To state this in physical terms: the Dirac particle cannot be changed from an ordinary particle at $T=-\infty$ to an antiparticle at $T=+\infty$, or vice versa.[9] Before this condition is expressed in mathematical form, some notation must be introduced. If the incident Dirac particle is in a positive-energy state, then its wave function may be expressed as[10]

$$\psi_{\text{inc}}=\exp(if\cdot x)[A_1U_1(f)+A_2U_2(f)],$$

while for a negative-energy state

$$\psi_{\text{inc}}=\exp(-if\cdot x)[A_3U_3(f)+A_4U_4(f)].$$

Here f is the four-momentum representing the physically measured energy and momentum of the Dirac particle. Thus $f_0>0$; and the space part of f has the same direction and sense as the incident velocity. Notice that f is not the relative momentum, like k, but rather the momentum in the basic reference frame. The four spinors $U_i(f)$ each have four components $U_{si}(f)$ which are given by[11]

$$U_{si}(f)=(\mp if\cdot\gamma_{si}+M)[2M(f_0+M)]^{-\frac{1}{2}}.$$

Here, and in what follows, the upper sign refers to indices $i=1$, 2 (positive-energy states) and the lower sign refers to $i=3$, 4 (negative-energy states). The covariant normalization condition,

$$U_i^\dagger(f)U_j(f)=U_i^*(f)\beta U_j(f)=\pm\delta_{ij},$$

is satisfied by these spinors. In this relativistic treatment an asterisk is used to denote complex conjugate transpose and $U_i^\dagger$ denotes $U_j^*\beta$, the adjoint of U_i. The $U_i(f)$ introduced above are easily seen to be solutions of the Dirac equation

$$(\pm if\cdot\gamma+M)U_i(f)=0.$$

It is now convenient to introduce for any four-vector v the symbol

$$\gamma(v)\equiv(\gamma\cdot v)/(v\cdot v)^{\frac{1}{2}}, \quad (2)$$

where the square root in the denominator is to be taken as positive or positive imaginary. The Dirac equation then becomes

$$\gamma(f)U_i(f)=\pm U_i(f). \quad (3)$$

When this relation is used, the hole-theory condition may be expressed by the equation

$$\mathcal{S}(f',t,f)=\gamma(f')\mathcal{S}(f',t,f)\gamma(f), \quad (4)$$

where $\mathcal{S}(f',t,f)$ denotes the S-matrix element between states in which the Dirac particle has the physical momenta f and f' in the initial and final states respectively. It will prove convenient, however, to cast the condition expressed by Eq. (4) into the form of a commutation relation. This may be done with the help of the operator

$$\gamma(u,w)\equiv\gamma(u|u\cdot u|^{-\frac{1}{2}}+w|w\cdot w|^{-\frac{1}{2}})\propto[\gamma(u)+\gamma(w)],$$

where the proportionality is valid if u and w are both timelike or both spacelike. Using the equations

$$\gamma(u)\gamma(u)=1=\gamma(w)\gamma(w),$$

one finds that

$$\gamma(u)\gamma(u,w)=\gamma(u,w)\gamma(w). \quad (5)$$

With the aid of this equation and $\mathcal{S}_q(k',t,k)$ defined by

$$\mathcal{S}(f',t,f)=\gamma(f',t)\mathcal{S}_q(k',t,k)\gamma(t,f), \quad (6)$$

the hole-theory condition may be expressed as

$$\mathcal{S}_q(k',t,k)=\gamma(t)\mathcal{S}_q(k',t,k)\gamma(t). \quad (7)$$

Since the S matrix and the $\gamma(u,w)$ have covariant forms, the $\mathcal{S}_q(k',t,k)$ must also be covariant and it may be written in the form given by Eq. (1) with the subscript p

[8] $\mathcal{S}_p(k',t,k)$ is a matrix *element* in momentum space and a matrix in spinor space. The subscript p distinguishes it from a symbol to be defined later.

[9] Cases in which *real* particles are created during a collision may be treated by an extension of the S-matrix formalism, but will not be considered here.

[10] Lower case italic letters without Greek subscripts will represent four-vectors. The dot product $f\cdot x$ represents $f_\mu x_\mu$ where repeated indices are to be summed from one to four if they are Greek and from one to three if italic. As usual, $f_4=if_0$, etc. Italic or numerical subscripts other than zero on lower case italic letters usually denote special four-vectors and not components of the four-vector. (The subscript four in this footnote is the sole exception.) Three-vectors will be in boldface type [*viz.*, $\mathbf{p}$] and scalars will be capitalized, as will most other nonvector quantities.

[11] $\hbar=c=1$; M=proton rest mass. The lower case Greek γ will represent the matrix four-vector $\{-i\beta\boldsymbol{\alpha},\beta\}$, and the Roman subscripts on it select a particular matrix element.

replaced now by q. The commutation relation, Eq. (7), may be used to restrict the coefficients in this expression for $\mathcal{S}_q$ to the forms

$$\begin{aligned} &B_\mu = -iN_B(Bt_\mu),\\ &C_{\mu\nu} = N_C C\left\{k_\mu k_\nu' - k_\nu k_\mu' - \frac{(M^2-M'^2)}{|t\cdot t|}\right.\\ &\qquad\left.\times[t_\mu(k_\nu'-k_\nu)-t_\nu(k_\mu'-k_\mu)]\right\},\\ &D_\mu = N_D D(-i)k_\lambda k_\rho' t_\sigma \epsilon_{\mu\lambda\rho\sigma} \equiv D n_\mu,\\ &E=0. \end{aligned} \tag{8}$$

Here the coefficients B, C, and D are scalar functions, M' is the rest mass of the second particle, and the normalization factors N_B, N_C, and N_D are chosen so that

$$B_\mu B_\mu = B^2, \quad C_{\mu\nu}C_{\mu\nu} = 2C^2, \quad D_\mu D_\mu = D^2. \tag{9}$$

The $\epsilon_{\mu\lambda\rho\sigma}$ is the antisymmetric symbol and n is a unit pseudovector which satisfies

$$(k\cdot n) = (k'\cdot n) = (t\cdot n) = (1-n\cdot n) = 0. \tag{10}$$

This pseudovector n is the four-dimensional generalization of $\mathbf{n}$, the three-dimensional vector normal to the plane of scattering. The auxiliary operator $\mathcal{S}_q(k',t,k)$, which has just been introduced, has a rather simple interpretation. To see this let Eq. (7) be substituted into Eq. (6) to give

$$\mathcal{S}(f',t,f) = (\gamma(f',t)\gamma(t))\mathcal{S}_q(k',t,k)(\gamma(t)\gamma(t,f)). \tag{6'}$$

The operator $(\gamma(t)\gamma(t,f))$ is closely related to the Lorentz transformation between the center-of-mass frame and the rest frame of the incident Dirac particle, and the operator $(\gamma(f',t)\gamma(t))$ is similarly related to the rest frame of the scattered particle. This may be seen by reducing the Lorentz transformation

$$L(f) \equiv \exp[-\tfrac{1}{2}\theta(\boldsymbol{\alpha}\cdot\mathbf{f})|\mathbf{f}|^{-1}]$$

to form

$$L(f) = \beta(-i\gamma\cdot f + M\beta)[2M(f_0+M)]^{-\frac{1}{2}}. \tag{11}$$

In the center-of-mass frame, in which $\gamma(t)=\beta$, one may immediately identify terms to obtain

$$\begin{aligned} &\gamma(t_1)\gamma(t_1 f_1) = L(f_1),\\ &\gamma(f_1',t_1)\gamma(t_1) = L^{-1}(f_1'), \end{aligned} \tag{12}$$

where the subscript one indicates the center-of-mass value. Thus

$$\mathcal{S}(f_1',t_1,f_1) = L^{-1}(f_1')\mathcal{S}_q(k_1',t_1,k_1)L(f_1).$$

This equation has the following interpretation: the S matrix in the center-of-mass frame may be decomposed into a product of two Lorentz transformations and a scattering matrix $\mathcal{S}_q$. The first factor is a Lorentz transformation which converts the spinors of the incident-wave function from their values in the center-of-mass frame to their values in a rest frame of the incident Dirac particle. It converts the spinors to their "proper" values, one might say. Then the unitary operator $\mathcal{S}_q$ gives the effect of the scattering upon the "proper" spinors, and finally a Lorentz transformation converts the "proper" spinors of the scattered particle back to their value as seen in the center-of-mass frame.

The form of $\mathcal{S}_q$ in the center-of-mass frame is particularly simple. Equations (8) give in this case

$$\begin{aligned} &B_\mu\gamma_\mu = B\beta,\\ &\tfrac{1}{2}C_{\mu\nu}\sigma_{\mu\nu} = C\sigma_i N_i, \quad (i=1,2,3),\\ &D_\mu i\gamma_5\gamma_\mu = D\beta\sigma_i N_i,\\ &E=0. \end{aligned} \tag{13}$$

Here the σ_i are the usual four-by-four Dirac matrices

$$\sigma_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix},$$

and $\mathbf{N}$ is the three-vector which is normal to the scattering plane in the center-of-mass frame. Combining these, one obtains

$$\mathcal{S}_q(k_1',t_1,k_1) = \begin{pmatrix} (F^+ + G^+\sigma_N) & 0 \\ 0 & (F^- - G^-\sigma_N) \end{pmatrix}, \tag{14}$$

where σ_N is the Pauli $\sigma_i N_i$ and

$$F^\pm = A \pm B, \quad G^\pm = D \pm C. \tag{15}$$

The F's and G's are scalar functions which completely describe the scattering. The upper two-by-two matrix operates only on the positive-energy "proper" spinors and the lower matrix operates only on negative-energy parts.

In the general frame, also, the $\mathcal{S}_q$ may be put into a form which clearly separates the parts referring to positive- and negative-energy states. The desired form is obtained by first writing

$$\begin{aligned} \tfrac{1}{2}C_{\mu\nu}\sigma_{\mu\nu} &\equiv \tfrac{1}{2}C_{\mu\nu}(-i/2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)\\ &= -(i/2)C_{\mu\nu}\gamma_\mu\gamma_\nu\\ &= -(i/4)C_{\mu\nu}\gamma_\sigma\gamma_\rho\epsilon_{\rho\sigma\mu\nu}\gamma_5. \end{aligned} \tag{16}$$

The condition that $\frac{1}{2}C_{\mu\nu}\sigma_{\mu\nu}$ commutes with $\gamma(t)$ requires that

$$t_\mu C_{\mu\nu} = -C_{\nu\mu}t_\mu = 0. \tag{17}$$

Using this relation, Eq. (16) may be written

$$\tfrac{1}{2}C_{\mu\nu}\sigma_{\mu\nu} = i\gamma(t)\gamma_5\gamma\cdot c, \tag{18}$$

where

$$c_\rho = \tfrac{1}{2}(-it_\sigma)\epsilon_{\rho\sigma\mu\nu}C_{\mu\nu}|t\cdot t|^{-\frac{1}{2}}. \tag{19}$$

If the expression for $C_{\mu\nu}$ from Eq. (8) is put into Eq. (19) and the definition of n from Eq. (8) is used, one obtains

$$c = Cn. \tag{20}$$

Equations (1), (8), and (18) now combine to give

$$\mathcal{S}_q(k',t,k)=A+B\gamma(t)+D(i\gamma_5\gamma\cdot n)+C\gamma(t)(i\gamma_5\gamma\cdot n). \quad (21)$$

With the introduction of the covariant projection operators

$$\Lambda^{\pm}(t)=\tfrac{1}{2}[1\pm\gamma(t)], \quad (22)$$

this reduces to

$$\mathcal{S}_q(k',t,k)=\textstyle\sum_{\pm}\Lambda^{\pm}(t)[F^{\pm}+G^{\pm}i\gamma_5\gamma\cdot n]. \quad (23)$$

In this form of $\mathcal{S}_q$, the $\sigma_{\mu\nu}$ type of term has been eliminated in favor of projection operators and terms of the $i\gamma_5\gamma_\rho$ type. Alternatively the $i\gamma_5\gamma_\rho$ may be eliminated in favor of projection operators and $\sigma_{\mu\nu}$'s. The form of the S matrix obtained by substituting Eq. (23) into Eq. (6′) is covariant and clearly separates the parts referring to the positive- and negative-energy states. This form will be used in the analysis of the polarization experiments in the third section. In the next section, the covariant form of the density matrix will be introduced and reduced in a manner quite similar to the reduction of the S matrix in this section.

II. COVARIANT DENSITY MATRIX

In the treatment of polarization phenomena it is necessary to consider mixtures of states, and a density matrix formulation is convenient. The expectation value of an operator A in the incident beam is expressed in terms of the density matrix $\rho(f)$ by the equation

$$\langle A\rangle_f=\mathrm{Tr}\rho(f)A/\mathrm{Tr}\rho(f). \quad (24)$$

For the scattered beam the corresponding equation is

$$\langle A\rangle_{f'}=\mathrm{Tr}\rho'(f')A/\mathrm{Tr}\rho'(f'). \quad (25)$$

The differential cross section is

$$I=\mathrm{Tr}\rho'(f')/\mathrm{Tr}\rho(f), \quad (26)$$

where the density matrices before and after the scattering are related by[12]

$$\rho'(f')=\mathcal{S}(f',t,f)\rho(f)\mathcal{S}^{\dagger}(f',t,f). \quad (27)$$

The adjoint $A^{\dagger}$ of an operator A is defined by the equation

$$(AU)^{\dagger}=U^{\dagger}A^{\dagger},$$

and thus

$$\mathcal{S}^{\dagger}=\beta\mathcal{S}^{*}\beta,$$

where the asterisk denotes complex conjugate transpose.

The covariant density matrices $\rho(f)$ and $\rho'(f')$ may be expressed in the forms

$$\begin{aligned}\rho(f)&=[\tfrac{1}{4}\,\mathrm{Tr}\rho(f)]\{1+\lambda_\mu\gamma_\mu+\tfrac{1}{2}s_{\mu\nu}\sigma_{\mu\nu}\\&\qquad+i\gamma_5\gamma_\mu p_\mu+J\gamma_5\},\\ \rho'(f')&=[\tfrac{1}{4}\,\mathrm{Tr}\rho'(f')]\{1+\lambda_\mu'\gamma_\mu+\tfrac{1}{2}s_{\mu\nu}'\sigma_{\mu\nu}\\&\qquad+i\gamma_5\gamma_\mu p_\mu'+J'\gamma_5\},\end{aligned} \quad (28)$$

where λ_μ, $s_{\mu\nu}$, p_μ, and J are respectively vector, antisymmetric tensor, pseudovector, and pseudoscalar, and similarly for the primed quantities. The condition that the Dirac particle must be definitely in a positive-energy state or definitely in a negative-energy state in the asymptotic region may be expressed by the equations

$$\begin{aligned}\rho(f)&=\gamma(f)\rho(f)\gamma(f),\\ \rho'(f')&=\gamma(f')\rho'(f')\gamma(f').\end{aligned} \quad (29)$$

By a treatment very similar to the reduction of the form of $\mathcal{S}_q$ in Sec. 1, the density matrices may now be reduced to the forms[13]

$$\begin{aligned}\rho(f)&=[\tfrac{1}{2}\,\mathrm{Tr}\rho(f)]\{\textstyle\sum_{\pm}\Lambda^{\pm}(f)\lambda^{\pm}(1+i\gamma_5\gamma\cdot p^{\pm})\},\\ \rho'(f')&=[\tfrac{1}{2}\,\mathrm{Tr}\rho'(f')]\{\textstyle\sum_{\pm}\Lambda^{\pm}(f')\lambda'^{\pm}(1+i\gamma_5\gamma\cdot p'^{\pm})\},\end{aligned} \quad (30)$$

where

$$p\cdot f=p'\cdot f'=0 \quad (31)$$

and

$$\lambda^{\pm}=\mathrm{Tr}[\rho(f)\Lambda^{\pm}(f)]/\mathrm{Tr}\rho(f)=\langle\Lambda^{\pm}(f)\rangle,$$

$$\lambda^{\pm}p^{\pm}=\frac{\mathrm{Tr}[\rho(f)\Lambda^{\pm}(f)i\gamma_5\gamma]}{\mathrm{Tr}[\rho(f)]}=\langle\Lambda^{\pm}i\gamma_5\gamma\rangle,$$

and similarly for the primed variables.

The value of $\lambda^{\pm}$ specifies the energy state. For a positive-energy particle $\lambda^{+}=1$ and $\lambda^{-}=0$, whereas for the negative-energy particle $\lambda^{+}=0$ and $\lambda^{-}=1$. The pseudovectors $p^{\pm}$ are the relativistic generalizations of the polarization vectors of the nonrelativistic treatment and describe the spin of the particle and antiparticle.

This form of the density matrix, used in conjunction with the form of the S matrix developed in Sec. 1, will give a covariant description of polarization phenomena. In the following section, this covariant treatment is applied to double- and triple-scattering experiments and relativistic corrections are obtained.

III. COVARIANT POLARIZATION FORMALISM

To find the state of polarization of a nucleon after a single scattering, one may put the expressions for $\mathcal{S}(f',t,f)$, $\rho(f)$, and $\rho'(f')$ given in Eqs. (6′), (23), and (30) into Eq. (27), which relates $\rho(f)$ and $\rho'(f')$. With the help of the relations $\gamma(u)=\gamma^{\dagger}(u)$ and $\gamma(u,w)=\gamma^{\dagger}(u,w)$ for timelike u and w, one then obtains

$$\begin{aligned}&\frac{\mathrm{Tr}\rho'(f')}{\mathrm{Tr}\rho(f)}\{\textstyle\sum_{\pm}\Lambda^{\pm}(f')\lambda'^{\pm}(1+i\gamma_5\gamma\cdot p'^{\pm})\}\\&\quad=(\gamma(f',t)\gamma(t))\{\textstyle\sum_{\pm}\Lambda^{\pm}(t)(F^{\pm}+iG^{\pm}\gamma_5\gamma\cdot n)\}\\&\qquad\times(\gamma(t)\gamma(t,f))\{\textstyle\sum_{\pm}\Lambda^{\pm}(f)\lambda^{\pm}(1+i\gamma_5\gamma\cdot p^{\pm})\}\\&\qquad\times(\gamma(f,t)\gamma(t))\{\textstyle\sum_{\pm}\Lambda^{\pm}(t)(F^{\pm *\frac{2}{3}}+iG^{\pm *\frac{2}{3}}\gamma_5\gamma\cdot n)\}\\&\qquad\qquad\times(\gamma(t)\gamma(t,f')).\end{aligned} \quad (32)$$

[12] See Appendix for a discussion of the covariant density matrix used here.

[13] This form has been used by L. Michel and A. S. Wightman. See reference 6.

By reducing the right-hand side of this equation to the form appearing on the left, one may obtain the polarization $p'^{\pm}$ of the final beam in terms of $p^{\pm}$, the initial polarization, and $F^{\pm}$ and $G^{\pm}$, the scattering parameters. At the same time the differential cross section

$$I=\mathrm{Tr}\rho'(f')/\mathrm{Tr}\rho(f)$$

will be obtained. Before performing this reduction, however, it is convenient to transform the equation into a simpler form. In particular the equation may be separated into two equations, each of which involves only two-by-two matrices and refers to a single type of particle. This not only simplifies computations but allows a more direct comparison to the nonrelativistic formulation.

To obtain this simplification, the relations

$$\gamma(u)\gamma(u)=\gamma(u,w)\gamma(u,w)=1$$

may be used to first transform Eq. (32) into

$$\begin{aligned}I&(\gamma(t)\gamma(f',t))\{\textstyle\sum\Lambda(f')\lambda'(1+i\gamma_5\gamma\cdot p')\}(\gamma(f',t)\gamma(t))\\&=\{\textstyle\sum\Lambda(t)(F+iG\gamma_5\gamma\cdot n)\}\\&\quad\times(\gamma(t)\gamma(f,t))\{\textstyle\sum\Lambda(f)\lambda(1+i\gamma_5\gamma\cdot p)\}(\gamma(f,t)\gamma(t))\\&\quad\times\{\textstyle\sum\Lambda(t)(F^*+iG^*\gamma_5\gamma\cdot n)\},\end{aligned}\tag{33}$$

where the $\pm$ signs are now to be understood. Using the Lorentz transformations $L(t)$ and $L^\dagger(t)=L^{-1}(t)$, we may write this

$$\begin{aligned}IL&(t)(\gamma(t)\gamma(f',t))\{\textstyle\sum\Lambda(f')\lambda'(1+i\gamma_5\gamma\cdot p')\}\\&\qquad\times(\gamma(f',t)\gamma(t))L^\dagger(t)\\&=L(t)\{\textstyle\sum\Lambda(t)(F+iG\gamma_5\gamma\cdot n)\}L^\dagger(t)\\&\quad\times L(t)(\gamma(t)\gamma(f,t))\{\textstyle\sum\Lambda(f)\lambda(1+i\gamma_5\gamma\cdot p)\}\\&\quad\times(\gamma(f,t)\gamma(t))L^\dagger(t)\\&\qquad\times L(t)\{\textstyle\sum\Lambda(t)(F^*+iG^*\gamma_5\gamma\cdot n)\}L^\dagger(t).\end{aligned}\tag{34}$$

The $L(t)$ has the property that

$$L(t)\gamma_\mu L^\dagger(t)=\mathfrak{L}_{\mu\nu}(t)\gamma_\nu,\tag{35}$$

where $\mathfrak{L}_{\mu\nu}(t)$ satisfies

$$\begin{aligned}x_\mu\mathfrak{L}_{\mu\nu}(t)&=(x_1)_\nu,\\x_\mu&=\mathfrak{L}_{\mu\nu}(t)(x_1)_\nu,\end{aligned}\tag{36}$$

$(x_1)_\mu$ being the components of any arbitrary vector x in the center-of-mass frame. Using Eqs. (35), (36), and (12), one finds

$$\begin{aligned}L(t)(\gamma(t)\gamma(x,t))L^\dagger(t)&=\gamma(t_1)\gamma(x_1,t_1)\\&=L(x_1),\end{aligned}\tag{37}$$

where x may be f or f'. Equation (34) may then be written

$$\begin{aligned}I&(L(f_1')L(t))\{\textstyle\sum\Lambda(f')\lambda(1+i\gamma_5\gamma\cdot p')\}(L^\dagger(t)L^\dagger(f_1'))\\&=L(t)\{\textstyle\sum\Lambda(t)(F+iG\gamma_5\gamma\cdot n)\}L^\dagger(t)\\&\quad\times(L(f_1)L(t))\{\textstyle\sum\Lambda(f)\lambda(1+i\gamma_5\gamma\cdot p)\}(L^\dagger(t)L^\dagger(f_1))\\&\qquad\times L(t)\{\textstyle\sum\Lambda(t)(F^*+iG^*\gamma_5\gamma\cdot n)\}L^\dagger(t).\end{aligned}\tag{38}$$

With the introduction of the pure space-rotation transformation

$$R(x_1)\equiv L(x_1)L(t)L^\dagger(x),\tag{39}$$

one obtains

$$\begin{aligned}IR&(f_1')L(f')\{\textstyle\sum\Lambda(f')\lambda'(1+i\gamma_5\gamma\cdot p')\}L^\dagger(f')R^\dagger(f_1')\\&=L(t)\{\textstyle\sum\Lambda(t)(F+iG\gamma_5\gamma\cdot n)L^\dagger(t)\\&\quad\times R(f_1)L(f)\{\textstyle\sum\Lambda(f)(1+i\gamma_5\gamma\cdot p)\}L^\dagger(f)R^\dagger(f_1)\\&\qquad\times L(t)\{\textstyle\sum\Lambda(t)(F^*+iG^*\gamma_5\gamma\cdot n)\}L^\dagger(t).\end{aligned}\tag{40}$$

Defining

$$\begin{aligned}\mathfrak{p}_\mu&\equiv p_\nu\mathfrak{L}_{\nu\mu}(f),\\\mathfrak{p}_\mu'&\equiv p_\nu'\mathfrak{L}_{\nu\mu}(f'),\\\mathfrak{n}_\mu&\equiv n_\nu\mathfrak{L}_{\nu\mu}(t),\\\Lambda^\pm(0)&\equiv\tfrac{1}{2}(1\pm\beta),\end{aligned}$$

and using equations similar to Eqs. (35) and (36), one obtains

$$\begin{aligned}IR&(f_1')\{\textstyle\sum\Lambda(0)\lambda'(1+i\gamma_5\gamma\cdot\mathfrak{p}')\}R^\dagger(f_1')\\&=\{\textstyle\sum\Lambda(0)(F+iG\gamma_5\gamma\cdot\mathfrak{n})\}\\&\qquad\times R(f_1)\{\textstyle\sum\Lambda(0)\lambda(1+i\gamma_5\gamma\cdot\mathfrak{p})\}R^\dagger(f_1)\\&\qquad\qquad\times\{\textstyle\sum\Lambda(0)(F^*+iG^*\gamma_5\gamma\cdot\mathfrak{n})\}.\end{aligned}\tag{41}$$

According to their definitions the $\mathfrak{p}$, $\mathfrak{p}'$, and $\mathfrak{n}$ are the values of p, p', and n in the Lorentz frame where f, f', and t, respectively, are purely timelike. Thus from the conditions

$$p\cdot f=p'\cdot f'=n\cdot t=0$$

the four-vectors $\mathfrak{p}$, $\mathfrak{p}'$, and $\mathfrak{n}$ must have vanishing fourth components. Considered as three-vectors, the vectors $\mathfrak{p}$ and $\mathfrak{p}'$ are, in fact, just the proper polarizations of the incident and final beams, and $\mathfrak{n}$ is the normal to the scattering plane as measured in the center-of-mass frame. With the definition

$$R(x_1)\gamma_iR^\dagger(x_1)\equiv r_{ij}(x_1)\gamma_j,\tag{42}$$

Eq. (41) reduces to

$$\begin{aligned}I&\textstyle\sum_\pm\Lambda^\pm(0)\lambda'^\pm(1+i\gamma_5\mathfrak{p}'_i{}^\pm r_{ij}(f_1')\gamma_j)\\&=\{\textstyle\sum_\pm\Lambda^\pm(0)(F^\pm+iG^\pm\gamma_5\gamma_i\mathfrak{n}_i)\}\\&\qquad\times\{\textstyle\sum_\pm\Lambda^\pm(0)\lambda^\pm(1+i\gamma_5\mathfrak{p}_i{}^\pm r_{ij}(f_1)\gamma_j)\}\\&\qquad\qquad\times\{\textstyle\sum_\pm\Lambda^\pm(0)(F^{\pm*}+iG^{\pm*}\gamma_5\gamma_i\mathfrak{n}_i)\},\end{aligned}\tag{43}$$

where i and j need be summed only from 1 to 3. Since

$$i\gamma_5\gamma_i=\beta\sigma_i\quad(i=1,2,3),$$

this equation splits into two parts, each of which is an equation in two-by-two matrices which refers to a single type of particle.

For the cases $\lambda^+=1$ or $\lambda^-=1$, the equations may be written

$$\begin{aligned}I^\pm(1+P_i'{}^\pm\sigma_i)&=(F^\pm\pm G^\pm N_i\sigma_i)\\&\qquad\times(1\pm P_i{}^\pm\sigma_i)(F^{\pm*}\pm G^{\pm*}N_i\cdot\sigma_i),\end{aligned}$$

thereby defining $I^\pm$. The σ_i are now the two-by-two

Pauli matrices and the vectors P and P' are defined by

$$P_i=\mathfrak{p}_j r_{ji}(f_1) \quad P_i'=\mathfrak{p}_j' r_{ji}(f_1'). \tag{44}$$

These equations are, except for a sign change in σ_i for the negative energy states, identical with the equations obtained from the nonrelativistic treatment, except that the vectors $\mathbf{P}$ and $\mathbf{P}'$ replace the polarization vectors of the nonrelativistic treatment. In the analysis of double- and triple-scattering experiments one may proceed much as in the nonrelativistic case, remembering, however, that it is the proper polarization vector $\mathfrak{p}$, rather than $\mathbf{P}$, which is the same in the outgoing beam of one scattering as in the incoming beam for the next. The connection between the $\mathbf{P}$ of one scattering and the $\mathbf{P}'$ of the preceding scattering is

$$P_i^{(n)}=P_j'^{(n-1)} r_{jk}^{-1}(f_{n-1}') r_{ki}(f_n), \tag{45}$$

where Eq. (44) has been used in conjunction with the identity $\mathfrak{p}_i'^{(n)}\equiv\mathfrak{p}_i^{(n-1)}$. The superscript (n) will denote the quantities referring to the nth scattering and the subscript n on the four-momenta denotes their center-of-mass values. The rotations appearing on the left of Eq. (45) will introduce certain differences between the relativistic and nonrelativistic treatments. These will be called the rotational corrections.

A second type of correction comes from the use of the relativistic transformation of momenta between the successive frames. Thus the relation between the incoming momentum for the nth scattering and the outgoing momentum for the preceding scattering as measured in their respective center-of-mass frames is

$$(f_n)_\mu=(f_{n-1}')_\lambda \mathfrak{L}_{\lambda\mu}^{-1}(t^{(n-1)})\mathfrak{L}_{\nu\mu}(t^{(n)}). \tag{46}$$

The major portion of the transformation appearing here will, except for extreme relativistic cases, be given by the nonrelativistic Galilean transformation. The remainder will be called the kinematical corrections.

To analyze double- and triple-scattering experiments, it appears most convenient to choose the laboratory system as the basic reference frame. Assuming the target particles to be at rest in the laboratory system, one notices that

$$\mathfrak{p}^{(n)}=\mathbf{P}^{(n)},$$

since the three Lorentz transformations that give the rotation

$$r_{\mu\nu}(f_n)\equiv\mathfrak{L}_{\mu\eta}^{-1}(f^{(n)})\,\mathfrak{L}_{\eta\lambda}(t^{(n)})\mathfrak{L}_{\lambda\nu}(f_n) \tag{47}$$

will be colinear and their product will be unity. For the scattered beam, however, the $\mathfrak{p}'$ and $\mathbf{P}'$ will differ. The formal manipulations in the relativistic treatment will, therefore, be identical with those of the nonrelativistic treatment except for the following two modifications: first, the connection between the momenta in the successive center-of-mass frames is given by Eq. (46); the second, an extra rotation $r_{\mu\nu}^{-1}(f_1')$ is applied to the polarization vector in the outgoing beam before it is interpreted as the incident polarization of the next scattering, or as the proper polarization. The rotation $r_{\mu\nu}^{-1}(f_1')$ is the effect of the three successive Lorentz transformations, which take a vector from its value in a rest frame of the scattered particle to the center-of-mass frame; then from center-of-mass to laboratory system; and finally from laboratory system back to a (new) rest frame of the scattered particle. This rotation may be specified by an axial vector $\mathbf{\Omega}$, whose magnitude is given by the equation

$$\sin|\mathbf{\Omega}|=|\mathbf{V}_a\times\mathbf{V}_b| \times\frac{1+(\gamma)^{(a)}+(\gamma)^{(b)}+(\gamma)^{(c)}}{[1+(\gamma)^{(a)}][1+(\gamma)^{(b)}][1+(\gamma)^{(c)}]}, \tag{48}$$

where $(\gamma)^{(a)}$, $(\gamma)^{(b)}$, and $(\gamma)^{(c)}$ are the Lorentz contraction factors associated with the three transformations listed above and $\mathbf{V}_a$, $\mathbf{V}_b$, and $\mathbf{V}_c$ are the space parts of the three relative relativistic velocities, respectively. The transformations and the corresponding rotation are schematically represented in Fig. 1, where $\theta^{(n)}$ and θ_n are the laboratory and center-of-mass scattering angles respectively. Since the rotation is about an axis perpendicular to the plane of scattering it may be neglected in the simple double-scattering experiments and in the depolarization experiments; in these experiments the polarization vector is always perpendicular to the scattering plane and the rotation will not affect it.

In triple-scattering experiments of the rotation category the polarization vector will have components in the plane of the second scattering. The asymmetry in the differential cross section after the third scattering will measure the component of proper polarization which is in the plane of the second scattering and which is perpendicular to the laboratory direction of the scattered beam. Both the kinematical and rotational effects will play a role. As an example, the important case in which the masses of the Dirac particle and the second target particle are equal will be treated. The considerations of the next section show that the results obtained here will be applicable to the case in which the second target is a Dirac particle.

Because of the kinematical corrections the second laboratory scattering angle $\theta^{(2)}$ is not $\theta_2/2$. The difference may be defined as

$$\alpha=\tfrac{1}{2}\theta_2-\theta^{(2)}=\tfrac{1}{2}\theta_{\text{c.m.}}-\theta_{\text{lab}}.$$

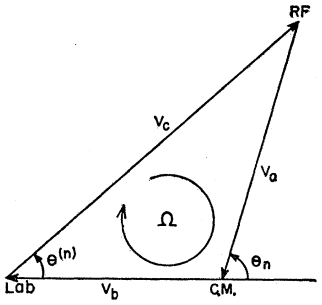

FIG. 1. Diagram showing the sense of rotation of the polarization vector caused by the relativistic effect. The vectors represent schematically the relative velocities of the center-of-mass frame, the laboratory frame, and the rest frame of the scattered particle.

Since it is the component of polarization perpendicular to the laboratory direction of the scattered beam that is measured, there will, for a fixed θ_2, be a kinematical correction of the direction that specifies the component of polarization which is measured by the angle α. There will also be a rotational correction which changes the direction of the polarization vector by the angle $\delta=|\mathbf{\Omega}|$. The effect of this second correction may be accounted for by letting the polarization vector remain fixed but rotating the direction of the component which is in effect measured, by the angle $-\delta$. Taking the various senses into account, one finds that the net effect of the two corrections is to rotate the direction of the effective component by $(\delta-\alpha)$ about the normal vector N. A calculation shows that $\delta=2\alpha$, and the rotational effect just reverses the kinematical correction. This has the simple physical consequence that the direction of the effective component makes an angle $\theta^{(2)}$ with the normal to the center-of-mass velocity. The relativistic expression for the rotation parameter[14] R in the $P-P$ system, therefore, takes the relatively simple form[15]

$$I_0R=(|a|^2-|m|^2)\cos(\theta_{\text{c.m.}}-\theta_{\text{lab}})-4\,\text{Re}[gh^*\cos(\theta_{\text{lab}})]+2\,\text{Re}[ic(a^*-m^*)\sin(\theta_{\text{c.m.}}-\theta_{\text{lab}})], \quad (49)$$

where $\theta_{\text{c.m.}}$ and θ_{lab} are the center-of-mass and laboratory angles at the second scattering. To obtain this last equation it was assumed that the prescription for extending the nonrelativistic formulas into the relativistic domain will continue to be valid when the target particle has internal coordinates. In the next section, the case in which the target is another Dirac particle is considered and this assumption is validated.

IV. POLARIZATION FORMALISM FOR TWO DIRAC PARTICLES

In the developments in the preceding sections, it was assumed that the target particle had no internal coordinates. The form of the results suggests that the relativistic corrections involving the spin state of the first particle would not be changed if the second particle were to possess internal coordinates. Indeed, one finds that the manipulations involving the first-particle spin state may be carried out almost unchanged if the second particle possesses spin. In this section, the important case in which the second particle is also a Dirac particle is considered and the expected generalization is obtained. In this treatment, it will be assumed that the two particles are distinguishable. Indistinguishable particles may then be treated by an appropriate antisymmetrization of the results.

The S matrix for the system of two Dirac particles may be expressed as a sum of terms, each of which is a product of an operator in the first spin space times an operator in the second spin space. Thus one may take all possible bilinear combinations of the matrices

$$(I^{(1)},\ \gamma_\mu^{(1)},\ \tfrac{1}{2}\sigma_{\mu\nu}^{(1)},\ i\gamma_5^{(1)}\gamma_\mu^{(1)},\ \gamma_5^{(1)};\ I^{(2)},\ \gamma_\mu^{(2)},\ \tfrac{1}{2}\sigma_{\mu\nu}^{(2)},\ i\gamma_5^{(2)}\gamma_\mu^{(2)},\ \gamma_5^{(2)}),$$

which are linear in the first and in the second subsets.

In exact analogy to the case treated above, the matrix $\mathcal{S}_q(k',t,k)$ may be defined by

$$\mathcal{S}(f',h',t,f,h)=(\gamma^{(1)}(f',t)\gamma^{(1)}(t))(\gamma^{(2)}(h',t)\gamma^{(2)}(t))\times\mathcal{S}_q(k',t,k)(\gamma^{(2)}(t)\gamma^{(2)}(h,t))\times(\gamma^{(1)}(t)\gamma^{(1)}(f,t)), \quad (50)$$

where h and h' are the initial and final momenta of the second Dirac particle. The hole-theory condition may be introduced and used in a manner analogous to the reduction to Eq. (7), with the results that

$$\gamma^{(1)}(t)\mathcal{S}_q(k',t,k)\gamma^{(1)}(t)=\mathcal{S}_q(k',t,k),$$
$$\gamma^{(2)}(t)\mathcal{S}_q(k',t,k)\gamma^{(2)}(t)=\mathcal{S}_q(k',t,k).$$

Consider now the term in $\mathcal{S}_q(k',t,k)$ of the form

$$C_{\mu\nu\sigma\rho}(\tfrac{1}{2}\sigma_{\mu\nu}^{(1)})(\tfrac{1}{2}\sigma_{\sigma\rho}^{(2)}).$$

The condition that this term commute with $\gamma^{(1)}(t)$ requires, in analogy to Eq. (17), that

$$t_\mu C_{\mu\nu\sigma\rho}=-t_\mu C_{\nu\mu\sigma\rho}=0.$$

Now, applying the arguments that led to Eq. (18), one obtains

$$C_{\mu\nu\sigma\rho}(\tfrac{1}{2}\sigma_{\mu\nu}^{(1)})=\gamma^{(1)}(t)(i\gamma_5^{(1)}\gamma_\lambda^{(1)})C_{\lambda;\sigma\rho},$$

where $t_\lambda C_{\lambda;\sigma\rho}=0$. The dependence on $\sigma_{\sigma\rho}^{(2)}$ may be similarly transformed to give

$$C_{\mu\nu\sigma\rho}(\tfrac{1}{2}\sigma_{\mu\nu}^{(1)})(\tfrac{1}{2}\sigma_{\sigma\rho}^{(2)})=C_{\lambda\eta}\gamma^{(1)}(t)(i\gamma_5^{(1)}\gamma_\lambda^{(1)})\gamma^{(2)}(t)(i\gamma_5^{(2)}\gamma_\eta^{(2)}),$$

where $C_{\lambda\eta}t_\eta=t_\lambda C_{\lambda\eta}=0$. Eliminating all terms containing $\sigma_{\mu\nu}$'s in a similar manner, one obtains

$$\begin{aligned}\mathcal{S}_q(k',t,k)=&F+F^{(1)}\gamma^{(1)}(t)+F^{(2)}\gamma^{(2)}(t)\\&+G_\lambda^{(1)}(i\gamma_5^{(1)}\gamma_\lambda^{(1)})+G_\lambda^{(2)}(i\gamma_5^{(2)}\gamma_\lambda^{(2)})\\&+C_\lambda^{(1)}\gamma^{(1)}(t)(i\gamma_5^{(1)}\gamma_\lambda^{(1)})+C_\lambda^{(2)}\gamma^{(2)}(t)(i\gamma_5^{(2)}\gamma_\lambda^{(2)})\\&+G_{\lambda\rho}(i\gamma_5^{(1)}\gamma_\lambda^{(1)})(i\gamma_5^{(2)}\gamma_\rho^{(2)})\\&+C_{\lambda\rho}\gamma^{(1)}(t)(i\gamma_5^{(1)}\gamma_\lambda^{(1)})\gamma^{(2)}(t)(i\gamma_5^{(2)}\gamma_\rho^{(2)})\\&+F^{(3)}\gamma^{(1)}(t)\gamma^{(2)}(t)\\&+E_\lambda^{(1)}\gamma^{(2)}(t)(i\gamma_5^{(1)}\gamma_\lambda^{(1)})\\&+E_\lambda^{(2)}\gamma^{(1)}(t)(i\gamma_5^{(2)}\gamma_\lambda^{(2)})\\&+D_\lambda^{(1)}\gamma^{(2)}(t)\gamma^{(1)}(t)(i\gamma_5^{(1)}\gamma_\lambda^{(1)})\\&+D_\lambda^{(2)}\gamma^{(1)}(t)\gamma^{(2)}(t)(i\gamma_5^{(2)}\gamma_\lambda^{(2)})\\&+H_{\lambda\rho}^{(1)}(i\gamma_5^{(1)}\gamma_\lambda^{(1)})\gamma^{(2)}(t)(i\gamma_5^{(2)}\gamma_\rho^{(2)})\\&+H_{\lambda\rho}^{(2)}(i\gamma_5^{(2)}\gamma_\lambda^{(2)})\gamma^{(1)}(t)(i\gamma_5^{(1)}\gamma_\rho^{(1)}).\end{aligned}$$

The coefficients appearing here are functions of k', t, k and are pseudovectors and tensors which are orthogonal

[14] This is the R parameter which is measured in experiments in which magnetic fields are not used to rotate the directions of polarization vectors. See reference 5.

[15] In the notation, used elsewhere by this author, where the M matrix is written $M=a+c(\sigma_{1N}+\sigma_{2N})+m(\sigma_{1N}\sigma_{2N})+g(\sigma_{1P}\sigma_{2P}+\sigma_{1K}\sigma_{2K})+h(\sigma_{1P}\sigma_{2P}-\sigma_{1K}\sigma_{2K})$, with the vectors N, P, and K as defined by Wolfenstein and Ashkin.

to t on all indices. Thus, for example, $t_\lambda H_{\lambda\rho}'=t_\rho H_{\lambda\rho}'=0$. Now the first two terms may be transformed into a more suitable form:

$$F+F^{(1)}\gamma^{(1)}(t)=\sum_\pm \tfrac{1}{2}[1\pm\gamma^{(1)}(t)]F^\pm,$$

where

$$\tfrac{1}{2}(F^++F^-)=F,\quad \tfrac{1}{2}(F^+-F^-)=F^{(1)}.$$

In the same way, the rest of the terms may be grouped in pairs to give

$$\begin{aligned}\mathcal{S}_q(k',t,k)=\sum_\pm \tfrac{1}{2}[1\pm\gamma^{(1)}(t)]\{&F^\pm+G_\lambda{}^{(1)\pm}i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)}\\&+F^{(2)\pm}\gamma^{(2)}(t)+G_\lambda{}^{(2)\pm}(i\gamma_5{}^{(2)}\gamma_\lambda{}^{(2)})\\&+C_\lambda{}^{(2)\pm}\gamma^{(2)}(t)(i\gamma_5{}^{(2)}\gamma_\lambda{}^{(2)})\\&+G_{\lambda\rho}{}^{\pm}(i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)})(i\gamma_5{}^{(2)}\gamma_\lambda{}^{(2)})\\&+H_{\lambda\rho}{}^{(1)\pm}(i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)})\gamma^{(2)}(t)(i\gamma_5{}^{(2)}\gamma_\rho{}^{(2)})\\&+E_\lambda{}^{(1)\pm}\gamma^{(2)}(t)(i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)})\}.\end{aligned}$$

Performing the analogous grouping relative to $\gamma^{(2)}(t)$, one obtains

$$\begin{aligned}\mathcal{S}_q(k',t,k)=\sum_{\pm\pm}\{&\tfrac{1}{2}[1\pm\gamma^{(1)}(t)]\tfrac{1}{2}[1\pm\gamma^{(2)}(t)]\}\\&\times\{F^{\pm\pm}+G_\lambda{}^{(1)\pm\pm}(i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)})\\&+G_\lambda{}^{(2)\pm\pm}(i\gamma_5{}^{(2)}\gamma_\lambda{}^{(2)})\\&+G_{\lambda\rho}{}^{\pm\pm}(i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)})(i\gamma_5{}^{(2)}\gamma_\rho{}^{(2)}),\end{aligned}$$

where

$$t_\lambda G_\lambda{}^{(1)\pm\pm}=t_\lambda G_\lambda{}^{(2)\pm\pm}=t_\lambda G_{\lambda\rho}{}^{\pm\pm}=G_{\lambda\rho}{}^{\pm\pm}t_\rho=0. \quad (51)$$

The $G_\lambda{}^{(1)\pm\pm}$ and $G_\lambda{}^{(2)\pm\pm}$ must be pseudovectors and may therefore be written

$$G_\lambda{}^{(1)\pm\pm}=G^{(1)\pm\pm}n_\lambda,\quad G_\lambda{}^{(2)\pm\pm}=G^{(2)\pm\pm}n_\lambda,$$

where n_λ are the components of the only available unit pseudovector, that is,

$$n_\lambda\propto k_\rho'k_\sigma t_\mu\epsilon_{\rho\sigma\mu\lambda}.$$

The tensors $G_{\lambda\rho}{}^{\pm\pm}$ are, on the other hand, not restricted to a single type of term. The classification of possible tensor terms is facilitated by introducing the normalized vectors

$$\begin{aligned}s_\lambda&=N_s[k_\lambda+k_\lambda'-t_\lambda\{t_\rho(k_\rho+k_\rho')\}(t\cdot t)^{-1}],\\d_\lambda&=N_d[k_\lambda-k_\lambda'].\end{aligned}$$

The vectors t, n, s, d form an orthogonal set. The condition in Eq. (51) limits the possible terms in the $G_{\lambda\rho}{}^{\pm\pm}$ to those bilinear in the components of n, s, and d. If the requirement of invariance under spatial reflections is invoked, the $G_{\lambda\rho}{}^{\pm\pm}$ reduce to the form

$$\begin{aligned}G_{\lambda\rho}{}^{\pm\pm}=&C^{\pm\pm}n_\lambda n_\rho+D^{\pm\pm}s_\lambda s_\rho+E^{\pm\pm}d_\lambda d_\rho\\&+G'^{\pm\pm}(s_\lambda d_\rho+d_\lambda s_\rho)+G^{\pm\pm}(s_\lambda d_\rho-d_\lambda s_\rho).\end{aligned}$$

Just as in the nonrelativistic case, the required invariance under time inversion removes the last two terms, since d_λ retains its sign under time inversion, whereas s_λ changes sign. Thus the $\mathcal{S}_q(k',t,k)$ finally takes the form

$$\begin{aligned}\mathcal{S}_q(k',t,k)=\sum_{\pm\pm}&(\Lambda^{(1)\pm}(t)\Lambda^{(2)\pm}(t))\\&\times[F^{\pm\pm}+G^{(1)\pm\pm}i\gamma_5{}^{(1)}\gamma^{(1)}\cdot n+G^{(2)\pm\pm}i\gamma_5{}^{(2)}\gamma^{(2)}\cdot n\\&+(C^{\pm\pm}n_\lambda n_\rho+D^{\pm\pm}s_\lambda s_\rho+E^{\pm\pm}d_\lambda d_\rho)\\&\times(i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)})(i\gamma_5{}^{(2)}\gamma_\rho{}^{(2)})].\end{aligned} \quad (52)$$

In a very similar way, the density matrix is reduced to the form

$$\begin{aligned}\rho(f,h)=\tfrac{1}{4}\,\mathrm{Tr}\rho(f,h)&[\textstyle\sum_{\pm\pm}(\Lambda^{(1)\pm}(f)\Lambda^{(2)\pm}(h))\lambda^{\pm\pm}\\&\times[1+i\gamma_5{}^{(1)}\gamma_\mu{}^{(1)}p_\mu{}^{(1)\pm\pm}+i\gamma_5{}^{(2)}\gamma_\mu{}^{(2)}p_\mu{}^{(2)\pm\pm}\\&+(i\gamma_5{}^{(1)}\gamma_\lambda{}^{(1)})(i\gamma_5{}^{(2)}\gamma_\eta{}^{(2)})c_{\lambda\eta}{}^{\pm\pm}],\end{aligned} \quad (53)$$

where $p_\mu{}^{(1)\pm\pm}$, $p_\mu{}^{(2)\pm\pm}$, and $c_{\lambda\eta}{}^{\pm\pm}$ are the polarization and correlation parameters for the four types of systems, and satisfy

$$p_\mu{}^{(1)\pm\pm}f_\mu=p_\mu{}^{(2)\pm\pm}h_\mu=c_{\lambda\rho}{}^{\pm\pm}h_\rho=f_\lambda c_{\lambda\rho}{}^{\pm\pm}=0.$$

These forms for ρ and $\mathcal{S}_q$ may now be substituted into

$$\rho'(f',h')=\mathcal{S}(f',h',t,f,h)\rho(f,h)\mathcal{S}^\dagger(f',h',t,f,h).$$

The transformations carried out in Sec. 3 may then be performed upon the matrices in the two spin spaces independently and the equation will split into four equations in the two-by-two matrices, each of which is identical in form to the nonrelativistic equations. The quantities appearing in the places of the nonrelativistic polarization and correlation components will be

$$\begin{aligned}P_i{}^{(1)}&=\mathfrak{p}_j{}^{(1)}r_{ji}(f_1),\\P_i{}^{(2)}&=\mathfrak{p}_j{}^{(2)}r_{ji}(h_1),\\P_i'{}^{(1)}&=\mathfrak{p}_j'{}^{(1)}r_{ji}(f_1'),\\P_i'{}^{(2)}&=\mathfrak{p}_j'{}^{(2)}r_{ji}(h_1'),\end{aligned} \quad (54)$$

$$\begin{aligned}C_{ij}&=\mathfrak{c}_{km}r_{ki}(f_1)r_{mj}(h_1),\\C_{ij}'&=\mathfrak{c}_{km}'r_{ki}(f_1')r_{mj}(h_1'),\end{aligned} \quad (55)$$

where now the superscripts refer to the first or second particle and the $r_{\mu\nu}(x_1)$ is defined as

$$r_{\mu\nu}(x_1)=\mathfrak{L}_{\mu\eta}{}^{-1}(x)\mathfrak{L}_{\eta\lambda}(t)\mathfrak{L}_{\lambda\nu}(x_1). \quad (56)$$

The modifications of the nonrelativistic formulas which the relativistic effects introduce are seen, now, to be completely parallel to those obtained when the target had no spin, and the assumption used at the end of Sec. 3 is valid.

In the treatment of the correlation experiments the relativistic effects on both particles must be considered. In the C_{NN} type of correlation experiment, where the components of polarization perpendicular to the scattering plane are measured, the rotations will again play no role. In the C_{KP} experiment the relativistic corrections will not vanish. The application of Eqs. (55) and (56) shows that the expression for the quantity measured in these experiments is, in the relativistic region,

$$\begin{aligned}I_0C_{KP}=&4\,\mathrm{Re}(ich^*)\\&-2\,\mathrm{Re}[g(a^*-m^*)\sin(\theta_{\mathrm{c.m.}}-2\theta_{\mathrm{lab}})].\end{aligned} \quad (57)$$

CONCLUSIONS

It has been shown that the covariant treatment of the polarization effects may be transformed into a form which separates the positive and negative energy states. The formulas for either energy state are in terms of two-by-two matrices and the theory is very similar in appearance to the nonrelativistic theory. The three-dimensional vector, which in these relativistic equations takes the place of the polarization vector of the Pauli treatment, is the *proper* polarization vector. The *proper* polarization vector is defined as the axial three-vector whose components are equal to the space components of the four-dimensional pseudovector p_μ (which specifies the polarization in the relativistic theory) when these components are measured in a particular rest frame of the particle in question. This particular rest frame is the one generated by transforming the basic reference frame (which is conveniently taken as the laboratory frame) by means of a single timelike Lorentz transformation into a frame in which the particle is at rest. (A timelike Lorentz transformation is defined to be one involving the time axis and only a single space direction. It is important to recognize that the rest frame which would be generated by a succession of timelike Lorentz transformations would in general have a different orientation of its spatial axes.) Except for a rotational effect to be described below, the manipulations in the relativistic case may be carried through exactly as in the nonrelativistic case; as in that case the successive collisions are described in their resecptive center-of-mass frames. The spatial orientations of these center-of-mass frames are again to be determined by transforming the basic reference frame to the appropriate velocity by means of a single timelike Lorentz transformation. Thus the transformation of the outgoing momentum vector for one collision into the incoming momentum of the next collision involves transforming from the value in the first center-of-mass frame to the laboratory frame and then to the second center-of-mass frame.

The relativistic treatment differs, however, in one important way from the nonrelativistic treatment. In a particular collision, rotations must be applied to the incident and final proper polarization vectors to transform them from those rest frames of the incident and final particles which are associated with the basic reference frame, by means of single timelike Lorentz transformations, to those rest frames which are associated with the center-of-mass frame in the same way; it is these *latter* three-vectors which are transformed from initial to final values by means of the two-by-two form of the S matrix. Thus, in a sequence of collisions, one must take into account the rotations induced by changing between the various rest frames. In the laboratory frame, the rotations associated with the incident beam vanish if the targets are at rest in the laboratory system, and the rotational effect reduces to an additional rotation of the polarization vector which must be added after each scattering. This rotation gives the effect on the components of a vector which is induced by transforming the coordinate system from a frame in which the particle is at rest to a frame in which the center-of-mass is at rest, then next transforming to a coordinate system in which the laboratory is at rest, and then finally transforming this coordinate system to the new coordinate system in which the particle is at rest. Each of these transformations is supposed to take place by means of a Lorentz transformation which involves the time axis and a single space direction. The effects of these transformations of coordinate systems upon the coordinates of the polarization vector may be expressed as the effect of a rotation of this vector with respect to a fixed frame. It is this rotation of the polarization vector whose magnitude and sense are given by Eq. (48) and the accompanying diagram. This is the additional rotation of the polarization vector which must be added after each scattering if the relativistic results are to be obtained from the nonrelativistic formalism.

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APPENDIX. COVARIANT DENSITY MATRIX

In situations in which statistical mixtures of states are considered, it is convenient to introduce the density matrix ρ, which in an appropriate representation may be written

$$\rho=\sum_\alpha|\psi_\alpha>W_\alpha<\psi_\alpha|,$$

where W_α is the probability that the system is in the state α, so that $\sum W_\alpha=1$. The probability of finding the system in a region R may be written

$$w(R)=\mathrm{Sp}\rho\mathcal{P},$$

where Sp is the trace over both coordinate and spin variables, and $\mathcal{P}$ is the operator that projects onto the region R. If R is taken as the three-dimensional momentum region $(d\mathbf{f})=df_1df_2df_3$, then

$$w(d\mathbf{f})=(d\mathbf{f})\ \mathrm{Tr}\rho_S(\mathbf{f}),$$

where Tr is the trace in spin space and

$$\rho_S(\mathbf{f})=\sum_\alpha|\,\mathfrak{A}_\alpha(\mathbf{f})\,|^2\{|\,U_\alpha(\mathbf{f})\rangle W_\alpha\langle U_\alpha^*(\mathbf{f})\,|\}.$$

The amplitude $\mathfrak{A}_\alpha(\mathbf{f})$ is a function of the three-momentum $\mathbf{f}$ defined in terms of $\psi_\alpha(\mathbf{f})$, the momentum-space wave function, by

$$\psi_\alpha(\mathbf{f})=\mathfrak{A}_\alpha(\mathbf{f})\,|\,U_\alpha(\mathbf{f})\rangle.$$

The $U_\alpha(\mathbf{f})$ are spinors which can be expressed as linear

combinations of the $U_i(f)$ of Sec. 1, and like the $U_i(f)$ they may be defined in terms of their values in the frame in which $\mathbf{f}=0$ by the equation

$$U_\alpha(\mathbf{f})=L^{-1}(f)U_\alpha(0).$$

Then

$$w(d\mathbf{f})=(d\mathbf{f})\,|\,\mathfrak{A}(\mathbf{f})\,|^2(\gamma)^f\equiv(d\mathbf{f})\{\sum W_\alpha|\,\mathfrak{A}_\alpha(\mathbf{f})\,|^2\}(\gamma)^f,$$

where $(\gamma)^f$ is the Lorentz contraction factor. Since $d\mathbf{f}/(\gamma)^f$ is an invariant, the required invariance of $w(d\mathbf{f})$ requires that $|\,\mathfrak{A}(\mathbf{f})(\gamma)^f|^2$ is unchanged in a Lorentz transformation.

Notice that the density matrix and the volume element are not invariants separately. If, however, the particle is definitely in a positive energy state or definitely in a negative energy state one may write (dropping now the summation signs),

$$\begin{aligned}\rho_S(\mathbf{f})&=|\,\mathfrak{A}_\alpha(\mathbf{f})\,|^2|\,U_\alpha(\mathbf{f})\rangle W_\alpha\langle U_\alpha{}^*(\mathbf{f})\,|\\&=|\,\mathfrak{A}_\alpha(\mathbf{f})\,|^2|\,U_\alpha(\mathbf{f})\rangle W_\alpha\langle U_\alpha{}^*(\mathbf{f})\,|\,U_\alpha(\mathbf{f})\rangle(\pm)\langle U_\alpha{}^\dagger(\mathbf{f})\,|\\&=|\,(\gamma)^f\mathfrak{A}_\alpha(\mathbf{f})\,|^2|\,U_\alpha(\mathbf{f})\rangle(\pm W_\alpha)\langle U_\alpha{}^\dagger(\mathbf{f})\,|/(\gamma)^f\\&\equiv\rho(f)/(\gamma)^f.\end{aligned}$$

The $[(\gamma)^f]^{-1}$ may now be put with the $(d\mathbf{f})$ to form an invariant. The matrix $\rho(f)$ defined by the foregoing equation will be called the covariant density matrix. Since its matrix elements

$$\rho_{ij}(f)\equiv\langle U_i{}^\dagger(f)\,|\,\rho(f)\,|\,U_j(f)\rangle$$

are invariants, it must be of the form

$$\rho(f)=[\tfrac{1}{4}\,\mathrm{Tr}\rho(f)][1+\lambda_\mu\gamma_\mu+\tfrac{1}{2}s_{\mu\nu}\sigma_{\mu\nu}+i\gamma_5\gamma_\mu p_\mu+J\gamma_5],$$

where the coefficients λ_μ, $s_{\mu\nu}$, p_μ, and J transform in the evident manner.

The expectation value of the operator A over measurements in the region $R=(d\mathbf{f})$ is

$$\begin{aligned}\langle A\rangle_{(d\mathbf{f})}&=\mathrm{Sp}\rho A\mathcal{P}/\mathrm{Sp}\rho\mathcal{P}\\&=\frac{[(\gamma)^f]^{-1}(d\mathbf{f})\,\mathrm{Tr}\rho(f)A}{[(\gamma)^f]^{-1}(d\mathbf{f})\,\mathrm{Tr}\rho(f)}.\end{aligned}$$

If the region R restricts also the three-momentum $\mathbf{h}$ of the second particle, then the element $(d\mathbf{h})$ will also appear in the invariant combination $d\mathbf{h}/(\gamma)^h$.

For the final state the matrix $\rho'(f')$ is defined in the analogous way. It is related to $\rho(f)$ by the equation

$$\rho'(f')=\mathcal{S}(f',t,f)\rho(f)\mathcal{S}^\dagger(f',t,f).$$

Here the invariant elements $d\mathbf{k}d\mathbf{h}/(\gamma)^k(\gamma)^t$ and $d\mathbf{t}/(\gamma)^t$ have been incorporated into the definitions of $\rho(f)$ and $\rho'(f')$ respectively and the trivial integration over $\mathbf{t}$ and $\mathbf{k}$ performed, allowing these variables to be considered as fixed and discrete. The condition $\mathrm{Sp}\rho=1$ becomes then $\mathrm{Tr}\rho(f)=1$, and the differential cross section is

$$I(f')=|\,\mathfrak{A}(\mathbf{f}')(\gamma)^f|^2=\mathrm{Tr}\rho'(f').$$