Dispersion Relations for Finite Momentum-Transfer Pion-Nucleon Scattering

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(Received June 4, 1956)

The field-theoretical derivation of dispersion relations for forward pion-nucleon scattering has been generalized to apply to the case of a fixed finite momentum transfer. The generalization is facilitated by use of the special Lorentz frame in which the sum of the momenta of the initial and final nucleons is zero. In this reference system the relations between dispersive and absorptive parts of the scattering amplitude are independent of momentum transfer and are similar in form to the forward-angle relations. At energies below the minimum energy necessary to allow a particular momentum transfer, the scattering amplitude has no direct physical meaning; it is interpreted as an analytic continuation of the physical amplitude to scattering angles corresponding to $\cos\theta < -1$. The resulting equations are expressed in terms of the amplitudes for individual angular momenta and are given in two forms, corresponding to the inclusion or neglect of nucleon recoil.

1. INTRODUCTION

RECENTLY, many authors[1–3] have investigated the consequences of causality for boson-fermion scattering problems. The requirement of causality in a scattering problem may be stated in the following manner: If the scattered wave at a space-time point $\mathbf{x}_1$, t_1 is dependent on the amplitude of the incoming wave at the point $\mathbf{x}_2$, t_2, then the time t_2 must be previous to t_1, as observed from any Lorentz system. (Lorentz systems in which the direction of time is reversed must be excluded from this definition, of course.) The Lorentz invariance of this requirement implies that the separation between the two points must be time-like; thus causality requires that the wave does not propagate with a speed exceeding that of light in a vacuum. In a field theory the condition may be imposed that field amplitudes corresponding to points separated by a space-like interval must commute; this condition is equivalent to the requirement that no disturbance may propagate with a velocity greater than c.

Gell-Mann, Goldberger, and Thirring[1] and Goldberger[2] have shown that the requirement of causality in a field theory may be used to derive useful dispersion relations for photon-nucleon scattering and pion-nucleon scattering. These equations relate the dispersive part $D(\omega)$ of the forward amplitude for elastic scattering to an energy integral of the absorptive part $A(\omega)$. If use is made of the well-known relation between $A(\omega)$ and the total cross section, i.e., $A(\omega)=(k/4\pi)\sigma_T$, the dispersion equations make possible the determination of the forward scattering amplitude from a knowledge of the total cross section at all energies. The equations essentially are equivalent to the classical dispersion relations of Kramers and Kronig.

It is reasonable to investigate whether or not the amplitude for finite-angle scattering satisfies a simple dispersion relation. One might attempt to generalize the forward-scattering relations by considering the energy dependence of the amplitude for a fixed, finite center-of-mass scattering angle. There are two important difficulties with such a procedure, however. First, such a finite-angle relation must depend on the size of the scattering region. This difficulty is especially discouraging in such problems as gamma-nucleon or pion-nucleon scattering, for which there is no definite boundary to the scattering region, and the extent of the region is not too well known. The second difficulty has to do with the fact that, as the energy of the bombarding particle varies, the energy of the target particle in the center-of-mass system varies also, giving rise to a complicated energy dependence of the scattering amplitude.

In this paper a generalization to finite angles is made by considering the energy dependence of the scattering amplitude for a fixed center-of-mass value of the momentum transfer. This procedure overcomes the above-mentioned difficulties. That a fixed momentum-transfer dispersion relation is independent of the size of the scattering region may be seen most easily in the scattering of a particle from a fixed potential of range a. In this case the quantity that satisfies a dispersion relation is $S\exp[2iak\sin(\frac{1}{2}\theta)]$, where S is the scattering matrix, $\hbar k$ is the momentum, and θ is the scattering angle. If the momentum transfer, $2\hbar k\sin(\frac{1}{2}\theta)$, is held constant as k is varied, the exponential factor is constant and S satisfies a dispersion relation which is independent of a. The second difficulty is overcome by expressing the scattering amplitude in a special Lorentz system,

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] Gell-Mann, Goldberger, and Thirring, Phys. Rev. **95**, 1612 (1954).

[2] M. L. Goldberger, Phys. Rev. **99**, 979 (1955).

[3] M. L. Goldberger, Phys. Rev. **97**, 508 (1955); Goldberger, Miyazawa, and Oehme, Phys. Rev. **99**, 986 (1955); Anderson, Davidon, and Kruse, Phys. Rev. **100**, 339 (1955); R. Oehme, Phys. Rev. **100**, 1503 (1955) and **102**, 1174 (1956). We are indebted to Dr. Oehme for sending preliminary copies of these manuscripts to us before publication. Y. Nambu, Phys. Rev. **98**, 803 (1955); **100**, 394 (1955). R. Karplus and M. Ruderman, Phys. Rev. **98**, 771 (1955).

defined by the condition that the initial and final momenta of the target particle are equal and opposite. If these momenta are held constant as the energy of the projectile varies, clearly the target particle energy remains constant. The difference between the final and initial target momentum, i.e., the momentum transfer, is equal to the momentum transfer in the center-of-mass system.[4]

The method used in this derivation is based upon the method of Goldberger,[2] and the assumptions made concerning the high-energy convergence of the scattering amplitude are identical to those in reference 2. The derivation is given for pion-nucleon scattering, though the method is applicable to other boson-fermion scattering problems.

The advantages of a fixed momentum-transfer dispersion relation over a fixed scattering-angle relation are partially nullified by an important disadvantage; namely, a minimum pion kinetic energy is necessary in order to transfer a specific amount of momentum to the nucleon. The scattering amplitude corresponding to energies less than this minimum energy must be determined by an analytic continuation process, if the dispersion relations are to be useful. In order to make this continuation, and in order to express the scattering amplitude in terms of convenient quantities, the amplitude is expanded in terms of waves of different orbital angular momenta. The analytic continuation into the nonphysical region is then made by the simple process of continuing the Legendre polynomials into the region $\cos\theta<-1$. It has been pointed out by Symanzik[5] that this continuation procedure is not rigorous in all cases. It is hoped, however, that the error will be unimportant in the low-energy applications of the relations.

The results express the dispersive part of the amplitude for a particular partial wave in terms of a sum over angular momenta of energy integrals of the absorptive parts of the various partial-wave amplitudes. The form of the dispersion relation depends on the asymptotic behavior of the scattering amplitude at high energies.

2. CAUSAL SCATTERING AMPLITUDE

Dispersion relations for scattering problems depend upon the principle that no disturbance may propagate with a velocity greater than that of light in a vacuum. Goldberger[2] has made use of this causal principle in giving a field-theoretical derivation of dispersion relations for pion-nucleon scattering in the forward direction. In this paper the method of Goldberger is generalized and applied to scattering at finite angles.

We shall consider a pion-nucleon scattering event[6] in which a pion of four-momentum k is scattered into a state k', the nucleon undergoing a transition from a state of momentum p to a state p'. The Greek subscripts α and α' are used to denote the charge states of the initial and final pion. The element of the scattering matrix corresponding to this event may be written in the form

$$S_{\alpha'\alpha}(k',p';k,p)$$
$$=\sum_n(-i)^n/(n!)\int dx_1\cdots dx_n\{\phi_{p'},a_{\alpha'}(k')$$
$$\times P[H(x_1),\cdots H(x_n)]a_\alpha{}^*(k)\phi_p\}. \quad (2.1)$$

The quantity $P[H(x_1),\cdots H(x_n)]$ denotes the time-ordered product of the operators $H(x_i)$, which represent the interaction Hamiltonian density at the space-time points x_i. The symbol ϕ_p or $\phi_{p'}$ represents a state of the nucleon with momentum p or p'. These state vectors are normalized by the equation

$$(\phi_{p'},\phi_p)=\delta(\mathbf{p}-\mathbf{p}'). \quad (2.2)$$

The symbol $a_{\alpha'}(k')$ denotes an annihilation operator for a pion of four-momentum k' and charge state α', while $a_\alpha{}^*(k)$ represents a creation operator for the state (k,α). The operators and state vectors have the time dependence of the interaction representation.

It is assumed that the Hamiltonian density representing the local interaction between the pion and nucleon fields may be written in the form

$$H(x)=\sum_\beta\phi_\beta(x)O_\beta(x), \quad (2.3)$$

where $\phi_\beta(x)$ is the pion field operator for the charge state β, and $O_\beta(x)$ is some nucleon field operator. In symmetrical, pseudoscalar meson theory with pseudoscalar coupling, $O_\beta(x)$ is given, in conventional notation, by

$$O_\beta(x)=ig\bar{\psi}(x)\gamma_5\tau_\beta\psi(x). \quad (2.4)$$

The method of Low[7] may be used to write the S matrix in terms of the operators $\mathbf{O}_\beta(x)$ in the Heisenberg representation.

[4] It has come to our attention that results quite similar to ours have been derived independently by several groups, *viz.*, Gell-Mann, Goldberger, Nambu, and Oehme (private communication); A. Salam, Nuovo cimento **3**, 424 (1956). The case of finite-angle, potential scattering has been considered by J. S. Toll and D. Y. Wong (private communication). The authors are indebted to Professor Y. Nambu for information on the results of the first group.

[5] K. Symanzik (private communication).

[6] Throughout this manuscript the ordinary italic letters k and p represent four vectors. The three-dimensional momenta corresponding to k and p are denoted by the boldface letters, **k** and **p**, while the symbols ω and E denote the corresponding energies. The spacelike and timelike components of the coordinate four-vector x are denoted by $\mathbf{x}$ and x_0. A four-vector inner product is written in the form $kx=\mathbf{k}\cdot\mathbf{x}-\omega x_0$. For convenience the constants $\hbar$ and c are taken to be unity.

[7] F. E. Low, Phys. Rev. **97**, 1392 (1955).

$$
\begin{aligned}
S_{\alpha'\alpha}&(k',p';k,p)\\
&=\delta_{\alpha'\alpha}\delta(\mathbf{k}'-\mathbf{k})\delta(\mathbf{p}'-\mathbf{p})+(-i)^2(2\pi)^{-3}(4\omega\omega')^{-\frac{1}{2}}\\
&\quad\times\int dx\int dy e^{-ik'x}e^{iky}\{\psi_{p'}P[\mathbf{O}_{\alpha'}(x),\mathbf{O}_{\alpha}(y)]\psi_p\}\\
&=\delta_{\alpha'\alpha}\delta(\mathbf{k}'-\mathbf{k})\delta(\mathbf{p}'-\mathbf{p})+(-i)^2\delta(k'+p'-k-p)\\
&\quad\times\pi(\omega\omega')^{-\frac{1}{2}}\int dze^{-\frac{1}{2}i(k+k')z}\\
&\qquad\times\{\psi_{p'},\,P[\mathbf{O}_{\alpha'}(\tfrac{1}{2}z),\,\mathbf{O}_{\alpha}(-\tfrac{1}{2}z)]\psi_p\}. \qquad (2.5)
\end{aligned}
$$

The symbols $\psi_{p'}$ and ψ_p represent exact nucleon eigenstates of the total Hamiltonian in the Heisenberg representation. When the nucleon current contains terms depending on the pion field, Eq. (2.5) must be modified to include other terms. This complexity is neglected here, since, as shown by Goldberger,[2] the extra terms do not alter the causal property of the scattering matrix.

A matrix U, similar to the U matrix of Møller,[8] may be defined by the equation

$$
S_{\alpha'\alpha}=\delta_{\alpha'\alpha}+i(2\pi)^{-1}\delta(k+p-k'-p')(k',p'|U_{\alpha'\alpha}|k,p).
$$

We shall define a scattering amplitude, which is invariant to Lorentz transformations, in terms of $U_{\alpha'\alpha}$,[8]

$$
\begin{aligned}
(k',p'|F_{\alpha'\alpha}|k,p)&=-m^{-1}(E'\omega)^{\frac{1}{2}}(k',p'|U_{\alpha'\alpha}|k,p)(E\omega)^{\frac{1}{2}}\\
&=i2\pi^2m^{-1}(E'E)^{\frac{1}{2}}\int dze^{-\frac{1}{2}i(k+k')z}\\
&\quad\times(\psi_{p'},\,P[\mathbf{O}_{\alpha'}(\tfrac{1}{2}z),\,\mathbf{O}_{\alpha}(-\tfrac{1}{2}z)]\psi_p), \qquad (2.6)
\end{aligned}
$$

where m is the mass of the nucleon.

Following the method of Goldberger, we obtain a causal scattering amplitude by replacing the time-ordered product $P[\mathbf{O}_{\alpha'}(\frac{1}{2}z),\,\mathbf{O}_{\alpha}(-\frac{1}{2}z)]$ by the quantity $\eta(z)[\mathbf{O}_{\alpha'}(\frac{1}{2}z),\,\mathbf{O}_{\alpha}(-\frac{1}{2}z)]$, where

$$
\begin{aligned}
\eta(z)&=1 \quad \text{for } z_0>0,\\
&=0 \quad \text{for } z_0<0.
\end{aligned}
\qquad (2.7)
$$

The modified scattering amplitude is given by

$$
\begin{aligned}
M_{\alpha'\alpha}(k',p';k,p)&=i2\pi^2(EE')^{\frac{1}{2}}m^{-1}\int dze^{-\frac{1}{2}i(k+k')z}\eta(z)\\
&\quad\times\{\psi_{p'}[\mathbf{O}_{\alpha'}(\tfrac{1}{2}z),\,\mathbf{O}_{\alpha}(-\tfrac{1}{2}z)]\psi_p\}. \qquad (2.8)
\end{aligned}
$$

[8] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys Medd. **23**, No. 1 (1945). The I matrix of Møller is related to the matrix F of Eq. (2.6) by the relation $F_{\alpha'\alpha}=-i(2\pi/m)I_{\alpha'\alpha}$, where m is the mass of the nucleon. The relation of $F_{\alpha'\alpha}$ to the differential cross section in any Lorentz frame is given by Møller as

$$
d\sigma=(m^2|F|^2/B)\int\delta(k'+p'-k-p)(d\mathbf{k}/\omega)(d\mathbf{p}/E),
$$

where B is the Lorentz-invariant quantity

$$
B=[(\mathbf{k}E-\mathbf{p}\omega)^2-(\mathbf{k}\times\mathbf{p})^2]^{\frac{1}{2}}.
$$

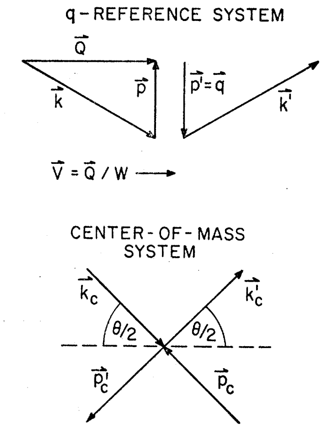

FIG. 1. Relative orientations of the pion and nucleon momentum vectors for a typical scattering event. The vector $\mathbf{V}$ denotes the velocity of the center of mass in the q-reference system, while W represents the total q-system energy. The subscript c refers to momenta in the center-of-mass system.

The two matrices $M_{\alpha'\alpha}$ and $F_{\alpha'\alpha}$ differ for negative energies, but not for positive energies. The modification of the amplitude causes negative-energy pions, as well as positive-energy pions, to propagate from past to future, thus assuring the causal nature of the scattering amplitude, $M_{\alpha'\alpha}$.

Though we have not considered the nucleon's spin coordinates, quantities such as $M_{\alpha'\alpha}$ depend on this variable. An alternative point of view, which is adopted here, is that $M_{\alpha'\alpha}$ is a matrix in the spin space of the nucleon. The Hermitian conjugate of this matrix is denoted by $M_{\alpha'\alpha}^{\dagger}$.

A physical scattering event corresponds to a positive value of pion energy. Thus, in order to derive a useful dispersion relation, we must find some symmetry property relating the negative-energy part of $M_{\alpha'\alpha}$ to the positive-energy part. Since M is expressed in terms of a matrix element between two nucleon states of momenta $\mathbf{p}$ and $\mathbf{p}'$, the symmetry properties of M may be expressed simply in the Lorentz system defined by the condition that the momentum $\mathbf{p}+\mathbf{p}'=0$. This system is called the q system, and the momentum $-\mathbf{p}=\mathbf{p}'$ is denoted by $\mathbf{q}$. Conservation of momentum and energy may be used to show that the vectors $\mathbf{k}+\mathbf{p}$ and $\mathbf{k}'+\mathbf{p}'$ are equal and are perpendicular to the vector $\mathbf{q}$. Thus we define two perpendicular vectors, $\mathbf{q}$ and $\mathbf{Q}$,

$$
\mathbf{q}=-\mathbf{p}=\mathbf{p}',\quad \mathbf{Q}=\mathbf{k}-\mathbf{q}=\mathbf{k}'+\mathbf{q}. \qquad (2.9)
$$

The orientation of these vectors for a typical scattering event, and the corresponding vectors in the center-of-mass system, are shown in Fig. 1. The momentum transferred to the nucleon during the collision is the same in either the q system or center-of-mass system, and is equal to $2\mathbf{q}$.

The magnitude of the vector $\mathbf{Q}$ depends on the pion energy ω,

$$
\mathbf{Q}=Q(\omega)\boldsymbol{\varepsilon}, \qquad (2.10)
$$

where $\boldsymbol{\varepsilon}$ is a unit vector. The function $Q(\omega)$ is given by

$$Q(\omega)=(\omega^2-\omega_q{}^2)^{\frac{1}{2}}, \tag{2.11}$$

where ω_q is defined in terms of $\mathbf{q}$ and the meson mass μ by the relation $\omega_q{}^2=(\mu^2+\mathbf{q}^2)^{\frac{1}{2}}$. The scattering amplitude may be expressed in terms of the variables of the q system,

$$M_{\alpha'\alpha}(\mathbf{q},\boldsymbol{\varepsilon},\omega)=i2\pi^2(E_q/m)\int dz\eta(z)\exp(-i\mathbf{Q}\cdot\mathbf{z}+i\omega z_0)$$
$$\times\{\psi_q[\mathbf{O}_{\alpha'}(\tfrac{1}{2}z),\mathbf{O}_\alpha(-\tfrac{1}{2}z)]\psi_{-q}\}, \tag{2.12}$$

where E_q is given by $E_q=(m^2+\mathbf{q}^2)^{\frac{1}{2}}$. The symbol ψ_{-q} denotes a nucleon state of momentum $-\mathbf{q}$ and energy E_q. The relation between $M_{\alpha'\alpha}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$ and the corresponding amplitude in the center-of-mass system is discussed in Sec. 5.

The variable ω in Eq. (2.12) may be considered as complex, thus defining $M_{\alpha'\alpha}$ for complex values of the energy. In the complex energy plane, the function $Q(\omega)$ has branch points at $\omega=\pm\omega_q$. The complex ω plane, including branch cuts, is illustrated in Fig. 2. We

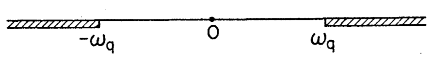

FIG. 2. The complex ω plane for the scattering amplitude $M_{\alpha'\alpha}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$.

define the function $Q(\omega)$ in the upper half ω plane by analytic continuation from the region corresponding to physical scattering, i.e., the region $\operatorname{Im}\omega=0$, $\operatorname{Re}\omega>\omega_q$. This leads to the result

$$Q(\omega)=-Q^*(-\omega^*). \tag{2.13}$$

For real values of ω, $Q(\omega)$ is positive when $\omega>\omega_q$, positive imaginary when $-\omega_q<\omega<\omega_q$, and negative when $\omega<-\omega_q$.

The implications of causality with respect to the analytic properties of $M_{\alpha'\alpha}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$ in the upper half ω plane are discussed in Sec. 4.

3. SYMMETRY PROPERTIES OF THE CAUSAL AMPLITUDE

If use is made of the Hermitian property of the operator $i[\mathbf{O}_{\alpha'}(\frac{1}{2}z),\mathbf{O}_\alpha(-\frac{1}{2}z)]$, the amplitude $M_{\alpha'\alpha}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$ defined by Eq. (2.12) may be shown to have the symmetry property

$$M_{\alpha'\alpha}{}^\dagger(\mathbf{q},\boldsymbol{\varepsilon},\omega)=M_{\alpha'\alpha}(-\mathbf{q},\,\boldsymbol{\varepsilon},\,-\omega). \tag{3.1}$$

This property permits us to write dispersion relations in terms of quantities corresponding to positive values of ω only.

The validity of Eq. (3.1) depends on the fact that the nucleon states ψ_q and ψ_{-q} are related by a reflection of the spatial coordinates; therefore the initial and final nucleon must be in the same charge state. For definiteness we assume this charge state to correspond to a proton; thus $M_{\alpha'\alpha}$ refers to the elastic scattering of pions by protons. The pair of indices $\alpha'\alpha$, which denote the charge states of the pions, may assume nine different values, since α and α' range from one to three. Charge conservation, however, limits the number of processes to three, $\pi^++P\rightarrow\pi^++P$, $\pi^0+P\rightarrow\pi^0+P$, and $\pi^-+P\rightarrow\pi^-+P$. We define three independent amplitudes which are simply related to these three processes:

$$\begin{aligned}M^{(1)}&=\tfrac{1}{2}(M_{11}+M_{22})=\tfrac{1}{2}(M_{\pi^+P}+M_{\pi^-P}),\\ M^{(2)}&=\tfrac{1}{2}i(M_{12}-M_{21})=\tfrac{1}{2}(M_{\pi^+P}-M_{\pi^-P}),\\ M^{(3)}&=M_{33}=M_{\pi^0P}.\end{aligned} \tag{3.2}$$

An important property of these amplitudes is their symmetry with respect to interchange of the indices α and α' of the quantities $M_{\alpha'\alpha}$. Under the transformation $\alpha\leftrightarrow\alpha'$, we have

$$M^{(\lambda)}\rightarrow\epsilon_\lambda M^{(\lambda)}, \tag{3.3}$$

where

$$\epsilon_\lambda=\begin{cases}1, & \lambda=1 \text{ or } 3,\\ -1, & \lambda=2.\end{cases} \tag{3.4}$$

In a charge-independent theory, there are only two independent amplitudes, corresponding to total isotopic spins $\frac{1}{2}$ and $\frac{3}{2}$. In such a theory,

$$\begin{aligned}M^{(1)}=M^{(3)}&=\tfrac{1}{3}(2M_{\frac{3}{2}}+M_{\frac{1}{2}}),\\ M^{(2)}=&\quad\tfrac{1}{3}(M_{\frac{3}{2}}-M_{\frac{1}{2}}).\end{aligned} \tag{3.5}$$

The quantities $M_{\alpha'\alpha}$ of Eq. (2.12) may be separated into dispersive and absorptive parts,

$$M_{\alpha'\alpha}=D_{\alpha'\alpha}+iA_{\alpha'\alpha}, \tag{3.6}$$

where D and A are defined by

$$D_{\alpha'\alpha}=i\pi^2(E_q/m)\int dz\epsilon(z)\exp(-i\mathbf{Q}\cdot\mathbf{z}+i\omega z_0)$$
$$\times\{\psi_q,\,[\mathbf{O}_{\alpha'}(\tfrac{1}{2}z),\,\mathbf{O}_\alpha(-\tfrac{1}{2}z)]\psi_{-q}\}, \tag{3.7}$$

$$A_{\alpha'\alpha}=\pi^2(E_q/m)\int dz\exp(-i\mathbf{Q}\cdot\mathbf{z}+i\omega z_0)$$
$$\times\{\psi_q,\,[\mathbf{O}_{\alpha'}(\tfrac{1}{2}z),\,\mathbf{O}_\alpha(-\tfrac{1}{2}z)]\psi_{-q}\}. \tag{3.8}$$

The function $\epsilon(z)$ of Eq. (3.7) is defined by the relation $\epsilon(z)=-1+2\eta(z)$.

Similarly, the amplitudes $M^{(\lambda)}$ may be written in terms of dispersive and absorptive parts, $M^{(\lambda)}=D^{(\lambda)}+iA^{(\lambda)}$, where $D^{(\lambda)}$ and $A^{(\lambda)}$ may be expressed in terms of the operators $\mathbf{O}_\alpha$ and $\mathbf{O}_{\alpha'}$ if use is made of Eqs. (3.2), (3.6), (3.7), and (3.8). Later it will be seen that this division of the quantities $M^{(\lambda)}$ corresponds to a separation of the entire scattering amplitude into Hermitian and anti-Hermitian parts.

If use is made of Eq. (3.2), the symmetry property, Eq. (3.1), may be written in terms of the quantities $D^{(\lambda)}$ and $A^{(\lambda)}$,

$$\begin{aligned}D^{(\lambda)\dagger}(\mathbf{q},\boldsymbol{\varepsilon},\omega)&=\epsilon_\lambda D^{(\lambda)}(-\mathbf{q},\,\boldsymbol{\varepsilon},\,-\omega),\\ A^{(\lambda)\dagger}(\mathbf{q},\boldsymbol{\varepsilon},\omega)&=-\epsilon_\lambda A^{(\lambda)}(-\mathbf{q},\,\boldsymbol{\varepsilon},\,-\omega).\end{aligned} \tag{3.9}$$

This symmetry condition alone is not enough to determine whether or not $D^{(\lambda)}$ and $A^{(\lambda)}$, which are matrices in nucleon spin-space, are Hermitian. Another useful property of $M^{(\lambda)}$ may be obtained, however, from the symmetry of the operator $\mathcal{O}_{\alpha'\alpha}(z)=[\mathbf{O}_{\alpha'}(\frac{1}{2}z), \mathbf{O}_{\alpha}(-\frac{1}{2}z)]$ with respect to the transformation $z\leftrightarrow -z$, i.e.,

$$\mathcal{O}_{\alpha'\alpha}(-z)=-\mathcal{O}_{\alpha\alpha'}(z). \tag{3.10}$$

From Eq. (3.10) and the symmetry properties of $M^{(\lambda)}$ with respect to the exchange $\alpha\leftrightarrow\alpha'$, Eq. (3.3), it can be shown that $D^{(\lambda)}$ and $A^{(\lambda)}$ satisfy the equations

$$\begin{aligned} D^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)&=\epsilon_\lambda D^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},-\omega) && \text{for } |\omega|>\omega_q\\ &\quad\epsilon_\lambda D^{(\lambda)}(\mathbf{q},-\boldsymbol{\varepsilon},-\omega) && \text{for } |\omega|<\omega_q,\\ A^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)&=-\epsilon_\lambda A^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},-\omega) && \text{for } |\omega|>\omega_q\\ &\quad-\epsilon_\lambda A^{(\lambda)}(\mathbf{q},-\boldsymbol{\varepsilon},-\omega) && \text{for } |\omega|<\omega_q. \end{aligned} \tag{3.11}$$

This symmetry condition is different in the two energy regions, $|\omega|>\omega_q$ and $|\omega|<\omega_q$, because the function $Q(\omega)=(\omega^2-\omega_q^2)^{\frac{1}{2}}$ is real and odd in ω for $|\omega|>\omega_q$, and is imaginary and even for $|\omega|<\omega_q$.

The symmetry properties of $M^{(\lambda)}$ may be more simply expressed, if the scattering amplitude is written as the sum of spin-independent and spin-dependent parts,

$$M^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)=\mathfrak{M}_N^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)\mathbf{1}+i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{Q}\mathfrak{M}_S^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega), \tag{3.12a}$$

$$D^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)=d_N^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)\mathbf{1}+i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{Q}d_S^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega), \tag{3.12b}$$

$$A^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)=a_N^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)\mathbf{1}+i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{Q}a_S^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega). \tag{3.12c}$$

Here $\boldsymbol{\sigma}$ is the nucleon spin matrix and $\mathbf{1}$ is the unit matrix. The quantities $\mathfrak{M}_{N,S}$, $d_{N,S}$ and $a_{N,S}$ are simple functions, rather than two-by-two matrices. Since the scattering amplitude used here is Lorentz invariant, the amplitude in the q system, $M^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$, must be invariant to spatial rotations and reflections. Therefore the functions $\mathfrak{M}_{N,S}$, $d_{N,S}$, and $a_{N,S}$ are invariant to spatial rotations and reflections. Since the vectors $\mathbf{q}$ and $\boldsymbol{\varepsilon}$ are orthogonal, these functions are quadratic in $\mathbf{q}$, and quadratic in $\boldsymbol{\varepsilon}$, and hence are functions of only the energy, and the magnitude q of the vector $\mathbf{q}$. In terms of these functions, the symmetry properties, Eq. (3.9) and Eq. (3.11), become

$$\begin{aligned} d_N^{(\lambda)*}(q,\omega)&=\epsilon_\lambda d_N^{(\lambda)}(q,-\omega),\\ a_N^{(\lambda)*}(q,\omega)&=-\epsilon_\lambda a_N^{(\lambda)}(q,-\omega),\\ d_S^{(\lambda)*}(q,\omega)&=-\epsilon_\lambda d_S^{(\lambda)}(q,-\omega),\\ a_S^{(\lambda)*}(q,\omega)&=\epsilon_\lambda a_S^{(\lambda)}(q,-\omega), \end{aligned} \tag{3.13}$$

$$\begin{aligned} d_N^{(\lambda)}(q,\omega)&=\epsilon_\lambda d_N^{(\lambda)}(q,-\omega),\\ a_N^{(\lambda)}(q,\omega)&=-\epsilon_\lambda a_N^{(\lambda)}(q,-\omega),\\ d_S^{(\lambda)}(q,\omega)&=-\epsilon_\lambda d_S^{(\lambda)}(q,-\omega),\\ a_S^{(\lambda)}(q,\omega)&=\epsilon_\lambda a_S^{(\lambda)}(q,-\omega). \end{aligned} \tag{3.14}$$

From Eqs. (3.13) and (3.14) we see that the functions $d_{N,S}^{(\lambda)}$ and $a_{N,S}^{(\lambda)}$ are all real. Thus the separation of $M^{(\lambda)}$ into dispersive and absorptive parts corresponds to a separation of the scattering amplitude into Hermitian and anti-Hermitian parts. The reality of the functions $d_{N,S}^{(\lambda)}$ and $a_{N,S}^{(\lambda)}$, together with either of the relations, Eq. (3.13) or Eq. (3.14), represent the symmetry properties of $M^{(\lambda)}$ in the form that is used in the derivation of the dispersion relations.

4. ANALYTICITY AND DISPERSION RELATIONS

The causality principle may be used to show that, for a fixed $\mathbf{q}$, the scattering amplitude $M_{\alpha'\alpha}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$ has certain analytic and boundedness properties in the region R_+, which denotes the upper half complex ω plane. The causal principle, that no disturbance propagates with a speed exceeding that of light in a vacuum, requires that the commutator $[\mathbf{O}_{\alpha'}(\frac{1}{2}z), \mathbf{O}_\alpha(-\frac{1}{2}z)]$ vanish for space-like values of the space-time variable z. Therefore, the factor $\eta(z)(\psi_q[\mathbf{O}_{\alpha'}(\frac{1}{2}z), \mathbf{O}_\alpha(-\frac{1}{2}z)]\psi_{-q})$ in Eq. (2.12) may be finite only for values of z satisfying the two inequalities,

$$z_0\geq 0 \quad\text{and}\quad z_0\geq |z|. \tag{4.1}$$

The amplitude $M_{\alpha'\alpha}$ in Eq. (2.12) depends on the complex energy ω only through the factor

$$\exp(-i\mathbf{Q}\cdot\mathbf{z}+i\omega z_0),$$

where $\mathbf{Q}$ is given in terms of ω by Eqs. (2.10) and (2.11). For a value of z in the region defined by Eq. (4.1), the exponential factor is bounded in R_+, i.e.,

$$\exp(-i\mathbf{Q}\cdot\mathbf{z}+i\omega z_0)\leq\exp(\omega_q z_0). \tag{4.2}$$

Since this bound is not uniform as a function of z_0, we must use the technique of Goldberger,[2] and interchange the order of a space-time integration, and an energy integration, in order to derive dispersion relations. A discussion of the justification of this exchange for forward scattering is given in reference 2. Intuitively, one expects a greater high-energy divergence problem for finite-angle scattering than for zero-angle scattering. However, if the momentum transfer is fixed, then as $\omega\to\infty$, the scattering angle approaches zero. Thus the convergence properties of $M(q,\omega)$ as $\omega\to\infty$ are similar to those of $M(0,\omega)$, the difference being that the "effective pion mass" is ω_q, rather than μ.

Instead of actually carrying out this exchange of integration order, we arrive at the same result more simply by treating $\exp(-i\mathbf{Q}\cdot\mathbf{z}+i\omega z_0)$ as if it were uniformly bounded. We may then apply a theorem of Titchmarsh,[9] to show that the amplitude $M^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$ of Eq. (2.12) is analytic in R_+, and that the divergence of $M(\omega)$ as the real part of ω approaches infinity is no worse above the real axis than it is for real values of the energy. The spin-flip and nonspin-flip amplitudes

[9] E. C. Titchmarsh, *Fourier Integrals* (Oxford University Press, New York, 1937), p. 119.

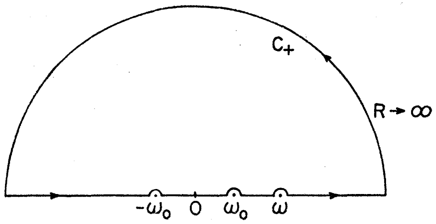

FIG. 3. The contour integral of Eq. (4.4) in the complex ω plane. The symbol R denotes the radius of the semicircle.

defined in Eq. (3.12a) must also be analytic in R_+, since they may be expressed in terms of $M^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)$ by the equations,

$$\begin{aligned}\mathfrak{M}_N^{(\lambda)}(q,\omega)&=\tfrac{1}{2}\,\mathrm{Tr}\{M^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)\},\\ q^2\mathfrak{M}_S^{(\lambda)}(q,\omega)&=\tfrac{1}{2}\,\mathrm{Tr}\{-i(\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{Q})M^{(\lambda)}(\mathbf{q},\boldsymbol{\varepsilon},\omega)\}/Q^2.\end{aligned}\tag{4.3}$$

The form of the dispersion relations depends upon the high-energy convergence of the amplitudes $\mathfrak{M}_{N,S}^{(\lambda)}$. If the Lesbegue integral $\int_\alpha^\infty|\mathfrak{M}/\omega^2|^2d\omega$ exists,[10] where $\mathfrak{M}$ is any of the six amplitudes $\mathfrak{M}_{N,S}^{(\lambda)}$, and α is any positive number, a dispersion relation may be obtained by considering the contour integral[11]

$$-\frac{i}{\pi}\int_{C_+}\frac{\mathfrak{M}(q,\omega')}{(\omega'-\omega)(\omega'^2-\omega_0^2)}d\omega'=0,\tag{4.4}$$

where the contour C_+ is shown in Fig. 3. The energy ω_0 is arbitrary and may be chosen for convenience.

If the scattering amplitude converges rapidly enough that the integral $\int_\alpha^\infty|\mathfrak{M}|^2d\omega$ exists, where α again is any positive constant, a stronger dispersion relation may be obtained by considering the contour integral,

$$-\frac{i}{\pi}\int_{C_+}\frac{\mathfrak{M}(q,\omega')}{\omega'-\omega}d\omega'=0.\tag{4.5}$$

Because of the boundedness property of $\mathfrak{M}$ in R_+, the contribution to the integrals of Eqs. (4.4) and (4.5) from the semicircle in C_+ will vanish as the radius approaches infinity. The two types of dispersion equations, those derived from the integral of Eq. (4.4) and those derived from Eq. (4.5), will be referred to as type A equations, and type B equations, respectively.

After suitable approximations have been made, the type B equations may be directly compared to Low's equation[7] for pion-nucleon scattering. Since it is questionable whether or not the high-energy convergence of $\mathfrak{M}(q,\omega)$ is sufficiently rapid to justify this procedure, we discuss the type A dispersion relations, which follow from Eq. (4.4). If use is made of the symmetry properties of the $\mathfrak{M}(q,\omega)$ functions, Eq. (3.14), the dispersion relations corresponding to $\epsilon_\lambda=1$ ($\lambda=1$ or 3) may be written

$$d_N^{(1,3)}(q,\omega)-d_N^{(1,3)}(q,\omega_0)=\frac{2(\omega^2-\omega_0^2)}{\pi}P\int_0^\infty\frac{\omega' d\omega' a_N^{(1,3)}(q,\omega')}{(\omega'^2-\omega_0^2)(\omega'^2-\omega^2)},\tag{4.6}$$

$$q^2d_S^{(1,3)}(q,\omega)-\frac{\omega}{\omega_0}q^2d_S^{(1,3)}(q,\omega_0)=\frac{2\omega(\omega^2-\omega_0^2)}{\pi}P\int_0^\infty\frac{d\omega' q^2a_S^{(1,3)}(q,\omega')}{(\omega'^2-\omega_0^2)(\omega'^2-\omega^2)},\tag{4.7}$$

where the symbol P denotes that the principal part of the integral is to be taken. Similar equations may be derived for the case $\epsilon_\lambda=-1$, which is discussed later.

The integral in these equations involves the absorptive part $a(q,\omega)$ as a function of energy for all energies in the range $0\leq\omega<\infty$. However, $a(q,\omega)$ may vanish for certain regions of ω in this range. In order to see this we expand the matrix element $\{\psi_q,\,[\mathbf{O}_{\alpha'}(\tfrac{1}{2}z),\mathbf{O}_\alpha(-\tfrac{1}{2}z)]\psi_q\}$ in $A_{\alpha'\alpha}$, Eq. (3.8), in a complete set of intermediate states, ψ_n, which we take to be eigenfunctions of the entire Hamiltonian,

$$\begin{aligned}A_{\alpha'\alpha}=\pi^2(E_q/m)\int dz\,\exp(-i\mathbf{Q}\cdot\mathbf{z}+i\omega z_0)\\ \times\sum_n[\{\psi_q,\mathbf{O}_{\alpha'}(\tfrac{1}{2}z)\psi_n\}\{\psi_n,\,\mathbf{O}_\alpha(-\tfrac{1}{2}z)\psi_{-q}\}\\ -\{\psi_q,\,\mathbf{O}_\alpha(-\tfrac{1}{2}z)\psi_n\}\{\psi_n,\mathbf{O}_{\alpha'}(\tfrac{1}{2}z)\psi_{-q}\}].\end{aligned}\tag{4.8}$$

If use is made of the relation $\mathbf{O}_\alpha(z)=e^{-iPz}\mathbf{O}_\alpha(0)e^{iPz}$, where P is the total momentum-energy operator, the space-time integral in Eq. (4.8) may be carried out, and $A_{\alpha'\alpha}$ may be written

$$\begin{aligned}A_{\alpha'\alpha}=(2\pi)^4\pi^2(E_q/m)\sum_n[\{\psi_q,\mathbf{O}_{\alpha'}(0)\psi_{n,Q}\}\\ \times\{\psi_{n,Q},\mathbf{O}_\alpha(0)\psi_{-q}\}\delta(\omega+E_q-E_{n,Q})\\ -\{\psi_q,\mathbf{O}_\alpha(0)\psi_{n,-Q}\}\{\psi_{n,-Q},\mathbf{O}_{\alpha'}(0)\psi_{-q}\}\\ \times\delta(\omega-E_q+E_{n,Q})],\end{aligned}\tag{4.9}$$

where $\psi_{n,\pm Q}$ denotes the state ψ_n with total momentum $\pm Q$, and $E_{n,Q}$ is the total energy of such a state.

Because of the energy delta functions in Eq. (4.9), the spectrum of $A_{\alpha'\alpha}$ depends simply on the spectrum of the states ψ_n. If the momentum-energy four vector, corresponding to the state $\psi_{n,P}$ is denoted by $(\mathbf{P},E_{n,P})$, the Lorentz invariant proper mss of the state ψ_n is given by $M_n=[(E_{n,P})^2-\mathbf{P}^2]^{\frac{1}{2}}$. We assume the following energy spectrum for the proper mass M_n of the states ψ_n: (i) a point spectrum at the energy $M_n=m$ corresponding to the real neutron or proton state; (ii) a continuous spectrum in the region $m+\mu<M_n<\infty$, corresponding to states consisting of a nucleon, plus one

[10] This condition is sufficient, but not necessary, for the validity of the procedure used here. For a brief discussion of convergence conditions, see Reinhard Oehme, Phys. Rev. **100**, 1503 (1955).

[11] It is assumed that $\mathfrak{M}$ is finite at all points in the region except at one point where $\mathfrak{M}$ has a simple pole, corresponding to the real nucleon state, which plays the role of a bound state of the pion-nucleon system. The contribution of the real nucleon state is discussed at the end of this section.

or more other "particles," where the term "particle" denotes either a pion or a nucleon pair.[12]

It has been assumed that no bound state of the nucleon-pion system exists. States involving no nucleons have been neglected, since they do not contribute to Eq. (4.9).

The energy spectrum of $A_{\alpha'\alpha}(\omega)$ may be determined from the spectrum of the states ψ_n. Because of the two terms of Eq. (4.9), a state of proper mass M_n will contribute to the spectrum of $A_{\alpha'\alpha}$ at two energies, one being the negative of the other. Though the integrals in Eqs. (4.6) and (4.7) involve only positive energies, it is useful to compute the spectrum of $A_{\alpha'\alpha}(\omega)$ in the entire energy region $-\infty<\omega<\infty$. When the index n refers to the real nucleon state, the quantity $E_{n,Q}$ in Eq. (4.9) is equal to $(m^2+Q^2)^{\frac{1}{2}}$. If use is made of the relation $\mathbf{Q}^2=\omega^2-\mu^2-\mathbf{q}^2$, it can be seen that the spectrum of $A_{\alpha'\alpha}(\omega)$ corresponding to the real nucleon intermediate state is given by

$$\omega=\pm\omega_b,$$
$$\omega_b=E_q-(m^2-\tfrac{1}{2}\mu^2)E_q^{-1}=(q^2+\tfrac{1}{2}\mu^2)E_q^{-1}. \tag{4.10}$$

The positive sign corresponds to the second term of Eq. (4.9). Note that ω_b must be a positive quantity, since $q^2\geq 0$.

The continuous spectrum of ψ_n contributes two continuous spectra to $A_{\alpha'\alpha}(\omega)$. The end points, ω_a and $-\omega_a$, of these spectra correspond to an intermediate state ψ_n of proper mass $m+\mu$. The determination of ω_a is analogous to that of ω_b, and yields the result

$$\omega_a=m(m+\mu)E_q^{-1}-E_q=(m\mu-q^2)E_q^{-1}. \tag{4.11}$$

Therefore, the two continuous spectra of $A_{\alpha'\alpha}(\omega)$ are given by

$$-\infty<\omega\leq-\omega_a \quad\text{and}\quad \omega_a\leq\omega<+\infty.$$

The complete spectrum of $A_{\alpha'\alpha}(\omega)$ is shown in Fig. 4.

The contribution of the real nucleon state to $A_{\alpha'\alpha}(\omega)$ or $A^{(\lambda)}(\omega)$ at the energy $\omega=\omega_b$ is denoted by

$$\delta(\omega-\omega_b)\mathfrak{A}_{\alpha'\alpha}(q,\omega_b) \quad\text{or}\quad \delta(\omega-\omega_b)\mathfrak{A}^{(\lambda)}(q,\omega_b).$$

The quantity $\mathfrak{A}^{(\lambda)}(q,\omega_b)$ may be expressed in spin-independent and spin-dependent parts, i.e.,

$$\mathfrak{A}^{(\lambda)}(q,\omega_b)=\pi[\Gamma_N^{(\lambda)}(q)+i\boldsymbol{\sigma}\cdot\mathbf{q}\times\mathbf{Q}_b\Gamma_S^{(\lambda)}(q)], \tag{4.12}$$

where the magnitude of $\mathbf{Q}_b$ is given by the relation $Q_b=(\omega^2-q^2-\mu^2)^{\frac{1}{2}}$. The quantities $\Gamma_{N,S}^{(\lambda)}(q)$ may be estimated from a specific meson theory.

The energy ω_a, which indicates the end points of the

[12] If a state ψ_n corresponds asymptotically to several particles having four momenta p_1, $p_2\cdots$, the mass corresponding to this state is $M_n=[(\sum_i E_i)^2-(\sum \mathbf{p}_i)^2]^{\frac{1}{2}}$. In the center-of-mass system we have $\sum \mathbf{p}_i=0$ and M_n is equal to the sum of the energies of the particles. Therefore, the lowest mass next to the nucleon mass m is clearly $m+\mu$, if states with no nucleons are neglected. For a discussion of this mass spectrum see, e.g., Y. Nambu, Phys. Rev. **100**, 394 (1955).

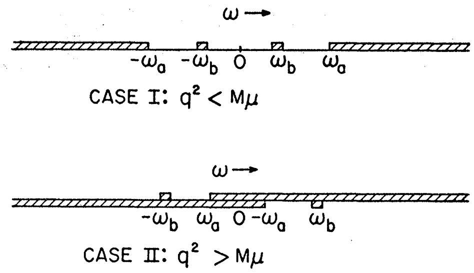

FIG. 4. The energy spectrum of the absorptive part of the scattering amplitude in the q system, shown in the two cases $q^2<m\mu$ and $q^2>m\mu$.

continuous spectra of $A(\omega)$, may be positive or negative. We shall consider the two cases separately.

Case I.—If $q^2<m\mu$, then $\omega_a>0$, and the absorption integrals in Eqs. (4.6) and (4.7) may be limited to the range $\omega_a<\omega<\infty$. If $A_{\alpha'\alpha}(\omega)$ is expressed in the form of Eq. (4.9), only the first term of this expression contributes to the absorption integrals.

Case II.—If $q^2>m\mu$, then $\omega_a<0$. In this case both terms of $A_{\alpha'\alpha}(\omega)$ contribute to the absorption integrals in the energy range $0<\omega<-\omega_a$, while only the first term contributes in the range $-\omega_a<\omega<\infty$. However, since the two terms of $A_{\alpha'\alpha}(\omega)$ are transformed into each other under the transformation $\omega\to-\omega$, the absorption integrals may be extended to the energy range $\omega_a<\omega<\infty$, provided that the contribution of the second term in $A_{\alpha'\alpha}(\omega)$, Eq. (4.9), is neglected.

From the above discussion it can be seen that the lower limit of the absorption integrals may be taken to be ω_a in either of the two cases $\omega_a>0$ or $\omega_a<0$, provided that $A(\omega)$ is properly interpreted in the anomalous region $\omega_a<\omega<-\omega_a$, while exists in the case $\omega_a<0$. If the arbitrary energy ω_0 is taken to be ω_a, and the real nucleon contribution is written in terms of the functions $\Gamma_{N,S}^{(\lambda)}(q)$, the dispersion relations, Eqs. (4.6) and (4.7), may be written in the form

$$d_N^{(1,3)}(q,\omega)-d_N^{(1,3)}(q,\omega_a)$$
$$=\frac{2(\omega^2-\omega_a^2)}{\pi}P\int_{\omega_a}^{\infty}\frac{\omega'd\omega' a_N^{(1,3)}(q,\omega')}{(\omega'^2-\omega_a^2)(\omega'^2-\omega^2)}+\frac{2\Gamma_N^{(1,3)}(q)\omega_b(\omega^2-\omega_a^2)}{(\omega_b^2-\omega_a^2)(\omega_b^2-\omega^2)}, \tag{I}$$

$$q^2d_S^{(1,3)}(q,\omega)-\frac{\omega}{\omega_a}q^2d_S^{(1,3)}(q,\omega_a)$$
$$=\frac{2\omega(\omega^2-\omega_a^2)}{\pi}P\int_{\omega_a}^{\infty}\frac{d\omega' q^2a_S^{(1,3)}(q,\omega')}{(\omega'^2-\omega_a^2)(\omega'^2-\omega^2)}+\frac{2q^2\Gamma_S^{(1,3)}(q)\omega(\omega^2-\omega_a^2)}{(\omega_b^2-\omega_a^2)(\omega_b^2-\omega^2)}. \tag{II}$$

The dispersion relations for the case $\epsilon_\lambda=-1$ may be derived in a similar fashion. The form of the equations is different in the two cases, $\epsilon_\lambda=\pm1$, since the symmetries in energy of $\mathfrak{M}^{(\lambda)}(q,\omega)$ are different [Eq. (3.14)]. The equations corresponding to the case $\epsilon_\lambda=-1(\lambda=2)$ are

$$d_N^{(2)}(q,\omega)-\frac{\omega}{\omega_a}d_N^{(2)}(q,\omega_a)$$
$$=\frac{2\omega(\omega^2-\omega_a^2)}{\pi}P\int_{\omega_a}^{\infty}\frac{d\omega' a_N^{(2)}(q,\omega')}{(\omega'^2-\omega_a^2)(\omega'^2-\omega^2)}+\frac{2\Gamma_N^{(2)}(q)\omega(\omega^2-\omega_a^2)}{(\omega_b^2-\omega_a^2)(\omega_b^2-\omega^2)}, \quad \text{(III)}$$

$$q^2d_S^{(2)}(q,\omega)-q^2d_S^{(2)}(q,\omega_a)$$
$$=\frac{2(\omega^2-\omega_a^2)}{\pi}P\int_{\omega_a}^{\infty}\frac{\omega' d\omega' q^2 a_S^{(2)}(q,\omega')}{(\omega'^2-\omega_a^2)(\omega'^2-\omega^2)}+\frac{2q^2\Gamma_S^{(2)}(q)\omega_b(\omega^2-\omega_a^2)}{(\omega_b^2-\omega_a^2)(\omega_b^2-\omega^2)}. \quad \text{(IV)}$$

The dispersion relations for forward scattering may be obtained from Eqs. (I), (II), (III), and (IV) by letting q^2 approach zero. An important distinction between the finite momentum-transfer equations and the zero-q limit is the existence of the energy region $\omega_a<\omega<\omega_q$, which shrinks to zero in the limit as $q\to0$. Since the momentum $\mathbf{Q}$ is imaginary in this energy region, the scattering amplitude cannot be directly related to physical processes. The interpretation of this nonphysical region is discussed in Sec. 5.

In order for the dispersion equations to be useful, some estimate must be made of the functions $\Gamma_{N,S}^{(\lambda)}(q)$, which represent the contribution to the equations of the real nucleon state. We shall assume symmetrical, pseudoscalar meson theory with pseudoscalar coupling, in which theory the operators $O_\alpha(x)$ are given by Eq. (2.4). If the operators $\mathbf{O}_+$, $\mathbf{O}_-$ and $\mathbf{O}_3$ are defined by the equations $\mathbf{O}_\pm=\frac{1}{2}\{\mathbf{O}_1(0)\pm i\mathbf{O}_2(0)\}$, $\mathbf{O}_3=\mathbf{O}_3(0)$, then Eq. (3.2) and Eq. (4.9) may be used to express the bound-state contributions $a^{(\lambda)}$ in the form

$$\mathfrak{a}^{(1)}(q,\omega_b)=\mathfrak{a}^{(2)}(q,\omega_b)=-(2\pi)^4\pi^2(E_q/m)(\psi_q,\mathbf{O}_+\psi_{-Q_b}{}^N)\times(\psi_{-Q_b}{}^N,\mathbf{O}_-\psi_{-q}), \quad (4.13)$$

$$\mathfrak{a}^{(3)}(q,\omega_b)=-(2\pi)^4\pi^2(E_q/m)\times(\psi_q,\mathbf{O}_3\psi_{-Q_b}{}^P)(\psi_{-Q_b}{}^P,\mathbf{O}_3\psi_{-q}).$$

The state $\psi_{-Q_b}{}^N$ corresponds to a real neutron of momentum $-\mathbf{Q}_b$, while $\psi_{-Q_b}{}^P$ corresponds to a real proton.

If terms of order $(\mu/m)^2$ are neglected, matrix elements of the operators $\mathbf{O}_+$, $\mathbf{O}_-$ and $\mathbf{O}_3$ between real nucleon states may be evaluated,[13] yielding the result,

$$(\psi_a{}^N,\mathbf{O}_-\psi_b{}^P)=(\psi_a{}^P,\mathbf{O}_+\psi_b{}^N)=(\psi_a{}^P,\mathbf{O}_3\psi_b{}^P)=i(2\pi)^{-\frac{3}{2}}(f/\sqrt{2}\pi\mu)[\boldsymbol{\sigma}\cdot(\mathbf{a}-\mathbf{b})], \quad (4.14)$$

[13] See reference 7, p. 1396.

where f is the renormalized coupling constant characteristic of the pseudovector interaction.[14] Therefore, in this no-recoil approximation the quantities $\mathfrak{a}^{(\lambda)}(q,\omega_b)$ are given by

$$\mathfrak{a}^{(1)}(q,\omega_b)=\mathfrak{a}^{(2)}(q,\omega_b)=\mathfrak{a}^{(3)}(q,\omega_b)=\pi(f^2/\mu^2)\{(q^2-Q_b^2)-2i[\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{Q}_b)]\}. \quad (4.15)$$

From this equation and the definition of $\Gamma_{N,S}^{(\lambda)}$, Eq. (4.12), the values of $\Gamma_{N,S}^{(\lambda)}$ to lowest order of $(\mu/m)^2$ in pseudoscalar meson theory with pseudoscalar coupling are

$$\Gamma_N^{(1)}=\Gamma_N^{(2)}=\Gamma_N^{(3)}=(f^2/\mu^2)(q^2-Q_b^2)=f^2[1+(2q^2/\mu^2)], \quad (4.16a)$$

$$\Gamma_S^{(1)}=\Gamma_S^{(2)}=\Gamma_S^{(3)}=-2f^2/\mu^2. \quad (4.16b)$$

5. PARTIAL WAVE ANALYSIS OF DISPERSION EQUATIONS

In order to express the dispersion relations in terms of experimentally measured quantities, we shall transform Eqs. (I), (II), (III), and (IV) into variables of the center-of-mass system. The q-system energy and momentum variables are related to center-of-mass quantities by the equations

$$E_qW=E_cW_c,$$
$$\omega=E_cW_cE_q^{-1}-E_q, \quad (5.1a)$$
$$2q^2=k_c^2(1-\cos\theta_c), \quad (5.1b)$$

where W and W_c represent the total energy in the q system and center-of-mass system, and k_c, θ_c, and E_c represent the center-of-mass values of the magnitude of the particle momentum, scattering angle, and nucleon energy. In general the subscript c will be used to denote a variable of the center-of-mass Lorentz system. The vector product $\mathbf{q}\times\mathbf{Q}$ is related to the scattering angle by the equation

$$\mathbf{q}\times\mathbf{Q}=qQ\mathbf{n}=(W_c/2E_q)(\mathbf{k}_c\times\mathbf{k}_c')=(W_c/2E_q)k_c^2(\sin\theta_c)\mathbf{n}, \quad (5.2)$$

where $\mathbf{n}$ represents a unit vector and $\mathbf{k}_c$ and $\mathbf{k}_c'$ are the initial and final values of the pion momentum in the center-of-mass system.

The center-of-mass scattering amplitude $M_c^{(\lambda)}(\mathbf{k}_c,\mathbf{k}_c')$ may be separated into spin-independent and spin-dependent parts in a manner similar to the separation of the q-system amplitude [Eq. (3.12a)],

$$M_c^{(\lambda)}(\mathbf{k}_c,\mathbf{k}_c')=\mathfrak{M}_{cN}^{(\lambda)}(\mathbf{k}_c,\mathbf{k}_c')\mathbf{1}+i\boldsymbol{\sigma}\cdot\mathbf{k}_c\times\mathbf{k}_c'\mathfrak{M}_{cS}^{(\lambda)}(\mathbf{k}_c,\mathbf{k}_c'). \quad (5.3)$$

Because of the invariance of $M_c^{(\lambda)}(\mathbf{k}_c,\mathbf{k}_c')$ under three-dimensional rotations and reflections, the functions $\mathfrak{M}_{cN}^{(\lambda)}$ and $\mathfrak{M}_{cS}^{(\lambda)}$ depend only on $\cos\theta_c$ and the energy ω_c. The center-of-mass quantities $\mathfrak{M}_{cN}^{(\lambda)}(\theta_c,\omega_c)$ and

[14] The quantity f^2 defined here is equal to that defined by G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956). As shown by these authors, f^2 is of the order $f^2\approx0.08$.

$\mathfrak{M}_{cS}^{(\lambda)}(\theta_c,\omega_c)$ may be related to the corresponding q-system amplitudes, if use is made of the fact that the entire scattering amplitude is Lorentz-invariant, i.e., the amplitude relating any specific initial and final states in the q system is equal to the amplitude relating the same states transformed into the center-of-mass system. The resulting transformation equations are

$$\begin{aligned}\mathfrak{M}_N^{(\lambda)}(q,\omega)&=\frac{E_q^2+mE_c}{E_q(m+E_c)}\mathfrak{M}_{cN}^{(\lambda)}(\theta_c,\omega_c)\\&\quad-2q^2\frac{E_q^2-E_c^2}{E_q(m+E_c)}\mathfrak{M}_{cS}^{(\lambda)}(\theta_c,\omega_c),\\ \mathfrak{M}_S^{(\lambda)}(q,\omega)&=\frac{1}{W_c}\left[-\frac{1}{m+E_c}\mathfrak{M}_{cN}^{(\lambda)}(\theta_c,\omega_c)\right.\\&\quad\left.+2\left(m+\frac{q^2}{m+E_c}\right)\mathfrak{M}_{cS}^{(\lambda)}(\theta_c,\omega_c)\right].\end{aligned}\tag{5.4}$$

These equations are derived in Appendix A. The complexity of the equations results from the complicated manner in which the Dirac spinors transform under Lorentz transformations.[15]

If the nucleon mass is considered to be large, and only terms of zero order and first order is an expansion in powers of m^{-1} are kept, then the relations between q-system quantities and center-of-mass quantities become very simple, i.e.,

$$\begin{aligned}\omega&=\omega_c+m^{-1}(k_c^2-q^2),\\ 2q^2&=k_c^2(1-\cos\theta_c),\\ \mathbf{q}\times\mathbf{Q}&=\tfrac{1}{2}(1+\omega_c/m)(\mathbf{k}_c\times\mathbf{k}_c').\end{aligned}\tag{5.5}$$

The Lorentz transformation does not mix the spin-independent and spin-dependent amplitudes in the limit of large m. The transformation equations reduce to the form

$$\begin{aligned}\mathfrak{M}_N^{(\lambda)}(q,\omega)&=\mathfrak{M}_{cN}^{(\lambda)}(\theta_c,\omega_c),\\ \mathfrak{M}_S^{(\lambda)}(q,\omega)&=2(1-\omega_c/m)\mathfrak{M}_{cS}^{(\lambda)}(\theta_c,\omega_c).\end{aligned}\tag{5.6}$$

It is interesting to notice how the center-of-mass values of the energy and scattering angle vary, as the momentum transfer is held fixed and the q-system energy varies between the limits of integration in the dispersion equations. From Eqs. (5.1) it is seen that as ω approaches infinity, the center-of-mass momentum becomes infinite and $\cos\theta_c$ approaches the limit 1. In this limit the scattering is in the forward direction. At the point $\omega=\omega_q$ the q system and the center-of-mass system are the same, and the quantities k_c and $\cos\theta_c$ are equal to q and -1. In the nonphysical region $\omega_a<\omega<\omega_q$, $\cos\theta_c$ is less than -1. The functions $Q(\omega)$ and $\sin\theta_c$ of Eq. (5.2) are imaginary in this region. As ω approaches ω_a, the center-of-mass momentum approaches zero, and $\cos\theta_c$ approaches $-\infty$. Thus the nonphysical energy region $\omega_a<\omega<\omega_q$ corresponds to center-of-mass scattering angles in the range $-\infty<\cos\theta_c<-1$.

[15] The fact that the spin-independent and spin-dependent amplitudes are mixed by the Lorentz transformation was pointed out to the authors by Dr. B. McCormick and by Professor M. L. Goldberger, independently.

In order to evaluate the contribution to the equations of the nonphysical region, we must remember that the scattering amplitude in this region is defined by analytic continuation from the physical region $\omega>\omega_q$. A convenient method for interpreting the center-of-mass scattering amplitude in both the physical and nonphysical regions is the method of expanding the amplitude in terms corresponding to different values of orbital angular momenta. In such a formalism, the analytic continuation may be made by analytically continuing the Legendre polynomials into the region $\cos\theta_c<-1$. This continuation method is quite simple; however, as is brought out later, it is not rigorous in all cases.

The magnitude of the relative orbital angular momentum, as well as the total angular momentum, is conserved in a pion-nucleon collision. For each value of the orbital angular momentum l (except $l=0$), there are two scattering states, corresponding to the two values of the total angular momentum, $j=l\pm\frac{1}{2}$. The amplitudes for these two terms are denoted by $f_{l\pm}^{(\lambda)}(k_c)$; they are related to the scattering phase shifts (which are complex if inelastic processes are possible at the momentum k_c) by the equation

$$f_{l\pm}^{(\lambda)}(k_c)=k_c^{-1}\exp[i\delta_{l\pm}^{(\lambda)}(k_c)]\sin\delta_{l\pm}^{(\lambda)}(k_c).\tag{5.7}$$

The amplitudes $f_{l\pm}^{(\lambda)}$ may be related to the spin-dependent and spin-independent amplitudes if use is made of the projection operators $B_{l\pm}$, defined by

$$\begin{aligned}B_{l+}&=(l+1+\boldsymbol{\sigma}\cdot\mathfrak{L})/(2l+1),\\ B_{l-}&=(l-\boldsymbol{\sigma}\cdot\mathfrak{L})/(2l+1),\end{aligned}\tag{5.8}$$

where $\mathfrak{L}$ is the orbital angular momentum operator. The operator B_{l+} or B_{l-}, when operating on a pion-nucleon state vector of orbital angular momentum l projects out the term corresponding to $j=l+\frac{1}{2}$ or $j=l-\frac{1}{2}$. If use is made of Eqs. (5.8), the scattering amplitude may be expanded in partial waves, i.e.,

$$\begin{aligned}&\mathfrak{M}_c^{(\lambda)}(\mathbf{k}_c,\mathbf{k}_c')\\&=(W_c/m)\textstyle\sum_l\{f_{l+}^{(\lambda)}[(l+1)P_l(\cos\theta_c)\\&\quad+k_c^{-2}i\boldsymbol{\sigma}\cdot(\mathbf{k}_c\times\mathbf{k}_c')\dot{P}_l(\cos\theta_c)]+f_{l-}^{(\lambda)}[lP_l(\cos\theta_c)\\&\quad-k_c^{-2}i\boldsymbol{\sigma}\cdot(\mathbf{k}_c\times\mathbf{k}_c')\dot{P}_l(\cos\theta_c)]\}.\end{aligned}\tag{5.9}$$

The functions $P_l(\cos\theta_c)$ are the Legendre polynomials, and $\dot{P}_l(\cos\theta_c)$ are their derivatives, i.e., $\dot{P}_l(z)=(d/dz)P_l(z)$. The expansions of $\mathfrak{M}_N$ and $\mathfrak{M}_S$ are given from a comparison of Eq. (5.3) and Eq. (5.9):

$$\mathfrak{M}_{cN}^{(\lambda)}(\theta_c,\omega_c)=(W_c/m)\textstyle\sum_l[(l+1)f_{l+}^{(\lambda)}+lf_{l-}^{(\lambda)}]P_l(\cos\theta_c),\tag{5.10a}$$

$$\mathfrak{M}_{cS}^{(\lambda)}(\theta_c,\omega_c)=(W_c/m)k_c^{-2}\times\textstyle\sum_l[f_{l+}^{(\lambda)}-f_{l-}^{(\lambda)}]\dot{P}_l(\cos\theta_c).\tag{5.10b}$$

The equations, Eqs. (5.4) and (5.10), may be used to expand the scattering amplitude in partial waves. The center-of-mass quantities $\mathfrak{M}_{cS}^{(\lambda)}$, $\mathfrak{M}_{cN}^{(\lambda)}$, and $f_{l\pm}^{(\lambda)}$ may be separated into real and imaginary parts, $\mathfrak{M}_{cN}^{(\lambda)}=d_{cN}^{(\lambda)}+ia_{cN}^{(\lambda)}$, $\mathfrak{M}_{cS}^{(\lambda)}=d_{cS}^{(\lambda)}+ia_{cS}^{(\lambda)}$, and $f_{l\pm}^{(\lambda)}=d_{l\pm}^{(\lambda)}+ia_{l\pm}^{(\lambda)}$. Since the coefficients of Eqs. (5.4) and (5.10) are real, these equations remain correct if the complex amplitudes are all replaced by their real parts, or by their imaginary parts.

The partial-wave expansion of the scattering amplitude provides a straightforward method of analytic continuation into the nonphysical region, since the functions $P_l(\cos\theta)$ and $\dot{P}_l(\cos\theta)$ are well defined for values of the argument in the range $-\infty<\cos\theta<-1$. The dispersion relations, Eqs. (I)–(IV), involve the threshold energy ω_a both in the dispersive terms and the absorptive terms; hence the behavior of the Legendre expansions must be examined in the limit $\omega\to\omega_a$. For values of ω close to ω_a, k_c is small, and the leading terms of the Legendre polynomials and their derivatives are given by

$$P_l(1-2q^2/k_c^2)\approx(-1)^l(2l)!(l!)^{-2}(q^2/k_c^2)^l,$$
$$\dot{P}_l(1-2q^2/k_c^2)\approx(-1)^{l-1}(2l-1)!\times[(l-1)!]^{-2}(q^2/k_c^2)^{l-1}. \tag{5.11}$$

For small k_c, the phase shifts $\delta_{l\pm}^{(\lambda)}(k_c)$ are real, and the leading terms of the real and imaginary parts of $f_l^{\pm}(k_c)$ are given by

$$d_{l\pm}^{(\lambda)}(k_c)=k_c^{-1}\cos\delta_{l\pm}^{(\lambda)}\sin\delta_{l\pm}^{(\lambda)}\approx\Delta_{l\pm}^{(\lambda)}k_c^{2l},$$
$$a_{l\pm}^{(\lambda)}(k_c)=k_c^{-1}\sin^2\delta_{l\pm}^{(\lambda)}\approx[\Delta_{l\pm}^{(\lambda)}k_c^{2l}]^2k_c, \tag{5.12}$$

where the symbols $\Delta_{l\pm}^{(\lambda)}$ represent constants with the dimensions of length to the power $(2l+1)$. From Eqs. (5.4), (5.10), (5.11), and (5.12) it can be seen that as ω approaches ω_a, the leading terms of $a_N^{(\lambda)}(q,\omega)$ and $a_S^{(\lambda)}(q,\omega)$ correspond to $l=0$ and are proportional to k_c. The quantities $d_N(q,\omega_a)$ and $d_S(q,\omega_a)$, on the other hand, are given by infinite series of finite terms, i.e.,

$$d_N^{(\lambda)}(q,\omega_a)=\frac{(m+\mu)(m^2+E_q^2)}{2E_qm^2}\sum_l(-1)^l\frac{(2l)![(l+1)\Delta_{l\pm}^{(\lambda)}+l\Delta_{l-}^{(\lambda)}]}{(l!)^2}q^{2l}$$
$$+\frac{(m+\mu)q^2}{2E_qm^2}\sum_l(-1)^l\frac{(2l)!}{l!(l-1)!}[\Delta_{l+}^{(\lambda)}-\Delta_{l-}^{(\lambda)}]q^{2l}, \tag{5.13a}$$

$$d_S^{(\lambda)}(q,\omega_a)=-\frac{1}{2m^2}\sum_l(-1)^l\frac{(2l)![(l+1)\Delta_{l+}^{(\lambda)}+l\Delta_{l-}^{(\lambda)}]}{(l!)^2}q^{2l}$$
$$-\left(1+\frac{q^2}{2m^2}\right)\sum_l(-1)^l\frac{(2l)!}{l!(l-1)!}[\Delta_{l+}^{(\lambda)}-\Delta_{l-}^{(\lambda)}]q^{2(l-1)}. \tag{5.13b}$$

In order to simplify the writing of the equations, we define partial nonspin-flip and spin-flip amplitudes by the equations

$$f_{lN}^{(\lambda)}(k_c)=(l+1)f_{l+}^{(\lambda)}(k_c)+lf_{l-}^{(\lambda)}(k_c), \tag{5.14a}$$
$$f_{lS}^{(\lambda)}(k_c)=f_{l+}^{(\lambda)}(k_c)-f_{l-}^{(\lambda)}(k_c), \tag{5.14b}$$
$$\Delta_{lN}^{(\lambda)}=(l+1)\Delta_{l+}^{(\lambda)}+l\Delta_{l-}^{(\lambda)}, \tag{5.14c}$$
$$\Delta_{lS}^{(\lambda)}=\Delta_{l+}^{(\lambda)}-\Delta_{l-}^{(\lambda)}. \tag{5.14d}$$

The real and imaginary parts of the amplitudes $f_{l\pm}^{(\lambda)}$, $f_{lN}^{(\lambda)}$, and $f_{lS}^{(\lambda)}$ are denoted by lower-case d's and a's with the proper subscripts and superscripts.

If use is made of Eqs. (5.1) and (5.4) the dispersion relations may be written in terms of center-of-mass quantities. Equations (5.10) may then be used to expand the scattering amplitude in terms of partial waves. Application of this procedure to Eq. (I) yields a rather lengthy equation; to simplify this equation we express some quantities in terms of the q-system energies ω and ω', which are related to the center-of-mass variables by the equations $\omega=E_cW_cE_q^{-1}-E_q$, and $\omega'=E_c'W_c'E_q^{-1}-E_q$. In terms of partial waves, Eq. (I) may be written

$$\sum_l d_{lN}^{(1,3)}(k_c)P_l(\cos\theta_c)-\frac{2q^2(E_q^2-E_c^2)}{k_c^2(E_q^2+mE_c)}\sum_l d_{lS}^{(1,3)}(k_c)\dot{P}_l(\cos\theta_c)-\frac{E_q(m+E_c)m}{(E_q^2+mE_c)W_c}d_N^{(1,3)}(q,\omega_a)$$
$$=\frac{2E_q(m+E_c)}{\pi(E_q^2+mE_c)}(\omega^2-\omega_a^2)\times P\int_\mu^\infty d\omega_c'\left\{\frac{W_c'^3}{E_qE_c'W_c}\frac{\omega'}{(\omega'^2-\omega^2)(\omega'^2-\omega_a^2)}\times\left[\frac{E_q^2+mE_c'}{E_q(m+E_c')}\sum_l a_{lN}^{(1,3)}(k_c')P_l(\cos\theta_c')-\frac{2q^2(E_q^2-E_c'^2)}{k_c'^2E_q(m+E_c')}\sum_l a_{lS}^{(1,3)}(k_c')\dot{P}_l(\cos\theta_c')\right]\right\}$$
$$+\frac{2\Gamma_N^{(1,3)}(q)\omega_b(\omega^2-\omega_a^2)}{(\omega_b^2-\omega_a^2)(\omega_b^2-\omega^2)}\frac{E_q(m+E_c)m}{(E_q^2+mE_c)W_c}, \tag{5.15}$$

where the scattering angles θ_c and θ_c' are determined by the momentum transfer and the respective center-of-mass momenta, k_c and k_c', by Eq. (5.1b). The functions $d_N^{(1,3)}(q,\omega_a)$ are given in terms of partial wave amplitudes by Eq. (5.13a).

Equations similar to Eq. (5.15) result from application of the above procedure to the dispersion relations, Eqs. (II)–(IV).

The procedure outlined above is not rigorous for all values of q. Symanzik[5] has pointed out that if the momentum transfer is large enough so that $q^2 > m\mu$, the Legendre polynomial method of analytic continuation is not justified. In this case the q-system threshold energy ω_a is negative, and the energy region $\omega_a < \omega < -\omega_a$ is anomalous because the causal amplitude and the Feynman amplitude are not identical in this region. The scattering amplitude has a branch point at the positive energy $-\omega_a$. At real energies below this branch point, the causal amplitude is determined by analytic continuation above the real axis, while the Feynman amplitude is determined by continuation below the real axis. This energy region is discussed in Sec. 4, where it is shown that, for $\omega < -\omega_a$, only part of the absorptive amplitude should be considered in analyzing Eqs. (I)–(IV). If the center-of-mass energy ω_c is held fixed, this branch point becomes a branch point in the q^2-plane at the value $q^2 = \frac{1}{2}(m\mu + \omega_c E_c + k_c^2)$. Thus, the Legendre polynomial expansion of the absorptive amplitudes appearing in the ω_c' integrals is not justified for $q^2 > \frac{1}{2}(m\mu + \omega_c' E_c' + k_c'^2)$. If $\omega_c' = \mu$, the branch point occurs at $q^2 = m\mu$, which implies that the expansions of Eqs. (5.13) are not justified, and may not converge, when $q^2 > m\mu$.

A further difficulty arises because of the pole in the scattering amplitude at the q-system energy ω_b. If the center-of-mass energy is held fixed, this corresponds to a pole in the q^2-plane at the point

$$q^2 = \tfrac{1}{2}(E_c\omega_c + k_c^2 - \tfrac{1}{2}\mu^2).$$

It is not clear whether or not the expansions of $\mathfrak{M}_{cN}$ and $\mathfrak{M}_{cS}$ are justified for q^2 larger than this value. If $q^2 = \frac{1}{2}(m\mu - \frac{1}{2}\mu^2) \pm \epsilon$, where ϵ is a small positive number, the pole at the q-system energy ω_a of the factor $(\omega^2 - \omega_a^2)^{-1}$, and the pole at ω_b of the scattering amplitude are close together. In this case the residue of the pole at ω_b contains the large factor $(\omega_b^2 - \omega_a^2)^{-1}$, as seen from Eqs. (I)–(IV). It appears that the quantities $d_N(q,\omega_a)$ and $d_S(q,\omega_a)$, which represent the residues of the poles at ω_a, may also be large, and the expansions of Eqs. (5.13) may not converge when $q^2 > \frac{1}{2}(m\mu - \frac{1}{2}\mu^2)$. It should be noted that if alternate dispersion relations were derived from the contour integral of Eq. (4.5), the troublesome factors $(\omega_b^2 - \omega_a^2)$ and $d_{N,S}(q,\omega_a)$ would not appear.

The nonrigorous nature of the present procedure for too large values of q^2 may be seen most clearly from the following considerations. The partial-wave analysis is made by expanding the various quantities of the dispersion relations in powers of $q^2 = \frac{1}{2}k_c^2(1-\cos\theta_c)$. In Sec. 7 it is shown that, in such an expansion, some of the quantities which refer to the anomalous energy region $\omega < |\omega_a|$ have radii of convergence of $q^2 = m\mu$ or $q^2 = \frac{1}{2}(m\mu - \frac{1}{2}\mu^2)$.

If the energy is low enough the above arguments do not apply, since we have $q^2 \leq k_c^2$ for scattering at any angle. In this case the quantities $d_{lN}(k_c)$ of an equation of the type of Eq. (5.15) may be separated by multiplying the equation by the Legendre polynomials and integrating over all angles. For energies such that $k_c^2 > m\mu$ one may still derive different dispersion relations by taking various derivatives of the quantities with respect to q^2 or $\cos\theta_c$, and evaluating in the forward direction.[16] The various dispersive amplitudes occurring in these equations may be separated only if it is a valid approximation to consider only a finite number of angular momenta. It is hoped that future research will clarify these points concerning the validity of various dispersion relations.

6. HEAVY-NUCLEON EQUATIONS

Since the analysis of equations of the type of Eq. (5.15) is long and complicated, we first discuss the simpler case in which the nucleon mass is considered to be large compared with the other energies involved. If the quantities of Eq. (5.15) are expanded in powers of m^{-1}, and terms of order higher than the first neglected, the nonspin-flip and spin-flip amplitudes are no longer mixed, and the coefficients in the equation are simplified. An equivalent method of obtaining this "heavy-nucleon limit" is to use Eqs. (5.5) and (5.6) in transforming the dispersion relations to the center-of-mass system. If, for convenience, both energy and momentum variables are used, the "heavy nucleon" limit of Eq. (5.15) may be written in the form

$$\begin{aligned}
&\sum_{l=0}^{\infty} d_{ln}^{(1,3)}(k_c)P_l(\cos\theta_c) \\
&\quad -\left(1-\frac{\omega_c-\mu}{m}\right)\sum_{l=0}^{\infty}(-1)^l\frac{(2l)!\Delta_{lN}^{(1,3)}}{(l!)^2}q^{2l} \\
&= \frac{2k_c^2}{\pi}P\int_\mu^\infty \frac{\omega_c' d\omega_c'}{k_c'^2(\omega_c'^2-\omega_c^2)}\left[1+\frac{(\omega_c'-\omega_c)(\omega_c\omega_c'+\mu^2)}{m\omega_c'(\omega_c+\omega_c')}\right. \\
&\quad \left. +\frac{q^2(\omega_c'-\omega_c)(\mu-\omega_c')(2\omega_c'+\omega_c+\mu)}{m\omega_c'(\omega_c'+\omega_c)(\omega_c+\mu)(\omega_c'+\mu)}\right]\sum_{l=0}^{\infty} a_{lN}^{(1,3)}(k_c') \\
&\quad \times P_l(\cos\theta_c')+\frac{2\Gamma_N^{(1,3)}(q)(q^2+\frac{1}{2}\mu^2)k_c^2}{m\mu^2\omega_c^2}. \qquad (6.1)
\end{aligned}$$

The dispersion equations would be more useful if the various partial-wave amplitudes were separated as much as possible. If Eq. (6.1) is valid for all values of q^2 in the range $0 < q < k_c$, a particular $d_{lN}^{(\lambda)}(k_c)$ may be

[16] A procedure similar to this has been used by R. Oehme, Phys. Rev. 100, 1503 (1955); Phys. Rev. 102, 1174 (1956).

separated from the $d_{l'N}^{(\lambda)}(k_c)$ corresponding to other angular momenta by multiplying the equation by $\frac{1}{2}P_l(\cos\theta_c)$ and integrating with respect to $\cos\theta_c$ between the limits -1 and 1.

In carrying out this procedure it is helpful to think of $\cos\theta_c=1-2(q^2/k_c^2)$ and k_c as the independent variables and to express q and $\cos\theta_c'$ in terms of k_c', k_c, and $\cos\theta_c$. If $\cos\theta_c$ and $\cos\theta_c'$ are denoted by z and z', z' is given as a function of z, k_c, and k_c' by the equation,

$$z'=1-(k_c/k_c')^2+(k_c/k_c')^2z. \tag{6.2}$$

The equation for the amplitude $d_{lN}^{(1,3)}(k_c)$, which results from the above procedure, is

$$\frac{d_{lN}^{(1,3)}(k_c)}{2l+1}-\left(1-\frac{\omega_c-\mu}{m}\right)\sum_{l=0}^{\infty}\alpha_{l,l'}{}^{\mathrm{I}}(k_c)\frac{\Delta_{l'N}^{(1,3)}}{2l'+1}k_c^{2l'}$$
$$=\frac{2k_c^2}{\pi}P\int_\mu^\infty\frac{\omega_c'd\omega_c'}{k_c'(\omega_c'^2-\omega_c^2)}\left[1+\frac{(\omega_c'-\omega_c)(\omega_c\omega_c'+\mu^2)}{m\omega_c'(\omega_c+\omega_c')}\right]$$
$$\times\sum_{l'=0}^{\infty}A_{l,l'}{}^{\mathrm{I}}(k_c,k_c')\frac{a_{l'N}^{(1,3)}(k_c')}{2l'+1}+\frac{f^2k_c^2}{m\omega_c^2}\eta_l{}^{\mathrm{I}}(k_c). \tag{I'}$$

The quantities $\alpha_{l,l'}{}^{\mathrm{I}}$, $A_{l,l'}{}^{\mathrm{I}}$, and $\eta_l{}^{\mathrm{I}}$ are given by the integrals

$$\alpha_{l,l'}{}^{\mathrm{I}}(k_c)=(-1)^{l'}\frac{(2l'+1)!}{2(l'!)^2}\int_{-1}^1 dzP_l(z)\left(\frac{1-z}{2}\right)^{l'}, \tag{6.3a}$$

$$A_{l,l'}{}^{\mathrm{I}}(k_c,k_c')=\frac{2l'+1}{2}\int_{-1}^1 dzP_l(z)P_{l'}(z')$$
$$\times\left[1+\frac{q^2(\omega_c'-\omega_c)(\mu-\omega_c')(2\omega_c'+\omega_c+\mu)}{m\omega_c'(\omega_c'+\omega_c)(\omega_c+\mu)(\omega_c'+\mu)}\right], \tag{6.3b}$$

$$\eta_l{}^{\mathrm{I}}(k_c)=\tfrac{1}{2}\int_{-1}^1 dzP_l(z)\left[1+\frac{2q^2}{\mu^2}\right]^2, \tag{6.3c}$$

where q^2 is equal to $\frac{1}{2}k_c^2(1-z)$ and the quantity z' of Eq. (6.3b) is given by Eq. (6.2).

It may be seen by inspection that the functions defined by Eqs. (6.3) satisfy certain "selection rules," i.e., $\alpha_{l,l'}{}^{\mathrm{I}}$ vanishes if $l'<l$, $A_{l,l'}{}^{\mathrm{I}}$ vanishes if $l'<l-1$, and $\eta_l{}^{\mathrm{I}}$ vanishes if $l>2$. If terms of order m^{-1} are neglected, $A_{l,l'}{}^{\mathrm{I}}$ and $\alpha_{l,l'}{}^{\mathrm{I}}$ vanish if $l'<l$. It is shown in Sec. 7 that, if higher orders in m^{-1} than the first are included, the functions that correspond to $\alpha_{l,l'}{}^{\mathrm{I}}$, $A_{l,l'}{}^{\mathrm{I}}$, and $\eta_l{}^{\mathrm{I}}$ do not satisfy such rigorous selection rules. In Appendix B the integrals in Eqs. (6.3) are evaluated for the smallest values of $|l'-l|$ and l.

An equation similar to Eq. (I') may be derived for the spin-flip amplitudes, $f_{lS}^{(1,3)}$. The derivation involves using Eqs. (5.5) and (5.6) to transform Eq. (II) to the center-of-mass system, and expanding in partial waves by means of Eqs. (5.10b), (5.14b), and (5.14d). The quantities $d_{lS}^{(1,3)}$ corresponding to different values of l may be separated by making use of the following orthogonality relation for the functions $\dot{P}_l(z)=dP_l(z)/dz$:

$$\tfrac{1}{2}(2l+1)\int_{-1}^1 dz(1-z^2)[dP_l(z)/dz][dP_{l'}(z)/dz]=l(l+1)\delta_{l,l'}.$$

The resulting dispersion relation for $f_{lS}^{(1,3)}(k_c)$, correct to first order in m^{-1}, is

$$d_{lS}^{(1,3)}(k_c)-\frac{\omega_c}{\mu}\left(1+\frac{k_c^2}{m\omega_c}\right)\sum_{l'=1}^{\infty}\alpha_{l,l'}{}^{\mathrm{II}}(k_c)\Delta_{l'S}^{(1,3)}k_c^{2l'}$$
$$=\frac{2\omega_ck_c^2}{\pi}P\int_\mu^\infty\frac{d\omega_c'}{k_c'^2(\omega_c'^2-\omega_c^2)}$$
$$\times\left[1+\frac{(\omega_c-\omega_c')(\mu^2+\omega_c^2+2\omega_c\omega_c')}{m\omega_c(\omega_c+\omega_c')}\right]\sum_{l'=1}^{\infty}A_{l,l'}{}^{\mathrm{II}}(k_c,k_c')$$
$$\times a_{lS}^{(1,3)}(k_c')-\frac{4f^2k_c^4}{\mu^4\omega_c}\left[1+\frac{\omega_c^2+\mu^2}{\omega_cm}\right]\eta_l{}^{\mathrm{II}}(k_c), \tag{II'}$$

where l and l' are both greater than zero. The functions $\alpha_{l,l'}{}^{\mathrm{II}}(k_c)$, $A_{l,l'}{}^{\mathrm{II}}(k_c,k_c')$, and $\eta_l{}^{\mathrm{II}}(k_c)$ are given by

$$\alpha_{l,l'}{}^{\mathrm{II}}(k_c)=(-1)^{l'+1}\frac{(2l'-1)!(2l+1)}{[(l'-1)!]^2l(l+1)}\int_{-1}^1 dz\left(\frac{1-z^2}{2}\right)\frac{dP_l}{dz}\left(\frac{1-z}{2}\right)^{l'-1}\left[1+\frac{q^2(\omega_c-\mu)}{m\omega_c\mu}\right], \tag{6.4a}$$

$$A_{l,l'}{}^{\mathrm{II}}(k_c,k_c')=\frac{2l+1}{2l(l+1)}\int_{-1}^1 dz(1+z)(1-z')\frac{dP_l(z)}{dz}\frac{dP_{l'}(z')}{dz'}\left[1+\frac{q^2(\omega_c-\omega_c')[\mu^2+\mu(\omega_c+\omega_c')+(3\omega_c'+2\omega_c)\omega_c]}{m\omega_c(\omega_c+\omega_c')(\omega_c+\mu)(\omega_c'+\mu)}\right], \tag{6.4b}$$

$$\eta_l{}^{\mathrm{II}}(k_c)=\frac{2l+1}{2l(l+1)}\int_{-1}^1 dz\left(\frac{1-z^2}{2}\right)\frac{dP_l(z)}{dz}\left[1+\frac{q^2(2\omega_c^2+\omega_c\mu+\mu^2)}{m\mu\omega_c(\omega_c+\mu)}\right]. \tag{6.4c}$$

From Eqs. (6.4) it may be seen that $\alpha_{l,l'}{}^{\mathrm{II}}$ and $A_{l,l'}{}^{\mathrm{II}}$ satisfy the same angular momentum selection rules as are satisfied by $A_{l,l'}{}^{\mathrm{I}}$. The quantity $\eta_l{}^{\mathrm{II}}$, on the other hand, behaves differently from $\eta_l{}^{\mathrm{I}}$, for $\eta_l{}^{\mathrm{II}}$ vanishes when $l>2$, and, if terms of order $1/m$ are neglected, $\eta_l{}^{\mathrm{II}}$ vanishes when $l\neq1$.

The partial scattering amplitudes $f_{lN}^{(2)}$ and $f_{lS}^{(2)}$, which represent the difference between the corresponding partial amplitudes for π^++p scattering and π^-+p scattering, satisfy equations similar to Eqs. (I') and (II'). The equations for $f_{lN}^{(2)}$, and $f_{lS}^{(2)}$ may be derived from Eqs. (III) and (IV) by following the same procedure as used for the cases $\lambda=1, 3$. The resulting equations are

$$\frac{d_{lN}^{(2)}(k_c)}{2l+1}-\frac{\omega_c}{\mu}\left[1+\frac{\mu(\omega_c-\mu)}{m\omega_c}\right]\sum_{l'=0}^{\infty}\alpha_{l,\,l'}{}^{\mathrm{III}}(k_c)\frac{\Delta_{l'N}{}^{(2)}}{2l'+1}k_c{}^{2l'}$$
$$=\frac{2\omega_c k_c{}^2}{\pi}P\int_{\mu}^{\infty}\frac{d\omega_c'}{k_c'^2(\omega_c'^2-\omega_c{}^2)}\left[1+\frac{(\omega_c\omega_c'+\mu^2)(\omega_c-\omega_c')}{m\omega_c(\omega_c+\omega_c')}\right]$$
$$\times\sum_{l'=0}^{\infty}A_{l,\,l'}{}^{\mathrm{III}}(k_c,k_c')\frac{a_{l'N}{}^{(2)}(k_c')}{2l'+1}+\frac{2f^2k_c{}^2}{\mu^2\omega_c{}^2}\left(1+\frac{\mu^2}{m\omega_c{}^2}\right)\eta_l{}^{\mathrm{III}}(k_c),\quad\text{(III')}$$

and

$$d_{lS}{}^{(2)}(k_c)-\sum_{l'=1}^{\infty}\alpha_{l,\,l'}{}^{\mathrm{IV}}(k_c)\Delta_{lS}{}^{(2)}k_c{}^{2l'}=\frac{2k_c{}^2}{\pi}P\int_{\mu}^{\infty}\frac{\omega_c'd\omega_c'}{k_c'^2(\omega_c'^2-\omega_c{}^2)}\left[1+\frac{(\omega_c'-\omega_c)(\mu^2-\omega_c'^2)}{m\omega_c'(\omega_c+\omega_c')}\right]$$
$$\times\sum_{l'=1}^{\infty}A_{l,\,l'}{}^{\mathrm{IV}}(k_c,k_c')a_{lS}{}^{(2)}(k_c')-\frac{2f^2k_c{}^4}{m\mu^2\omega_c{}^2}\eta_l{}^{\mathrm{IV}}(k_c).\quad\text{(IV')}$$

The functions $\alpha_{l,\,l'}{}^{\mathrm{III,IV}}$, $A_{l,\,l'}{}^{\mathrm{III,IV}}$, $\eta_l{}^{\mathrm{III,IV}}$ are similar to the corresponding functions of the case $\epsilon_\lambda=1$ ($\lambda=1$ or 3). They are defined by the integrals:

$$\alpha_{l,\,l'}{}^{\mathrm{III}}(k_c)=(-1)^{l'}\frac{(2l'+1)!}{2(l'!)^2}\int_{-1}^{1}dzP_l(z)\left(\frac{1-z}{2}\right)^{l'}\left[1+\frac{q^2(\omega_c-\mu)}{m\omega_c\mu}\right],\quad(6.5a)$$

$$A_{l,\,l'}{}^{\mathrm{III}}(k_c,k_c')=\frac{2l'+1}{2}\int_{-1}^{1}dzP_l(z)P_{l'}(z')\left[1+\frac{q^2(\omega_c-\omega_c')[\mu^2+\mu(\omega_c+\omega_c')+(3\omega_c'+2\omega_c)\omega_c]}{m\omega_c(\omega_c+\omega_c')(\omega_c+\mu)(\omega_c'+\mu)}\right],\quad(6.5b)$$

$$\eta_l{}^{\mathrm{III}}(k_c)=\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\left(1+\frac{2q^2}{\mu^2}\right)\left[1+\frac{q^2(2\omega_c{}^2+\omega_c\mu+\mu^2)}{m\mu\omega_c(\omega_c+\mu)}\right]\quad(6.5c)$$

and

$$\alpha_{l,\,l'}{}^{\mathrm{IV}}(k_c)=(-1)^{l'+1}\frac{(2l'-1)!(2l+1)}{[(l'-1)!]^2l(l+1)}\int_{-1}^{1}dz\left(\frac{1-z^2}{2}\right)\frac{dP_l}{dz}\left(\frac{1-z}{2}\right)^{l'-1},\quad(6.6a)$$

$$A_{l,\,l'}{}^{\mathrm{IV}}(k_c,k_c')=\frac{2l+1}{2l(l+1)}\int_{-1}^{1}dz(1+z)(1-z')\frac{dP_l(z)}{dz}\frac{dP_{l'}(z')}{dz'}\left[1+\frac{q^2(\omega_c-\omega_c')(\omega_c'-\mu)(2\omega_c'+\omega_c+\mu)}{m\omega_c'(\omega_c'+\omega_c)(\omega_c+\mu)(\omega_c'+\mu)}\right],\quad(6.6b)$$

$$\eta_l{}^{\mathrm{IV}}=\frac{(2l+1)}{2l(l+1)}\int_{-1}^{1}dz\left(\frac{1-z^2}{2}\right)\frac{dP_l(z)}{dz}\left(1-\frac{2q^2}{\mu^2}\right).\quad(6.6c)$$

The values of the angular integrals of Eqs. (6.3) through (6.6) corresponding to small values of $|l-l'|$ and l are given in Appendix B.

Equations (I') through (IV') represent the "heavy-nucleon limit" to the dispersion relations for pion-nucleon scattering, analyzed in terms of orbital angular momentum and spin dependence. If Eqs. (5.14) are used, these dispersion relations may be expressed in terms of the amplitudes $f_{l\pm}{}^{(\lambda)}=d_{l\pm}{}^{(\lambda)}+ia_{l\pm}{}^{(\lambda)}$, which correspond to orbital angular momentum l and total angular momentum $l\pm\frac{1}{2}$. The absorptive part of these equations cannot be expressed in terms of the total cross section, as can be done for the forward scattering dispersion relations. However, the imaginary parts $a_{l\pm}{}^{(\lambda)}(k_c)$ of the amplitudes $f_{l\pm}{}^{(\lambda)}(k_c)$ may be expressed in terms of partial cross sections by the equations

$$\begin{aligned}(l+1)a_{l+}{}^{(\lambda)}(k_c)&=(k_c/4\pi)\sigma_{l+}{}^{(\lambda)}(k_c),\\ la_{l-}{}^{(\lambda)}(k_c)&=(k_c/4\pi)\sigma_{l+}{}^{(\lambda)}(k_c).\end{aligned}\quad(6.7)$$

The symbols $\sigma_{l\pm}{}^{(\lambda)}$ represent partial cross sections for waves of orbital angular momentum l and total angular momentum $l\pm\frac{1}{2}$; they are total cross sections in the sense that they include both elastic and inelastic processes. The partial cross sections $\sigma_{l\pm}{}^{(1)}$ and $\sigma_{l\pm}{}^{(2)}$ are defined in terms of the corresponding quantities for π^++p scattering and π^-+p scattering by the equations

$$\begin{aligned}\sigma_{l\pm}{}^{(1)}&=\tfrac{1}{2}[\sigma_{l\pm}(\pi^++p)+\sigma_{l\pm}(\pi^-+p)],\\ \sigma_{l\pm}{}^{(2)}&=\tfrac{1}{2}[\sigma_{l\pm}(\pi^++p)-\sigma_{l\pm}(\pi^-+p)].\end{aligned}$$

If terms of order m^{-1} are neglected, the dispersion relations express the real parts $d_{l\pm}^{(\lambda)}(k_c)$ of the partial-wave amplitudes in terms of the partial-wave cross sections corresponding to angular momenta equal to or greater than l, since, in this approximation, the functions $A_{l,l'}$ of Eqs. (I′)–(IV′) vanish if $l'<l$. If terms of order m^{-1} are included, on the other hand, the partial cross sections $\sigma_{l'\pm}^{(\lambda)}$ may contribute to the equations for $d_{l\pm}^{(\lambda)}$ if $l'\geq l-1$. The generalization of this rule when terms of higher order are included is discussed in Sec. 7. Only the energy and momentum variables of Eqs. (I′)–(IV′) have been expanded in powers of m^{-1}, since the m dependence of the absorptive amplitudes depends on the nature of the meson theory used. In most simple meson theories, though, the partial cross sections $\sigma_{l\pm}^{(\lambda)}$ generally are smaller than $\sigma_{l'\pm}^{(\lambda)}$ by a factor of order $(k_c^2/m^2)^{l'-l}$. This relationship applies to pseudoscalar meson theory with pseudoscalar coupling, provided both l and l' are greater than zero. Hence, if a simple meson theory is used to expand the partial cross sections in powers of m^{-1}, and only the lowest order term is retained, approximate dispersion relations may be written which, in many cases involve only one angular momentum. However, if the experimentally measured partial cross sections $\sigma_{l\pm}^{(\lambda)}$ and $\sigma_{l'\pm}^{(\lambda)}$ (where $l'>l$), are of the same order of magnitude in a particular energy region, both partial cross sections should be included in the dispersion relations for $d_{l\pm}^{(\lambda)}$, of course.

It should be noted that if the sums over angular momenta are cut off at some finite number, all quantities appearing in Eqs. (I′)–(IV′) are finite even at energies such that $k_c^2>m\mu$. The energy range in which these "heavy nucleon" equations are approximately accurate is not known at present.

If the scattering amplitude is sufficiently convergent at high energies, dispersion relations of type B may be derived from the contour integral of Eq. (4.5). The partial-wave analysis of these relations leads to equations that are simpler than Eqs. (I′)–(IV′) in that they do not involve the threshold constants $\Delta_{lN}^{(\lambda)}$ and $\Delta_{lS}^{(\lambda)}$.

The partial-wave dispersion relations are most useful at low energies, where few angular momenta are important. Since only S and P waves seem to be important for low-energy pion-nucleon scattering, we study the form of the dispersion relations in the approximation that angular momenta of two or more units are neglected. Charge independence is assumed, so that the amplitudes $f_{0+}^{(\lambda)}$, $f_{1-}^{(\lambda)}$, and $f_{1+}^{(\lambda)}$ may be expressed in terms of scattering amplitudes for total isotopic spins $\frac{1}{2}$ and $\frac{3}{2}$ by equations similar to Eq. (3.5). At low energies elastic scattering is the dominant reaction process, so we neglect inelastic processes. In this approximation the scattering phase shifts are real and, if the spin and isotopic spin values of these phase shifts is denoted in the conventional manner, the relevant absorptive amplitudes are given by the equations

$$\begin{aligned}
k_ca_{0N}^{(1)}&=k_ca_{0N}^{(3)}=k_ca_{0+}^{(1,3)}=\tfrac{1}{3}(2\sin^2\delta_3+\sin^2\delta_1),\\
k_ca_{1N}^{(1,3)}&=k_c[2a_{1+}^{(1,3)}+a_{1-}^{(1,3)}]\\
&=\tfrac{1}{3}(4\sin^2\delta_{33}+2\sin^2\delta_{31}+2\sin^2\delta_{13}+\sin^2\delta_{11}),\\
k_ca_{1S}^{(1,3)}&=k_c[a_{1+}^{(1,3)}-a_{1-}^{(1,3)}]\\
&=\tfrac{1}{3}(2\sin^2\delta_{33}-2\sin^2\delta_{31}+\sin^2\delta_{13}-\sin^2\delta_{11}),\\
k_ca_{0N}^{(2)}&=k_ca_{0+}^{(2)}=\tfrac{1}{3}(\sin^2\delta_3-\sin^2\delta_1),\\
k_ca_{1N}^{(2)}&=k_c[2a_{1+}^{(2)}+a_{1-}^{(2)}]\\
&=\tfrac{1}{3}(2\sin^2\delta_{33}+\sin^2\delta_{31}-2\sin^2\delta_{13}-\sin^2\delta_{11}),\\
k_ca_{1S}^{(2)}&=k_c[a_{1+}^{(2)}-a_{1-}^{(2)}]\\
&=\tfrac{1}{3}(\sin^2\delta_{33}-\sin^2\delta_{31}-\sin^2\delta_{13}+\sin^2\delta_{11}).
\end{aligned}\tag{6.8}$$

Formulas for the corresponding dispersive amplitudes may be obtained from the above equations by replacing the functions $\sin^2\delta_i$ by $\frac{1}{2}\sin(2\delta_i)$, i.e.,

$$k_cd_{0N}^{(1,3)}=k_cd_{0+}^{(1,3)}=\tfrac{1}{6}(2\sin2\delta_3+\sin2\delta_1),\tag{6.9}$$

and so forth.

If the above equations are used to express the scattering amplitudes in terms of phase shifts, and the values of the functions $A_{l,l'}^{\mathrm{I-IV}}$, $\alpha_{l,l'}^{\mathrm{I-IV}}$, and $\eta_l^{\mathrm{I-IV}}$ corresponding to angular momenta of 0 and 1 are taken from Appendix B, six equations for the six S and P phase shifts may be obtained from Eqs. (I′)–(IV′). Since this procedure is straightforward, we do not list the six equations here, but list instead the corresponding type B equations which follow from the contour integral of Eq. (4.5). These equations are given below, expressed in terms of the quantities of Eqs. (6.8) and (6.9):

$$\begin{aligned}
k_cd_{0N}^{(1)}(k_c)=&\frac{2k_c}{\pi}P\int_0^\infty\frac{dk_c'}{(k_c'^2-k_c^2)}\left[1+\frac{(\omega_c'-\omega_c)(4\omega_c'^2+\omega_c^2+6\omega_c\omega_c'+\mu^2)}{2m\omega_c'(\omega_c'+\omega_c)}\right]k_c'a_{0N}^{(1)}(k_c')\\
&+\frac{2k_c}{\pi}P\int_0^\infty\frac{dk_c'}{k_c'^2}\left[1+\frac{(12\omega_c'^2+6\mu^2+18\omega_c\omega_c')(\omega_c'^2-\omega_c^2)}{6m\omega_c'(\omega_c'+\omega_c)^2}+\frac{k_c^2(3k_c'^2-4k_c^2)}{6m\omega_c'(\omega_c'+\omega_c)^2}\right]k_c'a_{1N}^{(1)}(k_c')\\
&-\frac{f^2\mu^2k_c}{m\omega_c^2}\left[1+2\frac{k_c^2}{\mu^2}+\frac{4k_c^4}{3\mu^4}\right];
\end{aligned}\tag{6.10}$$

$$k_c d_{1N}^{(1)}(k_c)=\frac{2k_c^3}{\pi}P\int_0^\infty \frac{dk_c'}{k_c'^2(k_c'^2-k_c^2)}\left[1+\frac{(\omega_c'-\omega_c)(3\omega_c'^2+2\omega_c^2+6\omega_c\omega_c'+\mu^2)}{2m\omega_c'(\omega_c'+\omega_c)}\right]k_c'a_{1N}^{(1)}(k_c')$$
$$-\frac{k_c^3}{\pi m}\int_0^\infty \frac{dk_c'}{\omega_c'(\omega_c'+\omega_c)^2}k_c'a_{0N}^{(1)}(k_c')+\frac{2f^2k_c^3}{m\mu^2}; \quad (6.11)$$

$$k_c d_{1S}^{(1)}(k_c)=\frac{2\omega_c k_c^3}{\pi}P\int_0^\infty \frac{dk_c'}{\omega_c'k_c'^2(k_c'^2-k_c^2)}\left[1+\frac{k_c^2(\omega_c'-\omega_c)}{2m\omega_c(\omega_c'+\omega_c)}\right]k_c'a_{1S}^{(1)}(k_c')+\frac{2f^2k_c^3}{\mu^2\omega_c}\left(1-\frac{k_c^2}{2m\omega_c}\right); \quad (6.12)$$

$$k_c d_{0N}^{(2)}(k_c)=\frac{2\omega_c k_c}{\pi}P\int_0^\infty \frac{dk_c'}{\omega_c'(k_c'^2-k_c^2)}\left[1+\frac{(\omega_c'-\omega_c)(3\omega_c^2+2\omega_c\omega_c'-\mu^2)}{2m\omega_c(\omega_c+\omega_c')}\right]k_c'a_{0N}^{(2)}(k_c')$$
$$+\frac{2\omega_c k_c}{\pi}P\int_0^\infty \frac{dk_c'}{\omega_c'k_c'^2}\left[1+\frac{6(\omega_c^2+\omega_c\omega_c')(\omega_c'^2-\omega_c^2)+k_c^2(3k_c'^2-2k_c^2)}{6m\omega_c(\omega_c+\omega_c')^2}\right]k_c'a_{1N}^{(2)}(k_c')$$
$$-\frac{2f^2\omega_c k_c}{\mu^2}\left[1-\frac{\mu^4}{m\omega_c^3}\left(1+\frac{5}{2}\frac{k_c^2}{\mu^2}+\frac{4}{3}\frac{k_c^4}{\mu^4}\right)\right]; \quad (6.13)$$

$$k_c d_{1N}^{(2)}(k_c)=\frac{2\omega_c k_c^3}{\pi}P\int_0^\infty \frac{dk_c'}{\omega_c'k_c'^2(k_c'^2-k_c^2)}\left[1+\frac{(\omega_c'-\omega_c)(3\omega_c^2+2\omega_c\omega_c'-\mu^2)}{2m\omega_c(\omega_c+\omega_c')}\right]k_c'a_{1N}^{(2)}(k_c')$$
$$+\frac{k_c^3}{\pi m}P\int_0^\infty \frac{dk_c'}{\omega_c'(\omega_c'+\omega_c)^2}[k_c'a_{0N}^{(2)}(k_c')]+\frac{2f^2k_c^3}{\mu^2\omega_c}\left[1+\frac{\mu^2-2\omega_c^2}{2m\omega_c}\right]; \quad (6.14)$$

$$k_c d_{1S}^{(2)}(k_c)=\frac{2k_c^3}{\pi}P\int_0^\infty \frac{dk_c'}{k_c'^2(k_c'^2-k_c^2)}\left[1+\frac{(\omega_c'-\omega_c)(2\omega_c'^2+\omega_c^2+4\omega_c\omega_c'+\mu^2)}{2m\omega_c'(\omega_c+\omega_c')}\right]k_c'a_{1S}^{(2)}(k_c')+\frac{f^2k_c^3}{m\mu^2}. \quad (6.15)$$

If Eqs. (6.8) and (6.9) are used to express the dispersive and absorptive amplitudes in terms of the phase shifts, the above equations represent six simultaneous nonlinear integral equations for the six phase shifts, δ_1, δ_3, δ_{11}, δ_{13}, δ_{31}, and δ_{33}. Although there is not a unique solution to these equations,[17] they may be useful in analyzing the low-energy pion-nucleon scattering data.

Equations (6.10) through (6.15) become particularly simple if terms of order m^{-1} are neglected. In such a limit Eqs. (6.11), (6.12), (6.14), and (6.15) involve only P-wave amplitudes; these equations have previously been derived by Low[7] and Oehme,[18,19] and are known as Low's equations for P-wave scattering. To zero order in m^{-1}, Eqs. (6.10) and (6.13) involve both S and P waves. Equations which involve only S waves may be derived, however, if use is made of the following facts. If the scattering amplitude converges rapidly enough at high energies so that Eqs. (6.10) and (6.13) are valid, dispersion relations of type A are also valid; in particular the forward angle equations of Goldberger,[2] which are of type A, are valid. It may be shown that in the low-energy approximation used here, (neglect of inelastic processes and orbital angular momenta greater than one) these forward-angle equations may be combined linearly with Eqs. (6.11) and (6.14) to give equations which, to zero order in m^{-1}, involve only S waves and are identical with S-wave equations derived by Oehme.[19]

The fact that the S and P amplitudes satisfy separate dispersion relations to zero order in m^{-1} does not mean that these amplitudes are independent to this order. Equations (6.10) and (6.13) express relations between the S and P amplitudes that must be satisfied if the assumptions made in this section are correct. The solutions of the S- and P-wave dispersion relations are not unique[17]; Eqs. (6.10) and (6.13) may be considered as additional conditions on these solutions.

Important examples of low-energy dispersion relations, which illustrate the interdependence of orbital angular momenta zero and one, may be obtained if Eqs. (6.10) and (6.13) are divided by k_c, and k_c is set equal to zero. The resulting equations express the scattering lengths for isotopic spins $\frac{1}{2}$ and $\frac{3}{2}$ in terms of energy integrals of S- and P-wave phase shifts. Reference to the experimental data shows that the P-wave contributions to these equations are quite important.

The fact that the low-energy dispersion equations, Eqs. (6.10)–(6.15), are integral equations for the phase shifts results from the neglect of inelastic processes. The situation would be quite different if the methods of the present paper were applied to the problem of the scat-

[17] These equations are of the same type as those derived by Low in reference 7. The multiplicity of the solutions to Low's equations has been discussed by Castillejo, Dalitz, and Dyson, Phys. Rev. **101**, 453 (1956).
[18] Reinhard Oehme, Phys. Rev. **100**, 1503 (1955).
[19] Reinhard Oehme, Phys. Rev. **102**, 1174 (1956).

tering from nucleons of gamma rays of energies in the range 50 Mev to 300 Mev. Because of the dominance of the meson-production cross section, which results when gamma rays of sufficient energy are used to bombard nucleons, it is an excellent approximation to neglect the elastic-scattering contribution to the cross sections which appear in the absorption integrals of the dispersion relations. Gell-Mann, Goldberger, and Thirring[1] have used the forward-direction dispersion relation to determine approximately the energy behavior of the coherent amplitude for forward photon-proton scattering from the experimental data on the energy dependence of the total cross section for photopion production from protons. The method of this paper could be used to determine the general nature of the angular dependence of the elastic-scattering amplitude, at various energies, from experimental data on the angular dependence of the photoproduction cross sections.

7. DISPERSION EQUATIONS AT ENERGIES COMPARABLE TO THE NUCLEON MASS

If terms of order higher than the first in an expansion in powers of m^{-1} are included, the nonspin-flip and spin-flip amplitudes are mixed by the transformation to the center-of-mass system, and the equations for the partial-wave amplitudes are quite complicated. Perhaps the most useful procedure, in this case, is to combine linearly Eqs. (I) and (II) or Eqs. (III) and (IV) in such a way that the center-of-mass dispersive amplitudes, $d_{cN}^{(\lambda)}(k_c)$ and $d_{cS}^{(\lambda)}(k_c)$, do not appear in the same dispersion equation after the transformation to center-of-mass quantities has been made. This procedure results in extremely lengthy equations; therefore we follow the alternate procedure of analyzing the first dispersion relation, Eq. (I), in terms of partial-wave amplitudes. The effect of the inclusion of higher order terms in m^{-1} is then studied.

Equation (5.15) represents the expansion of Eq. (I) into partial waves, correct to all orders of m^{-1}. The functions $d_{lN}^{(1,3)}$ corresponding to different values of l may be separated by the same method as used in Sec. 6, multiplication by the set of Legendre polynomials and integration over the scattering angle. The angular integrals are complicated because the functions q^2, ω, and ω' all depend on the center-of-mass scattering angle θ_c, as well as on k_c or k_c'. Because of the dependence on θ_c of the q-system energies ω and ω', it is useful to define two functions ν and ν', which are independent of $\cos\theta_c$, by the equations

$$\nu=m^{-1}(k_c^2+E_c\omega_c),$$
$$\nu'=m^{-1}(k_c'^2+E_c'\omega_c').$$

Since $q^2=0$ corresponds to $\cos\theta_c=1$, it may be seen from these equations and Eq. (5.1a) that when $\cos\theta_c=1$, the q-system energies ω and ω' are equal to ν and ν'. If Eq. (5.15) is multiplied by $\frac{1}{2}P_l(\cos\theta_c)$ and integrated over $\cos\theta_c$ between the limits 1 and -1, the result may be written

$$\frac{d_{lN}^{(1,3)}(k_c)}{2l+1}+\frac{k_c^2}{m(m+E_c)}\sum_{l'}\gamma_{l,l'}d_{l'S}^{(1,3)}(k_c)-\left(\frac{m+u}{W_c}\right)\sum_{l'}\left[\alpha_{l,l'}(k_c)\frac{\Delta_{l'N}^{(1,3)}}{2l'+1}+\frac{k_c^2}{m^2}\beta_{l,l'}(k_c)\Delta_{l'S}^{(1,3)}\right]k_c^{2l'}$$
$$=\frac{2}{\pi}(\nu^2-\mu^2)P\int_\mu^\infty d\omega_c'\left\{\frac{W_c'^3}{mW_cE_c'}\frac{\nu'}{(\nu'^2-\mu^2)(\nu'^2-\nu^2)}\sum_{l'}\left[A_{l,l'}(k_c,k_c')\frac{a_{lN}^{(1,3)}(k_c')}{2l'+1}\right.\right.$$
$$\left.\left.+\frac{k_c'^2}{m(m+E_c')}B_{l,l'}(k_c,k_c')a_{lS}^{(1,3)}(k_c')\right]\right\}+\frac{2f^2(\mu^2/2m)(\nu^2-\mu^2)}{[(\mu^2/2m)^2-\mu^2][(\mu^2/2m)^2-\nu^2]}\frac{m}{W_c}\eta_l(k_c). \quad \text{(I}''\text{)}$$

The coefficients $\gamma_{l,l'}$, $\alpha_{l,l'}$, $\beta_{l,l'}$, $A_{l,l'}$, $B_{l,l'}$, and η_l are defined in terms of complicated angular integrals. The writing of these integrals is simplified by defining $z=\cos\theta_c$, $z'=\cos\theta_c'$, and by expressing some quantities in terms of the functions $q^2=\frac{1}{2}k_c^2(1-z)$, $\omega=E_cW_cE_q^{-1}-E_q$, and $\omega'=E_c'W_c'E_q^{-1}-E_q$. The coefficients of Eq. (I″) are given by

$$\gamma_{l,l'}(k_c)=\frac{1}{2}\int_{-1}^1 dzP_l(z)\frac{dP_{l'}(z)}{dz}\left(\frac{1-z^2}{2}\right)\frac{m(m+E_c)}{E_q^2+mE_c}, \quad (7.1a)$$

$$\alpha_{l,l'}(k_c)=(-1)^{l'}\frac{(2l'+1)!}{2(l'!)^2}\int_{-1}^1 dzP_l(z)\left(\frac{1-z}{2}\right)^{l'}\left[\frac{(m^2+E_q^2)E_q(m+E_c)}{2mE_q(E_q^2+mE_c)}\right], \quad (7.1b)$$

$$\beta_{l,l'}(k_c)=(-1)^{l'}\frac{(2l'-1)!}{2[(l'-1)!]^2}\int_{-1}^1 dzP_l(z)\left(\frac{1-z}{2}\right)^{l'+1}\frac{m(m+E_c)}{E_q^2+mE_c}, \quad (7.1c)$$

$$A_{l,l'}(k_c,k_c')=\frac{2l'+1}{2}\int_{-1}^1 P_l(z)P_{l'}(z')dz\left[\frac{E_q(m+E_c)m}{(E_q^2+mE_c)E_q}\frac{(\omega^2-\omega_a^2)\omega'(\nu'^2-\nu^2)(\nu'^2-\mu^2)(E_q^2+mE_c')}{(\nu^2-\mu^2)\nu'(\omega'^2-\omega^2)(\omega'^2-\omega_a^2)E_q(m+E_c')}\right], \quad (7.1d)$$

$$B_{l,l'}(k_c,k_c')=\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\frac{dP_{l'}(z')}{dz'}\left(\frac{1-z'^2}{2}\right)\left[\frac{E_q(m+E_c)m^2}{(E_q^2+mE_c)E_q^2}\,\frac{(\omega^2-\omega_a^2)\omega'(\nu'^2-\nu^2)(\nu'^2-\mu^2)}{(\nu^2-\mu^2)\nu'(\omega'^2-\omega^2)(\omega'^2-\omega_a^2)}\right], \tag{7.1e}$$

$$\eta_l(k_c)=\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\frac{\Gamma_N^{(1,3)}\omega_b}{f^2(\mu^2/2m)}\left[\frac{(\omega^2-\omega_a^2)\{(\mu^2/2m)^2-\mu^2\}}{(\nu^2-\mu^2)(\omega_b^2-\omega_a^2)}\,\frac{\{(\mu^2/2m)^2-\nu^2\}E_q(m+E_c)}{(\omega_b^2-\omega^2)(E_q^2+mE_c)}\right]. \tag{7.1f}$$

If the quantities of Eq. (I″) are expanded in powers of $(1/m)$ and only the zero-order and first-order terms retained, the spin-dependent terms vanish and the equation reduces to Eq. (I′). To this order the functions $\alpha_{l,l'}$, $A_{l,l'}$, and η_l are equal to the functions $\alpha_{l,l'}^{\mathrm{I}}$, $A_{l,l'}^{\mathrm{I}}$, and η_l^{I} defined in Eqs. (6.3).

In order to investigate the nature of the angular momentum selection rules that apply when terms of higher order in $(1/m)$ are retained, we assume that the energy is low enough so that the integrands of Eqs. (7.1) may be expanded in powers of $q^2=\frac{1}{2}k_c^2(1-z)$. To illustrate this expansion we choose $A_{l,l'}(k_c,k_c')$ as a representative example, and write this coefficient in the form

$$A_{l,l'}(k_c,k_c')=\tfrac{1}{2}(2l'+1)\int_{-1}^{1}P_l(z)P_{l'}(z')dz\,\mathfrak{A}(k_c,k_c',q^2).$$

If the function $\mathfrak{A}(k_c,k_c',q^2)$ is expanded in powers of q^2, it is found that the coefficient of $(q^2)^n$ is of order n or higher in the quantity $(1/m)$. Therefore, if $A_{l,l'}$ is expanded in powers of $(1/m)$, i.e.,

$$A_{l,l'}=\textstyle\sum_i A_{l,l'}{}^i(1/m)^i,$$

then we have

$$A_{l,l'}{}^i(k_c,k_c')=0\quad\text{if}\quad l'<l-i.$$

The functions $\alpha_{l,l'}$, $\beta_{l,l'}$, $\gamma_{l,l'}$, $B_{l,l'}$, and η_l may be expanded in similar series, i.e., $\alpha_{l,l'}=\sum_i\alpha_{l,l'}{}^i m^{-i}$, etc. It may be seen that the coefficients of these series satisfy the following angular momentum selection rules:

$$\alpha_{l,l'}{}^i=0\quad\text{if}\quad l'<l-i, \tag{7.2}$$

$$\beta_{l,l'}{}^i,\quad\gamma_{l,l'}{}^i,\quad\text{and}\quad B_{l,l'}{}^i=0\quad\text{if}\quad l'<l-i-1, \tag{7.3}$$

$$\eta^i=0\quad\text{if}\quad l>2+i.$$

Equations similar to Eq. (I″) may be derived from Eqs. (II)–(IV). The coefficients in $(1/m)$ expansions of the functions occurring in these equations that are analogous to the functions $A_{l,l'}$ and $\alpha_{l,l'}$ satisfy the rule,[20] Eq. (7.2). On the other hand the coefficients analogous to $\beta_{l,l'}{}^i$, $\gamma_{l,l'}{}^i$, and $B_{l,l'}{}^i$, which represent the mixing of the spin-independent and spin-dependent amplitudes, satisfy the rule,[20] Eq. (7.3). Thus, if the partial-wave dispersion equations are expanded in powers of m^{-1} and terms of order higher than n are neglected, the dispersive amplitude $d_{l,N}{}^{(\lambda)}(k_c)$ depends on $a_{l'N}{}^{(\lambda)}$ [or $\sigma_{l'N}{}^{(\lambda)}$] and $\Delta_{l'N}{}^{(\lambda)}$ only if $l'\geq l-n$. Furthermore, $d_{lN}(k_c)$ depends on $a_{l'S}$ and $\Delta_{l'S}$ only if $n\geq 2$ and $l'\geq l-n+1$. Similar relations hold if the roles of the spin-dependent and spin-independent amplitudes are exchanged.

The above conclusions are based on the assumption that the momentum transfer is small enough that the quantities occurring in the angular integrals may be expanded in powers of m^{-1}. If $k_c<m$, so that $q<m$, it may be shown that most of the quantities occurring in the angular integrals may be expanded in convergent series in powers of $(q^2/\omega_c m)$ and (q^2/m^2). Some functions occur in the angular integrals which have smaller radii of convergence, however. Two examples of such functions are $(\omega'^2-\omega_a^2)^{-1}$ and $(\omega_b^2-\omega_a^2)^{-1}$. The first has a radius of convergence of $q^2=\frac{1}{2}(m\mu+k_c'^2+E_c'\omega_c')$, which, in the limit as $\omega_c'\to\mu$, becomes $q^2=m\mu$. The factor $(\omega_b^2-\omega_a^2)$ has a radius of convergence of $q^2=\frac{1}{2}(m\mu-\frac{1}{2}\mu^2)$. At energies such that $k_c^2>\frac{1}{2}(m\mu-\frac{1}{2}\mu^2)$, the integrand of Eq. (7.1f) is infinite at the point $z=1-k_c^{-2}(m\mu-\frac{1}{2}\mu^2)$. Therefore, at these energies, one cannot use the analysis of this section. One may use the alternative procedure of taking successive derivatives with respect to q^2 of equations of the type of Eq. (5.15) and evaluating in the forward direction, or one may use the heavy-nucleon equations of Sec. 6, consider only a finite number of angular momenta, and hope that the resulting equations are accurate even at energies such that $k_c^2>\frac{1}{2}(m\mu-\frac{1}{2}\mu^2)$.

8. CONCLUSIONS

The principle of causality is used in a derivation of dispersion relations for pion-nucleon scattering in the case of finite momentum transfer between the particles. If the relations are analyzed into partial waves, the resulting equations express the real parts of the scattering amplitudes corresponding to different values of the orbital and total angular momenta in terms of energy integrals of either the various partial cross sections or the imaginary parts of the various amplitudes. In general, the real part of a particular amplitude is dependent on the partial-wave cross sections corresponding to both spin-dependent and spin-independent scattering, and also to all values of the orbital angular momentum.

If the various functions of the particle momenta and energies are expanded in powers of $(1/m)$, where m is the nucleon mass, the dispersion relations are simplified. The spin-dependent and spin-independent amplitudes do ot occur in the same equation if terms of higher order than $(1/m)$ are neglected. If terms of order higher than $(1/m)^n$ are neglected, the real part of one of the

[20] Some of these functions, such as the coefficients $\gamma_{l,l'}{}^i$, $\alpha_{l,l'}{}^i$ and $\beta_{l,l'}{}^i$ in the expansion of Eqs. (7.1a)–(7.1c), satisfy more strict selection rules, but Eqs. (7.2) and (7.3) are the general rules.

amplitudes corresponding to angular momentum l can depend on partial cross sections of angular momentum l' only if $l' \geq l-n$.

The derivation of the dispersion relations depends on certain assumptions concerning the rate of convergence of the scattering amplitude at high energies, and the rate of convergence of an expansion of the amplitude in terms of partial waves. Comparison of the results derived here with experimental data will provide a partial test of these assumptions. The fact that the forward-scattering dispersion relation is consistent with the low-energy experimental data[21] may be evidence that the assumptions made here are justified.

The method used here may be generalized to other boson-fermion scattering problems. It is likely that useful results could be obtained from an application to the scattering of gamma rays from nucleons.

APPENDIX A. LORENTZ TRANSFORMATION OF THE SCATTERING AMPLITUDE

If the scattering amplitude M expressed in terms of four-by-four Dirac matrices, the Lorentz invariance of the amplitude implies that the quantity $\bar{u}(\mathbf{p}',\alpha') \times M u(\mathbf{p},\alpha)$ is invariant, where $u(\mathbf{p},\alpha)$ and $\bar{u}(\mathbf{p}',\alpha')$ are four-component Dirac spinors which represent the nucleon in initial and final states of positive energy, momenta $\mathbf{p}$ and $\mathbf{p}'$ and spin directions α and α'. The scattering amplitudes of this paper have been written in terms of the two-by-two matrices $\boldsymbol{\sigma}$ and $\mathbf{1}$, which operate between the two-component spinor functions $\chi(\alpha)$. A nucleon is defined as having spin up (or down) with respect to an axis in the direction of a unit vector $\mathbf{n}$, if, in its own rest system, it is an eigenfunction of the four-by-four Dirac operator $\boldsymbol{\sigma}\cdot\mathbf{n}$ with eigenvalue one (or minus one). Therefore, the scattering amplitudes of this paper may be expressed in terms of four-by-four spin matrices in the following manner:

$$\bar{\chi}(\alpha')M\chi(\alpha)=\bar{u}(0,\alpha')[\mathfrak{M}_N\mathbf{1}+i\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{Q})\mathfrak{M}_S]u(0,\alpha), \tag{A1}$$

$$\bar{\chi}(\alpha')M_c\chi(\alpha)=\bar{u}(0,\alpha')[\mathfrak{M}_{cN}\mathbf{1}+i\boldsymbol{\sigma}\cdot(\mathbf{k}_c\times\mathbf{k}_c')\mathfrak{M}_{cS}]u(0,\alpha). \tag{A2}$$

The matrices in Eqs. (A1) and (A2) may be considered as two-by-two matrices since the small components of the spinors $u(0,\alpha)$ and $\bar{u}(0,\alpha')$ vanish. These spinors $u(0,\alpha')$ and $\bar{u}(0,\alpha')$, which represent nucleons at rest, are related to $u(\mathbf{p},\alpha)$ and $\bar{u}(\mathbf{p}',\alpha')$ by

$$u(\mathbf{p},\alpha)=\frac{\Lambda_+(p)}{[\frac{1}{2}(1+E/m)]^{\frac{1}{2}}}u(0,\alpha),\qquad \bar{u}(\mathbf{p}',\alpha')=\bar{u}(0,\alpha')\frac{\Lambda_+(p')}{[\frac{1}{2}(1+E'/m)]^{\frac{1}{2}}}, \tag{A3}$$

where the projection operator Λ_+ is defined by the equation

$$\Lambda_+(p)=(m-\gamma p)/2m. \tag{A4}$$

The scattering amplitude may be expressed as a linear combination of two Lorentz-invariant scalar quantities. For this purpose we define the invariant quantities,

$$I_1=\bar{u}(\mathbf{p}',\alpha')u(\mathbf{p},\alpha),\qquad I_Q=\bar{u}(\mathbf{p}',\alpha')(\gamma Q/m)u(\mathbf{p},\alpha), \tag{A5}$$

where γ represents the four Dirac gamma matrices and the four-vector Q is defined in terms of the four-momenta of the initial and final pions by the equation

$$Q=\tfrac{1}{2}(k+k'). \tag{A6}$$

If use is made of Eqs. (A3), the quantities I_1 and I_Q may be written in terms of the spinors $u(0,\alpha)$ and $\bar{u}(0,\alpha')$ and the variables of the q system, i.e.,

$$I_1=\bar{u}(0,\alpha')(E_q/m)u(0,\alpha),\qquad I_Q=\bar{u}(0,\alpha')\left\{-\frac{\omega}{m}+\frac{i\boldsymbol{\sigma}\cdot(\mathbf{q}\times\mathbf{Q})}{m^2}\right\}u(0,\alpha). \tag{A7}$$

The corresponding quantities, expressed in terms of center-of-mass variables, are

$$I_1=\bar{u}(0,\alpha')\left\{1+\frac{q^2}{m(m+E_c)}+\frac{i\boldsymbol{\sigma}\cdot(\mathbf{k}_c\times\mathbf{k}_c')}{2m(m+E_c)}\right\}u(0,\alpha),$$

$$I_Q=\bar{u}(0,\alpha')\left\{-\frac{\omega_cE_c+k_c^2}{m^2}+\frac{q^2}{m^2}\left(1+\frac{\omega_c}{m+E_c}\right)+\frac{i\boldsymbol{\sigma}\cdot(\mathbf{k}_c\times\mathbf{k}_c')}{2m^2}\left(1+\frac{\omega_c}{m+E_c}\right)\right\}u(0,\alpha). \tag{A8}$$

Since only positive-energy states are involved, the scattering amplitude may be written as a linear combination of the quantities I_1 and I_Q,

$$\bar{u}(p',\alpha')Mu(p,\alpha)=\xi I_1+\eta I_Q, \tag{A9}$$

where ξ and η are spin-independent scalar quantities. The quantity $\bar{u}Mu$ is Lorentz-invariant; hence it refers to the scattering amplitudes in both the q system and the center-of-mass system. If use is made of Eqs. (A1) and (A7), the q-system amplitudes $\mathfrak{M}_N$ and $\mathfrak{M}_S$ may be expressed in terms of ξ and η, i.e.,

$$\mathfrak{M}_N=\frac{E_q}{m}\xi-\left(\frac{E_cW_c}{E_qm}-\frac{E_q}{m}\right)\eta,\quad \mathfrak{M}_S=\frac{1}{m^2}\eta. \tag{A10}$$

In a similar fashion, Eqs. (A2) and (A8) may be com-

[21] Anderson, Davidon, and Kruse, Phys. Rev. 100, 339 (1955).

bined to give corresponding equations for $\mathfrak{M}_{cN}$ and $\mathfrak{M}_{cS}$,

$$\mathfrak{M}_{cN}=\left[1+\frac{q^2}{m(m+E_c)}\right]\xi-\left[\frac{\omega_c E_c+k_c{}^2}{m^2}-\frac{q^2}{m^2}\left(1+\frac{\omega_c}{E_c+m}\right)\right]\eta, \tag{A11}$$

$$\mathfrak{M}_{cS}=\frac{1}{2m(m+E_c)}\xi+\frac{1}{2m^2}\left(1+\frac{\omega_c}{m+E_c}\right)\eta.$$

The relation between the center-of-mass amplitudes and the q-system amplitudes may be obtained by solving Eqs. (A11) for the constants ξ and η, and substituting these values in Eqs. (A10). The resulting equations are identical with Eqs. (5.4).

APPENDIX B. VALUES OF VARIOUS COEFFICIENTS

In this appendix the values of the functions defined in Eqs. (6.3)–(6.6) are given for low values of $|l-l'|$ and l. Since all quantities refer to the center-of-mass system, the subscript c of k_c and ω_c is omitted. The ratio $(k_c/k_c')^2$ is denoted by Λ. The integrals calculated here are denoted by such symbols as $A_{l,l'}{}^{\mathrm{I},0}$ and $A_{l,l'}{}^{\mathrm{I},1}$, where the second superscript denotes the order of the integral in the parameter $(1/m)$:

$$\begin{aligned}\alpha_{l,l'}{}^{\mathrm{I}}=\alpha_{l,l'}{}^{\mathrm{I},0}&=(-1)^{l'}\frac{(2l'+1)!}{2(l'!)^2}\times\int_{-1}^{1}dzP_l(z)\left(\frac{1-z}{2}\right)^{l'}\\&=1, && l'=l;\\&=-(2l+3), && l'=l+1;\\&=(2l+5)(l+2), && l'=l+2.\end{aligned}\tag{B1}$$

$$\begin{aligned}A_{l,l'}{}^{\mathrm{I},0}&=\frac{2l'+1}{2}\int_{-1}^{1}dzP_l(z)P_{l'}(z')\\&=\Lambda^l, && l'=l;\\&=(2l+3)\Lambda^l(1-\Lambda), && l'=l+1;\\&=(2l+5)\Lambda^l(1-\Lambda)\times[(l+1)-(l+2)\Lambda], && l'=l+2.\end{aligned}\tag{B2}$$

$$\begin{aligned}A_{l,l'}{}^{\mathrm{I},1}&=\frac{2l'+1}{2}\int_{-1}^{1}dzP_l(z)P_{l'}(z')\frac{q^2}{m}\\&=-\tfrac{1}{2}(k^2/m)\Lambda^{l-1}[l/2(2l+1)], && l'=l-1;\\&=\tfrac{1}{2}(k^2/m)\Lambda^{l-1}[(l+1)\Lambda-l], && l'=l;\\&=-\tfrac{1}{2}(k^2/m)\Lambda^{l-1}[(l+2)^2\Lambda^2-(l+1)(2l+3)\Lambda+(2l+3)l^2/(2l+1)], && l'=l+1.\end{aligned}\tag{B3}$$

$$\begin{aligned}\eta_l{}^{\mathrm{I},0}&=\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\left[1+\frac{2q^2}{\mu^2}\right]^2\\&=(\omega/\mu)^4+\tfrac{1}{3}(k/\mu)^4, && l=0;\\&=-\tfrac{2}{3}\omega^2k^2/\mu^4, && l=1;\\&=(2/15)(k^4/\mu^4), && l=2;\\&=0, && l>2.\end{aligned}\tag{B4}$$

$$\begin{aligned}\alpha_{l,l'}{}^{\mathrm{II},0}&=(-1)^{l'+1}\frac{(2l'-1)!(2l+1)}{[(l'-1)!]^2l(l+1)}\times\int_{-1}^{1}dz\frac{(1-z^2)}{2}\frac{dP_l}{dz}\left(\frac{1-z}{2}\right)^{l'-1}\\&=1, && l'=l;\\&=-(2l+1), && l'=l+1;\\&=(2l+1)(l+2), && l'=l+2.\end{aligned}\tag{B5}$$

$$\begin{aligned}\alpha_{l,l'}{}^{\mathrm{II},1}&=(-1)^{l'+1}\frac{(2l'-1)!(2l+1)}{[(l'-1)!]^2l(l+1)}\times\int_{-1}^{1}dz\left(\frac{1-z^2}{2}\right)\frac{dP_l}{dz}\left(\frac{1-z}{2}\right)^{l'-1}\frac{q^2}{m}\\&=-\tfrac{1}{2}(k^2/m)[(l-1)/(2l-1)], && l'=l-1;\\&=\tfrac{1}{2}(k^2/m)l, && l'=l;\\&=-\tfrac{1}{2}\left(\frac{k^2}{m}\right)\frac{(2l+1)(l+1)(l+2)}{2l+3}, && l'=l+1.\end{aligned}\tag{B6}$$

$$\begin{aligned}A_{l,l'}{}^{\mathrm{II},0}&=\frac{(2l+1)}{2l(l+1)}\int_{-1}^{1}dz(1+z)(1-z')\times\frac{dP_l(z)}{dz}\frac{dP_{l'}(z')}{dz'}\\&=\Lambda^l, && l'=l;\\&=(2l+1)\Lambda^l(1-\Lambda), && l'=l+1;\\&=(2l+1)\Lambda^l(1-\Lambda)\times[(l+1)-(l+2)\Lambda], && l'=l+2.\end{aligned}\tag{B7}$$

$$A_{l,l'}{}^{\mathrm{II},1}=\frac{(2l+1)}{2l(l+1)}\int_{-1}^{1}dz(1+z)(1-z')\times\frac{dP_l(z)}{dz}\frac{dP_{l'}(z')}{dz'}\frac{q^2}{m}$$

$$=-\tfrac{1}{2}\left(\frac{k^2}{m}\right)\Lambda^{l-1}\left(\frac{l-1}{2l-1}\right),\qquad l'=l-1;$$

$$=\tfrac{1}{2}(k^2/m)\Lambda^{l-1}(\Lambda l-l+1),\qquad l'=l;\qquad\text{(B8)}$$

$$=-\tfrac{1}{2}\left(\frac{k^2}{m}\right)\Lambda^{l-1}\left[\frac{\Lambda^2(2l+1)(l+1)(l+2)}{2l+3}-\Lambda l(2l+1)+l(l-1)\right],\qquad l'=l+1.$$

$$\eta_l{}^{\mathrm{II},0}=\frac{(2l+1)}{2l(l+1)}\int_{-1}^{1}dz\left(\frac{1-z^2}{2}\right)\frac{dP_l(z)}{dz}\qquad\text{(B9)}$$

$$=\tfrac{1}{2},\qquad l=1;$$

$$=0,\qquad l\neq 1.$$

$$\eta_l{}^{\mathrm{II},1}=\frac{2l+1}{2l(l+1)}\int_{-1}^{1}dz\left(\frac{1-z^2}{2}\right)\frac{dP_l(z)}{dz}\frac{q^2}{m}$$

$$=\tfrac{1}{4}k^2/m,\qquad l=1;\qquad\text{(B10)}$$

$$=-(1/12)k^2/m,\qquad l=2;$$

$$=0,\qquad l>2.$$

$$\eta_l{}^{\mathrm{III},0}=\tfrac{1}{2}\int_{-1}^{1}dzP_l(z)\left(1+\frac{2q^2}{\mu^2}\right)$$

$$=\omega^2/\mu^2,\qquad l=0;\qquad\text{(B11)}$$

$$=-\tfrac{1}{3}k^2/\mu^2,\qquad l=1;$$

$$=0,\qquad l>1.$$

$$\eta_l{}^{\mathrm{III},1}=\tfrac{1}{2}\int_{-1}dzP_l(z)\left(1+\frac{2q^2}{\mu^2}\right)\frac{q^2}{m}$$

$$=\tfrac{1}{2}(k^2/m)[1+(4k^2)/(3\mu^2)],\qquad l=0;$$

$$=-\tfrac{1}{6}(k^2/m)(1+2k^2/\mu^2),\qquad l=1;\qquad\text{(B12)}$$

$$=(1/15)(k^2/m)(k^2/\mu^2),\qquad l=2;$$

$$=0,\qquad l>2.$$

$$\eta_l{}^{\mathrm{IV},0}=\frac{(2l+1)}{2l(l+1)}\int_{-1}^{1}dz\left(\frac{1-z^2}{2}\right)\times\frac{dP_l(z)}{dz}\left(1+\frac{2q^2}{\mu^2}\right)$$

$$=\tfrac{1}{2}\omega^2/\mu^2,\qquad l=1;\qquad\text{(B13)}$$

$$=-\tfrac{1}{6}k^2/\mu^2,\qquad l=2;$$

$$=0,\qquad l>2.$$

The angular momentum indices l and l' in the integrals referring to the spin-flip equations may assume all positive integral values, but not zero. The angular integrals that appear in the expression for $\alpha_{l,l'}{}^{\mathrm{III}}$, $\alpha_{l,l'}{}^{\mathrm{IV}}$, $A_{l,l'}{}^{\mathrm{III}}$, and $A_{l,l'}{}^{\mathrm{IV}}$ are identical with those in the expressions for α^{I}, α^{II}, A^{I}, and A^{II}.