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Nuclear Moments of Inertia due to Nucleon Motion in a Rotating Well*

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A model of a deformed nucleus in which the spheroidal collective field is steadily cranked about a fixed axis, as introduced in previous papers, serves as a convenient approximation expected to reproduce some of the dynamic inertial properties of the collective motion. The independent-nucleon behavior in a rotating harmonic oscillator potential, deformed by the presence of the open-shell nucleons, gives the rigid-rotation moment of inertia and is discussed here with an attempt at graphic clarity. This result is much larger than observed and attention is here focused on the shortcomings of the harmonic oscillator approximation, although, as suggested by Bohr and Mottleson, the discrepancy may also be largely due to the internucleon interactions which have not been calculated adequately and are here neglected. The $1d-2s$ shell is treated as a tractable illustrative case. The characteristic harmonic-oscillator degeneracy of the undeformed levels within the magic-number groups, such as the $1d$ and $2s$ levels, profoundly affects the perturbation calculation of the moment of inertia through the energy denominators. Removing this degeneracy by lowering the states of high l (as required in heavier nuclei for the magic numbers) has the effect of increasing the calculated moment of inertia above the rigid-rotation value in most cases near the beginning of the shell and reducing it in most cases near the end of the shell. The preponderance of prolate deformations is also discussed.

THE effective moment of inertia of a distorted nucleus may conveniently be investigated by constraining the fictitious potential (first approximation to a self-consistent field) defining the wave functions to rotate with constant angular velocity Ω about a fixed axis in space.[1] This procedure gives a rotational angular momentum of the form[1]

$$\hbar\langle L_x\rangle_0=\Omega\hbar^2 \sum_i |(i|L_x|0)|^2/(E_0^{(0)}-E_i^{(0)}), \qquad (1)$$

arising from the admixture of excited states i in a perturbation theory by a relatively small Coriolis term $-\hbar L_x\Omega$ in the Hamiltonian. The essential approximation in this model is the neglect of "recoil" fluctuations in the angular velocity of the distorted effective potential contributed by the collective behavior of the many nucleons, leaving the angular momentum not strictly a constant of the motion. This may be accepted as an alternative to the approximation involved in assuming the "interaction" or "fluctuations" to be small in other treatments which in other manners formulate the separation of the internal and collective aspects of the motion.[2–4]

Early calculations[1,2] with this model have employed three-dimensional harmonic oscillator potentials and wave functions, elongated or flattened along an axis perpendicular to the axis of rotation. The method was first applied to artificially distorted closed-shell nuclei, and it was shown[1] that the closed-shell nucleons in this approximation contribute only the small moment of inertia characteristic of irrotational fluid flow.[3] In actual nuclei the distortion is due to the presence of additional open-shell nucleons and the observed moments of inertia are, in most cases, about five times that large so it appeared on this basis that the closed-shell nucleons contribute much less than their proportionate share of the moment of inertia.

Extending the treatment to include the contribution of open-shell nucleons, Bohr and Mottelson[2] (without pausing to present an explicit derivation, which is indeed quite simple, as we shall see) give the very interesting expression for the moment of inertia

$$\mathcal{J}_x=\frac{\hbar}{2\omega_2\omega_3}\left\{\frac{(\omega_2-\omega_3)^2}{\omega_2+\omega_3}\sum(m+n+1)+\frac{(\omega_2+\omega_3)^2}{\omega_2-\omega_3}\sum(n-m)\right\}. \qquad (2)$$

* Work performed under the auspices of the U. S. Atomic Energy Commission.
[1] D. R. Inglis, Phys. Rev. **96**, 1059 (1954). *Erratum:* For a correction in the method of deriving the expression for the rotational energy, see Appendix.
[2] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 1 (1955). See also S. A. Moszkowski, Phys. Rev. **103**, 1328 (1956); G. Lüders (to be published).
[3] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952); *Rotational States of Atomic Nuclei* (E. Munskgaards Forglag, Copenhagen, 1954).
[4] H. A. Tolhoek, Physica **21**, 1 (1955); F. Coester, Phys. Rev. **99**, 170 (1955); Bull. Am. Phys. Soc. Ser. II, **1**, 194 (1956); Lipkin, de-Shalit, and Talmi, Nuovo cimento **2**, 773 (1955); S. Tomonaga, Progr. Theoret. Phys. Japan **13**, 467 (1955); F. Villars (to be published). The related vibration problem is treated by J. M. Araujo, Nuclear Phys. **1**, 259 (1956).

Here l, m, and n are the quantum numbers in the three dimensions defining the single-nucleon states, ω_2 is the oscillator frequency in the x and y directions and ω_3 the oscillator frequency in the z direction. The last sum containing $\sum(n-m)$ is zero for distorted closed shells ($\omega_2 \neq \omega_3$) and contains the small energy denominator $\hbar(\omega_2-\omega_3)$—which vanishes for a spherical nucleus—corresponding to admixture of nearby states excited up in one dimension and down in the other, making a large contribution of the open-shell nucleons. The first sum corresponds to the contribution of more remote states excited up (or down) in both relevant directions, and gives the aforementioned irrotational-flow result for distorted closed shells.

If the distortion is attributed to a few open-shell nucleons by minimization of the total energy of the oscillators with volume-preserving distortion, the distortion increases with additional nucleons through the first half of the shell, and the contributions of the individual terms in the second sum decrease through increase of the energy denominators which depend on the distortion. In this independent-nucleon approximation with oscillator functions, one finds the somewhat surprising result that these large contributions from the last few nucleons (which are individually larger the fewer the nucleons) bring the angular momentum and hence the moment of inertia up to the very large value corresponding to rigid rotation of the whole nucleus, closed-shell nucleons and all.

In one sense, the rotational angular momentum is thus not a collective property at all. It is contributed almost entirely by the individual enterprise of the last few nucleons outside closed shells, which are, to be sure, moving in a collective environment.

By the extreme assumption of independent nucleons in an oscillator potential, one has thus overshot the experimental result, the rigid result being from two to five times larger than observed. It seems very plausible, especially as one looks at the details of the derivation, that any tendency to limit the freedom of independent motion of the nucleons would limit their freedom to make these large contributions. It is thus not disturbing that the simple result should be too large, but it is of interest to investigate the deviations from this simple behavior.

Bohr and Mottelson[2] have suggested as one, and perhaps the most important, limitation on the freedom of the nucleons, the interaction between the pairs of open-shell nucleons such as give rise to energy separations within the ground configuration in a spherical nucleus. In a brief discussion of the very special case of two p nucleons, which is simple and yet qualitatively representative, they introduce the parameter

$$v = U/\hbar\omega_0,$$

where U is the energy separation between $J=2$ and $J=0$ states induced by the pairwise interaction and ω_0 is the oscillator frequency of the undistorted nucleus, and find that the moments of inertia observed in much more complicated cases correspond to the result obtained for this simple case with $v \approx \frac{1}{3}$. It thus seems rather likely that interactions of a reasonable magnitude may come close to modifying the simple result enough to account for the observed results.

SINGLE-NUCLEON BEHAVIOR IN A ROTATING HARMONIC OSCILLATOR POTENTIAL

The treatment of reference 1 applies explicitly to a many-nucleon system, summations over excited states extending over only those states not excluded by the Pauli principle, in which individual nucleons are excited upward in energy. The Coriolis perturbation (3) is a sum of single-nucleon terms and the nucleon states thus contribute individually to the moment of inertia. It has been suggested by Villars that one may instead first treat in a similar manner the Coriolis influence on individual-nucleon properties, and then make up the wave function of the system from these modified single-nucleon wave functions. This simplifies the formulation considerably.

As in reference 1, we consider an ellipsoidal zeroth-order potential V symmetrical about the z' axis cranked so as to rotate slowly about the coincident x and x' axes with angular speed Ω. The Coriolis perturbation term in the wave equation is

$$\mathcal{H}^{(1)} = -(\hbar\mathbf{l})\cdot\mathbf{\Omega} = -\hbar l_x \Omega, \quad (3)$$

and for V we take the harmonic-oscillator simplification

$$V = \tfrac{1}{2}\hbar[\omega_2(\xi'^2+\eta'^2)+\omega_3\zeta'^2], \quad (4)$$

where $\xi' = (M\omega_2/\hbar)^{\frac{1}{2}}x'$, etc. The distortion from spherical shape is carried out without change of volume, $\omega_2^2\omega_3 = \omega_0^3 = \text{const.}$ [In reference 1, a single distortion parameter $\alpha = (\omega_2/\omega_3)^{1/6}$ was used.] The single-nucleon wave functions $u_{lmn}(\xi',\eta',\zeta')$ are then products of three factors $u_l(\xi') = H_l(\xi')\exp(-\xi'^2/2)$, etc., involving Hermite polynomials. By virtue of the familiar matrix elements of ξ and $\partial/\partial\xi$, Eqs. (10) and (11) of reference 1, the angular momentum operator

$$l_x = -i[(\omega_3/\omega_2)^{\frac{1}{2}}\eta'\partial/\partial\zeta' - (\omega_2/\omega_3)^{\frac{1}{2}}\zeta'\partial/\partial\eta'] \quad (5)$$

has the matrix elements

$$(l,\, m+1,\, n+1|l_x|lmn) = -\tfrac{1}{2}i(m+1)^{\frac{1}{2}}(n+1)^{\frac{1}{2}}(\omega_2-\omega_3)/(\omega_2\omega_3)^{\frac{1}{2}} \quad (6)$$

and

$$(l,\, m+1,\, n-1|l_x|lmn) = -\tfrac{1}{2}i(m+1)^{\frac{1}{2}}n^{\frac{1}{2}}(\omega_2+\omega_3)/(\omega_2\omega_3)^{\frac{1}{2}}. \quad (7)$$

The type (6) connects states differing in energy by $(\omega_2+\omega_3)\hbar$, while the type (7) connects states differing by an excitation upward in one dimension and downward in the other so that their energy difference $(\omega_2-\omega_3)\hbar$ is relatively small. With the perturbed wave function written $\psi_a = u_a + \sum c_{\beta a}u_\beta$, where a denotes l, m, n, and

$c_{\beta a}=-\hbar\Omega(\beta|l_x|a)/(E_a-E_\beta)$, we may evaluate l_x:

$$\langle l_x\rangle_a=\int\psi_a{}^*l_x\psi_a d\tau=2\hbar\Omega\sum(a|l_x|\beta)(\beta|l_x|a)/(E_\beta-E_a)$$
$$=(\Omega/2\omega_2\omega_3)\{[(m+1)(n+1)-mn]\times(\omega_2-\omega_3)^2/(\omega_2+\omega_3)+[(m+1)n-m(n+1)]\times(\omega_2+\omega_3)^2/(\omega_2-\omega_3)\}$$
$$=(\Omega/2\omega_2\omega_3)\{(m+n+1)(\omega_2-\omega_3)^2/(\omega_2+\omega_3)+(n-m)(\omega_2+\omega_3)^2/(\omega_2-\omega_3)\}. \quad (8)$$

We now define the moment of inertia by equating $\hbar\langle l_x\rangle_a=\mathscr{I}_x\Omega$, and have the result (2) when applied to a single nucleon.

For a system of many nucleons, we might write a wave function as an antisymmetric sum of products of the ψ_a with the sets a chosen to describe a number of closed shells and a few other nucleon states to make up the composite ground state, for example. The composite angular momentum operator L_x is a sum of the individual l_x, and its expectation value L_x is thus a sum of the $\langle l_x\rangle_a$, as given by (8), with sums over the quantum numbers, $\sum(m+n+1)$ and $\sum(n-m)$, as in (2). The first sum then contains the many small terms that add up to the irrotational result, as in reference 1, for distorted closed shells, and the second sum contains the large terms with small energy denominators.

A rigid rotation of the mass distribution described by these functions has the classical moment of inertia

$$\mathscr{I}_{\text{rigid}}=M\sum(y'^2+z'^2)_{\text{Av}}=\hbar[\sum(\eta'^2)_{\text{Av}}/\omega_2+\sum(\zeta')_{\text{Av}}/\omega_3]$$
$$=\hbar[\sum(m+\tfrac{1}{2})/\omega_2+\sum(n+\tfrac{1}{2})/\omega_3]. \quad (9)$$

Let us see in detail how (8) gives this result. The motivation for the distortion is to be found in an asymmetric distribution of the open-shell nucleons—more excitation of the quantum numbers in one direction than the others. Opposing the distortion is the energy it costs to distort the closed shells. We write for the energy

$$E=[2\omega_2\sum(m+\tfrac{1}{2})+\omega_3\sum(n+\tfrac{1}{2})], \quad (10)$$

with the assumption that the states are populated to preserve the symmetry about z', that is, $\sum l=\sum m$. In a distortion with $\omega_2^2\omega_3=$ constant to conserve volume, we have $\partial\omega_2/\partial\omega_3=-\omega_2/2\omega_3$, and with this condition minimization of the energy,[5] $dE/d\omega_3=0$, gives

$$\sum(m+\tfrac{1}{2})=(\omega_3/\omega_2)\sum(n+\tfrac{1}{2}), \quad (11)$$

whence

$$\sum(m+n+1)=\sum(n+\tfrac{1}{2})(\omega_2+\omega_3)/\omega_2,$$
$$\sum(n-m)=\sum(n+\tfrac{1}{2})(\omega_2-\omega_3)/\omega_2. \quad (12)$$

These sums in the last line of (8) lead to some interesting cancellation, which in the latter term means, as suggested above, that a very small excess population in the z direction $[\sum(n-m)]$ leaves the energy denominator so small that the individual terms are large enough to contribute a large total angular momentum:

$$\hbar\sum_a\langle l_x\rangle_a=(\hbar\Omega/2\omega_2^2\omega_3)\sum(n+\tfrac{1}{2})\times[(\omega_2+\omega_3)(\omega_2-\omega_3)^2/(\omega_2+\omega_3)+(\omega_2-\omega_3)(\omega_2+\omega_3)^2/(\omega_2-\omega_3)]$$
$$=(\hbar\Omega/\omega_2^2\omega_3)\sum(n+\tfrac{1}{2})(\omega_3^3+\omega_2^2)$$
$$=\Omega\hbar[\sum(m+\tfrac{1}{2})/\omega_2+\sum(n+\tfrac{1}{2})/\omega_3]=\Omega\mathscr{I}_{\text{rigid}}. \quad (13)$$

Thus, when the deformation is attributed to the expansiveness of the last nucleons outside closed shells, the moment of inertia has the large rigid value. It arises both from the remote states through (6), contributing to all the nucleons the small irrotational-flow result, and from the nearby states which through (7) give nothing for closed shells, but make exorbitantly large contributions for the few open-shell nucleons. The separation into small terms from (6) and large terms from (7) has been lost in the final expression (13), but the angular momentum still comes mainly from the large terms and from the open-shell nucleons.

CLASSICAL TREATMENT

A graphic description of this striking effect may be derived from the classical equations of motion,

$$\ddot{y}'+\omega_2^2y'=2\Omega\dot{z}',$$
$$\ddot{z}'+\omega_3^2z'=-2\Omega\dot{y}', \quad (14)$$

where the inhomogeneous terms on the right represent the familiar Coriolis acceleration $2\mathbf{v}\times\mathbf{\Omega}$. A model for a particle moving in such a rotating distorted harmonic-oscillator potential may be made by hanging a pendulum bob from a Y-shaped string suspension, as for a demonstration of Lissajou figures, but hung from a horizontal stick cranked steadily about a vertical axis through its center. Equations (14) are the same as for coupled oscillators except for a phase shift of 90° introduced by the time derivative in the coupling terms on the right, and the method of solution is just the same. Without

[5] The definition (10) for the energy in the oscillator is at best quite arbitrary, and justified only as a simple model. It even has the wrong sign, and one thinks of subtracting a constant to make it correspond more nearly to what one would get if one could satisfactorily calculate the energy with, say, phenomenological nuclear interactions [as suggested on pp. 704–705 of a recent paper: D. R. Inglis, Phys. Rev. **97**, 701 (1955)]. The potential V of Eq. (4) contains parameters which could be used to minimize an energy so calculated, rather than to minimize (10). Equation (10) taken literally implies that we consider it to give the average potential plus kinetic energy within an additive constant. Nilsson (reference 6) points out that this counts the interactions twice, if V_i is the average potential felt by one nucleon, i, due to all the others, $V_i=\sum_j(V_{ij})_{\text{Av}}$, since each term appears again in $\sum_i V_i$. Since oscillators have mean kinetic equal to mean potential energy, this suggests a factor $\frac{3}{4}$ on the right side of Eq. (10), as giving perhaps a better approximation to the dependence of the energy on the parameters, as it should be calculated. The argument is complicated by considerations of saturation. The factor $\frac{3}{4}$ would not affect the minimization leading to (11), which depends on competition between oscillator energies only. It would require a change of scale of the competing energy introduced in Eq. (19) in the modification below, but is ignored there because D is arbitrary.

rotation the solutions have the double periodicity of Lissajou figures, but with rotation we seek the singly-periodic solutions corresponding to normal modes, with

$$y' = A\cos\omega t, \quad z' = B\sin\omega t, \tag{15}$$

whence $(\omega_2^2-\omega^2)A-2\Omega\omega B=0$, etc. The secular determinant set equal to zero then gives for the normal frequencies

$$\begin{aligned}\omega^4-(\omega_2^2+\omega_3^2+4\Omega^2)\omega^2+\omega_2^2\omega_3^2=0,\\ \omega^2\approx(\omega_2^2+\omega_3^2+4\Omega^2)/2\pm[(\omega_2^2-\omega_3^2)/2\\ +2\Omega^2(\omega_2^2+\omega_3^2)/(\omega_2^2-\omega_3^2)],\end{aligned} \tag{16}$$

in the approximation in which the rate of rotation Ω is considered small. With the plus sign, we have a normal frequency somewhat larger than the larger frequency ω_2, say, as is familiar for coupled oscillators,

$$\omega^2\approx\omega_2^2[1+4\Omega^2/(\omega_2^2-\omega_3^2)],$$

in which the ratio of the amplitudes is

$$B/A\approx-2\Omega\omega_2/(\omega_2^2-\omega_3^2), \tag{17}$$

rather small, and (15) corresponds to a retrogressive motion near the y' axis as in Fig. 1(a). With the minus

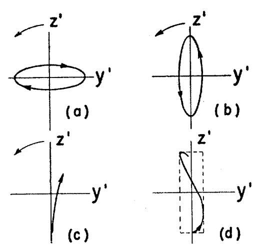

FIG. 1. Classical orbits for a particle in a rotating spheroidal harmonic oscillator field, (a) with higher frequency and retrogressive circulation; (b) with lower frequency and forward circulation; (c) Coriolis deflection with rotation but without restoring force; (d) Lissajou figure without rotation of the system.

sign in (16), we have instead

$$\begin{aligned}\omega^2\approx\omega_3^2[1-4\Omega^2/(\omega_2^2-\omega_3^2)],\\ B/A\approx(\omega_2^2-\omega_3^2)/2\Omega\omega_3,\end{aligned} \tag{18}$$

which represents an orbit circulating in a forward sense near the z' axis as in Fig. 1(b).

We may go one step further in *anschaulichkeit* and dispense with even these simple equations. Consider various ways we might start the particle at the bottom of Fig. 1(b). If we start it up along the z' axis, the Coriolis force alone makes it curve toward the right (corresponding to motion along a straight line in space), Fig. 1(c). If without the rotation we start it toward the right along the orbit drawn, it would not follow that orbit but, because of the relatively stronger restoring force in the y' direction, would cross the z' axis before reaching the top of the figure, describing the first loop of a Lissajou figure, Fig. 1(d). The greater the y' amplitude, the greater is the force involved in this rapid return to the z' axis. With rotation, it is possible to select the initial velocity $\dot{y}'$ and thus to select the y' amplitude in such a way that the Coriolis force arising from the z' motion just compensates the excess of restoring force in the y' direction, thus permitting the return to the z' axis to coincide with the upper extreme of the z' motion and bringing about an elliptical motion as drawn in Fig. 1(b). (This compensation is exact during the entire half-cycle and not just an average effect because of the simple harmonic way in which the y' displacement and the z' velocity vary, in phase with each other.) Since this approximates the lower-frequency original oscillation along the z' axis, this is the lower-frequency normal mode and when quantized is the first one to be filled beyond a closed shell. Its forward rotation (in the same sense as Ω) thus contributes additional angular momentum to the system, and it is clear that the orbit is fatter and the contribution greater, the smaller the disparity between the y' and z' restoring forces. It is similarly required that the higher-frequency motion approximating the y' oscillation be retrogressive, in order that the Coriolis force may make up for the deficit of restoring force in the z' direction, as in Fig. 1(a).

DEPARTURE FROM OSCILLATOR POTENTIAL, REMOVING THE *d-s* DEGENERACY

The insight given by the classical description makes it clear that the possibility of obtaining so striking a result depends sensitively on the simplifying assumption that the effective nuclear potential may be approximated by a three-dimensional harmonic-oscillator potential. An orbit bouncing from the walls of an elliptical box is not so simple to modify as a Lissajou figure, for example. In wave mechanics one may have the feeling that the wave functions are not very sensitive to the detailed shape of the potential, so the shape should not make much difference. We have seen, however, that the interplay of angular momentum operators and energy denominators is crucial to the derived result. The energy denominators are sensitively affected by the simple degeneracies introduced by the harmonic-oscillator assumption.

A complete departure from the harmonic-oscillator shape entails greatly complicated analysis, but a simple departure which in the spherical limit merely removes the l degeneracy is instructive. Following a suggestion made by Nilsson in another problem,[6] we introduce an additional term in the Hamiltonian:

$$\mathcal{H}^{(2)}=Dl^2 \tag{19}$$

for each nucleon. D will ordinarily be taken as negative to depress the levels with highest l, as required in the magic-number scheme for the heavier nuclei of principal interest. The additional term, although velocity-de-

[6] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 16 (1955). See also K. Gottfried, Phys. Rev. **103**, 1017 (1956).

pendent, has an effect similar to a lowering of the bottom of the potential function near the outside edge of the nucleus, leaving it in effect intermediate between harmonic oscillator and square well.

To see this, consider a depression of the potential in the form of a thin spheroidal shell of the form $V=-B\delta(\rho'-\rho_0)$, with $\rho'=[\xi'^2+\eta'^2+\eta'^2]^{\frac{1}{2}}$. This has matrix elements $(200|V|200)=[(4/5)\rho_0^4-(4/3)\rho_0^2+1]$ and $(200|V|020)=[(4/15)\rho_0^4-(4/3)\rho_0^2+1]$, each with a factor $2\pi^{-\frac{1}{2}}B\rho_0^3\exp(-\rho_0^2)$ omitted. These have the same ratio as the corresponding matrix elements of $\mathcal{H}^{(2)}$, namely, $4/(-2)$ as listed in Eq. (21) and used in the subsequent calculation, for $\rho_0=(3/2)^{\frac{1}{2}}$. On the same scale of length, $u_2(\xi)$ has its node at $\xi=(1/2)^{\frac{1}{2}}$. The ratio remains negative between the values $\rho_0=0.96$ and $\rho_0=2.02$, and a broad depression of the potential in this general region would be expected to give results qualitatively similar to those that follow.

In order to illustrate the type of influence that this effect can have on the moment of inertia, let us consider the simplest relevant case (even though it comes in a region of light nuclei where rotational states are not observed), the originally degenerate $1d$ and $2s$ functions, formed of the oscillator functions having $l+m+n=2$.

For these states the energy without rotation is given by

$$\begin{aligned}(i|\mathcal{H}^{(0)}+\mathcal{H}^{(2)}|j)&=\hbar[(l+m+1)\omega_2+(n+\tfrac{1}{2})\omega_3]\delta_{i,j}+D(i|l^2|j)\\&=\hbar(3\omega_2+\tfrac{1}{2}\omega_3)\delta_{i,j}+D(i|nd+l^2|j),\end{aligned}\tag{20}$$

with $d=\hbar(\omega_3-\omega_2)/D$. The term in $\hbar$ in the last line is the same for all six states, and the matrix of the second term is, to a sufficient approximation,[7] in the simple l, m, n representation:

$$l^2+nd=\begin{array}{c|ccc|cccccc} & l & m & n & u_f & u_e & u_d & u_c & u_b & u_a\\ \hline u_f & 0 & 2 & 0 & 4 & -2 & 0 & 0 & 0 & -2\\ u_e & 2 & 0 & 0 & -2 & 4 & 0 & 0 & 0 & -2\\ u_d & 1 & 1 & 0 & 0 & 0 & 6 & 0 & 0 & 0\\ u_c & 0 & 1 & 1 & 0 & 0 & 0 & 6+d & 0 & 0\\ u_b & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 6+d & 0\\ u_a & 0 & 0 & 2 & -2 & -2 & 0 & 0 & 0 & 4+2d\end{array}\,.\tag{21}$$

The states $a\cdots f$ are arranged in order of ascending energy, the high states at the top, for the prolate case in which $\omega_2>\omega_3$ and d is positive, D being negative.

The only nondiagonal elements connect u_a, u_e, and u_f. As a first step in diagonalizing, one may remove the degeneracy of u_e and u_f, and obtain for this part of the matrix:

$$l^2+nd=\begin{array}{r|ccc}\psi_f=(u_e+u_f)/\sqrt{2} & 2 & 0 & -2\sqrt{2}\\ \psi_e=(u_e-u_f)/\sqrt{2} & 0 & 6 & 0\\ \psi_a=u_a & -2\sqrt{2} & 0 & 4+2d\end{array}\,,\tag{22}$$

all other nondiagonal elements remaining zero. We then eliminate the remaining nondiagonal element by setting $\phi=c_a\psi_a+c_f\psi_f$. The resulting two-row determinant gives the diagonal elements

$$\lambda^{\pm}=3+d\pm(d^2+2d+9)^{\frac{1}{2}},\tag{23}$$

and the coefficients are

$$\begin{aligned}c_a^{\pm}&=[8/(\lambda^{\pm}-2)^2+1]^{-\frac{1}{2}},\\ c_f^{\pm}&=\mp[1+(\lambda^{\pm}-2)^2/8]^{-\frac{1}{2}}.\end{aligned}\tag{24}$$

We let the plus sign in $\lambda^{\pm}$ define $\phi_a=c_a^{+}\psi_a+c_f^{+}\psi_f$, which is the lower state (the factor D being negative) in a prolate deformation or the depressed $1d$ state having $\lambda^{+}=l(l+1)=6$ without deformation, and the minus sign define ϕ_f, which is the $2s$ state with $\lambda^{-}=0$ in the appropriate limit. With $\phi_e=\psi_e$ and the other three $\phi_i=u_i$, we then have a set of six states ϕ_i in which the energy is diagonal and for which the matrix of l_x, according to Eq. (6), is given by

$$l_x/i\rho=\begin{array}{c|cccccc}\phi_f & 0 & 0 & 0 & c^{-} & 0 & 0\\ \phi_e & 0 & 0 & 0 & -1 & 0 & 0\\ \phi_d & 0 & 0 & 0 & 0 & -1 & 0\\ \phi_c & -c^{-} & 1 & 0 & 0 & 0 & -c^{+}\\ \phi_b & 0 & 0 & 1 & 0 & 0 & 0\\ \phi_a & 0 & 0 & 0 & c^{+} & 0 & 0\end{array}\tag{25}$$

Here $\rho=(\omega_2+\omega_3)/2(\omega_2\omega_3)^{\frac{1}{2}}$ and $c^{\pm}=\sqrt{2}c_a^{\pm}-c_f^{\pm}$. The corresponding energies are

$$\begin{aligned}E_a=\lambda^{+}D,\quad E_b=E_c&=(6+d)D,\\ E_d=E_e=6D,&\quad E_f=\lambda^{-}D.\end{aligned}\tag{26}$$

In the evaluation of the angular momentum $\langle l_x\rangle_i$ in Eq. (8), we found small contributions from the states excited upward (or downward) in two dimensions, and large contributions from the states excited upward in one dimension and downward in the other, that were originally degenerate before the deformation. In order to isolate the principal modification introduced by $\mathcal{H}^{(2)}$, we may confine our attention to the large contributions by summing only within this group of six "originally degenerate" states:

$$\langle l_x\rangle_i=\int\phi_i^{*}l_x\phi_i d\tau=2\hbar\Omega\sum|(i|l_x|\beta)|^2/(E_\beta-E_i).\tag{8'}$$

In this approximation the angular momenta (divided by $\hbar$) for the six individual states are

$$\begin{aligned}\langle l_x\rangle_f&=A(\sqrt{2}c_a^{-}-c_f^{-})^2/[3+(d^2+2d+9)^{\frac{1}{2}}],\\ \langle l_x\rangle_e&=\langle l_x\rangle_d=A/d,\\ \langle l_x\rangle_c&=-[\langle l_x\rangle_a+\langle l_x\rangle_e+\langle l_x\rangle_f],\\ \langle l_x\rangle_b&=-\langle l_x\rangle_d=-A/d,\\ \langle l_x\rangle_a&=A(\sqrt{2}c_a^{+}-c_f^{+})^2/[3-(d^2+2d+9)^{\frac{1}{2}}].\end{aligned}\tag{27}$$

Here $A=2\Omega\rho^2/D$ and again $d=\hbar(\omega_3-\omega_2)/D$.

The solid lines in Fig. 2 display the angular momenta (27) for the six states a, b, c, d, e, f as functions of the ratio d of distortion energy to $1d-2s$ splitting, the left side corresponding to large distortion energy $\hbar(\omega_2-\omega_3)$, the right side to large D. It is apparent from the defini-

[7] The matrix for l_x^2 may be constructed from Eq. (7) and for l_y^2 and l_z^2 by permutation, then l^2 as the sum. The terms from l_x^2 and l_y^2, but not l_z^2, are multiplied by $(1+\epsilon+\cdots)$, with $\epsilon=(\Delta/2\omega)^2=[(\omega_3-\omega_2)/2\omega]^2$, from the factor $(\omega_2+\omega_3)^2/4\omega_2\omega_3$ in Eq. (7). The small terms in ϵ are neglected in this section.

tions of $\langle l_x\rangle_b$ and $\langle l_x\rangle_c$ in (27) that the total angular momentum of the closed shell consisting of the sum of these states is zero, just as for the case with $D=0$, the large term containing $\sum(n-m)$ in (2) vanishing.

The small contribution which arises from summing the terms in $(m+n+1)$ in the last line of Eq. (8) and is not included in our calculation, is indeed small enough to be neglected. These terms are not expected to be very sensitive to D, because of their already large energy denominators. The ratio of the sum of the small terms to the sum of the large terms, as given by (8) for $D=0$, is

$$[(\omega_2-\omega_3)/(\omega_2+\omega_3)]^2=[\sum(n-m)/\sum(m+n+1)]^2, \quad (28)$$

which for one nucleon in the shell (O^{17}) is $(2/27)^2=0.005$ and for the worst case of a half-filled shell (Si^{28} with four nucleons in each of states a, b, and c) is $(12/56)^2=0.05$.

The $D=0$ values on the left side of Fig. 2 thus give the rigid-rotation result derived from Eq. (8) within 5% or less (which we neglect), and the departure of the curves in Fig. 2 from their $D=0$ limit indicates the ratio by which they deviate from the rigid-rotation moment of inertia, insofar as we may neglect the influence of D on the distortion [a refinement discussed further below and formulated in Eq. (35)].

In the comparison of the prolate and oblate cases, the

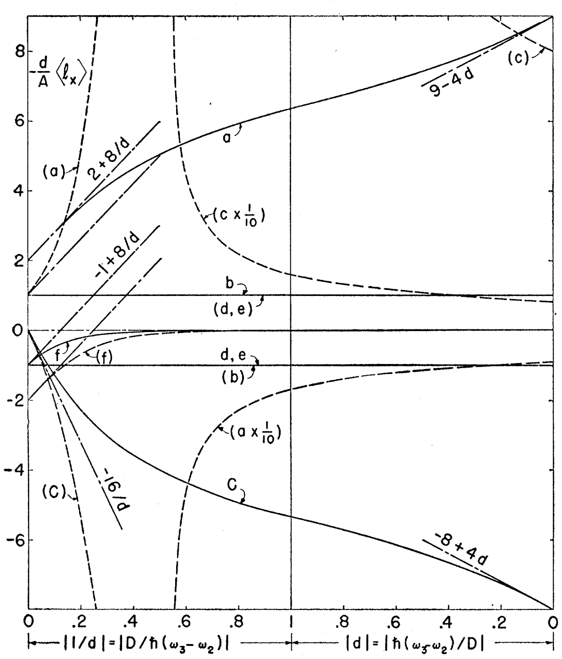

FIG. 2. Angular momentum of a nucleon in the $1d+2s$ shell, as dependent on the ratio of the deformation to the depression of the $1d$ below the $2s$ states, which are degenerate at the left side. To give the component of angular momentum along the axis of rotation, the ordinate is to be multiplied by the constant $(-\hbar A/d)=[\Omega\hbar/2(\omega_2\omega_3)][(\omega_2+\omega_3)^2/(\omega_2-\omega_3)]$ which is positive for the prolate case shown by full lines. The six states of the shell are labeled $a\cdots f$ in order of ascending energy for the prolate case. The broken lines, with labels in parentheses $(a)\cdots(f)$, refer to an oblate spheroidal deformation.

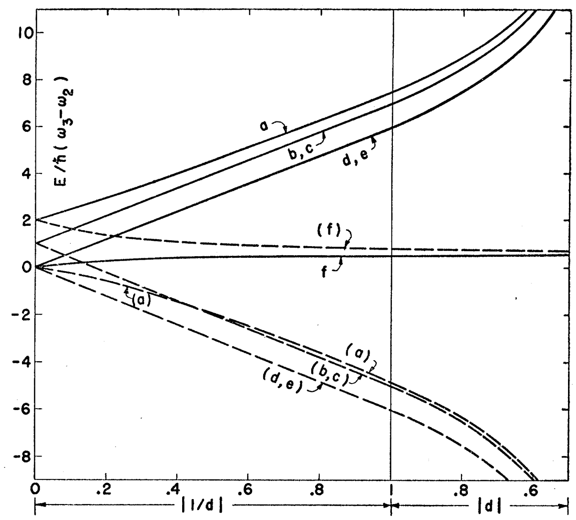

FIG. 3. The energies as functions of the same abscissa as appears in Fig. 2. The broken lines again refer to the oblate case. The energy scale is inverted for the prolate case $\omega_2>\omega_3$ and upright for the oblate case.

energies (26) play an important role and are shown in Fig. 3, with solid lines for the prolate case and with dotted lines and labels in parentheses for the oblate case. The energy differences vary only gently in the prolate case, but there is a disturbing crossover of curves (a) and (b,c) in the oblate case at $D=\frac{1}{2}\hbar(\omega_2-\omega_3)$. Accordingly, the curves for $\langle l_x\rangle_{(a)}$ and $\langle l_x\rangle_{(c)}$ blow up, as shown in Fig. 2, because of the vanishing of an energy denominator in Eq. (8′). In our present approximation this would correspond to enormous values of the moment of inertia over a rather wide range of parameters, if the state (a) or (c) should be filled with an oblate deformation. But at the same time our present approximation, being a perturbation treatment with the Coriolis perturbation assumed smaller than the energy denominators, becomes invalid. There enter large terms in the angular momentum not only proportional to the first power but also to higher powers of Ω, thus making the concept of an effective moment of inertia inapplicable, and the proportionality of the rotational levels to $l(l+1)$, as observed so strikingly in many cases, would not be expected. This proportionality is thus not obvious and one should be alert for similar difficulties in other approximations.

ORDER OF FILLING THE *d-s* SHELL

The simple deformed oscillator model with $D=0$ gives the result that prolate nuclei are stable in the first part of the shell and oblate for the more-than-half-filled shell. At first sight this seems contrary to what one might believe in view of the simple expectation that a single particle (with its orbit concentrated near a plane normal to $\mathbf{I}$) should give a negative quadrupole moment, either alone or in the deformation it induces, and similarly that a single particle lacking from a shell

TABLE I. Energy comparison for prolate and oblate partly-filled shell.

No. of neutrons in shell	g_1	Prolate shell l,	m,	n	$\mathcal{B}$	$\mathcal{A}/2$	$-k_1$	Oblate shell l,	m,	n	$\mathcal{B}_1$	$\mathcal{A}/2$	$-k_1$
12	78	0	2	0	26	26		0	0	2	26	26	
10	71	2	0	0	25	23	0.0008	1	0	1	21	25	0.0032
8	69	1	1	0	24	20	0.0039	0	1	1	18	23	0.0061
6	57	0	1	1	23	17	0.01108	0	2	0	15	21	0.01108
4	50	1	0	1	20	15	0.010	2	0	0	14	18	0.0064
2	43	0	0	2	17	13	0.0086	1	1	0	13	15	0.0022

should give a positive moment. However, because of the degeneracy, l is not a constant of the motion. For projection quantum number $k=0$, the d and s states mix giving a low state that is filled first and favors the prolate shape.

First we minimize the energy E to obtain the stable intrinsic deformations. In the simple oscillator model in which we leave the $1d$ and $2s$ states degenerate by putting $D=0$ in (20), we have, approximately,

$$E/\hbar=\mathcal{A}\omega_2+\mathcal{B}\omega_3, \tag{29}$$

with

$$\mathcal{A}=\sum(l+m+1) \quad \text{and} \quad \mathcal{B}=\sum(n+\tfrac{1}{2}).$$

By putting $\omega_2^2\omega_3=\omega^3$ and $\omega_3-\omega_2=\Delta$, we have

$$\begin{aligned} E/\hbar&=\tfrac{1}{3}g_1(2\omega_2+\omega_3)+g_2\Delta \\ &=g_1\omega[1+(\Delta/3\omega)^2]+g_2\Delta, \\ g_1&=\mathcal{A}+\mathcal{B}, \quad g_2=(2\mathcal{B}-\mathcal{A})/3. \end{aligned} \tag{30}$$

By minimizing with respect to Δ, we obtain

$$\begin{aligned} E&=\hbar\omega g_1(1+k_1), \\ k_1&=-(3g_2/2g_1)^2=-[(\mathcal{B}-\mathcal{A}/2)/(\mathcal{A}+\mathcal{B})]^2. \end{aligned} \tag{31}$$

The results of filling the shell with a pair of neutrons, say, in each state, with no protons, are shown in the columns headed $-k_1$ in Table I. It is seen that with one state filled the distortion contributes four times as much to the stability of the prolate shape as of the oblate shape, and that for just one state empty the opposite tendency is just as pronounced. For two states filled (or empty) the tendency is not nearly so strong, one exceeding the other by about 50%, and at the half-filled shell the prolate and oblate shapes are degenerate. These tendencies extend also to cases with both protons and neutrons present. This example is typical of harmonic-oscillator results in the larger shells of heavier nuclei.[6]

The filling of the $d-s$ shell may be considered to proceed as indicated by the examples listed in Table II. The comparison with the rigid-rotation result is indicated by the ratio $\mathcal{I}_{\text{calc}}/\mathcal{I}_{\text{rigid}}$ listed in the last two columns for the sample $2d-1s$ separations given by $D=\frac{1}{2}$ and $D=1$, corresponding to values taken from the left half and the middle of Fig. 2, as compared to values at the left edge. For $D=0$ the ratio is, of course, unity for all cases.

The reason for filling the degenerate states b and c equally in Table II, with one neutron in each rather than two in one of them, is that this corresponds to filling states characterized by a definite projection $K=\pm 1$, formed by taking the combinations $\phi'_{d,c}=(u_d\pm u_c)/\sqrt{2}$. Such states may be filled without violating the requirements for axial symmetry as represented by our assumption that $\sum l=\sum m$. With these states, the matrix (21) has more nondiagonal elements, but the result is the same.

The two cases marked (prolate), in parentheses, are cases in which the simple approximation of Eq. (31) and Table I indicates an oblate shape as more stable. The next approximation indicates that the prolate shape is stable in these cases also, as is shown in Eq. (34) and Table III. We thus avoid encountering the divergence difficulty characteristic of certain oblate cases such as the next-to-last line of Table II, involving states (a) and (c). The difficulty is perhaps similarly avoided in heavy nuclei.

DEFORMATION AS INFLUENCED BY LIFTING THE DEGENERACY

We have discussed the influence of lifting the degeneracy on the energy denominators and thus on the calculated angular momentum, for a given deformation. We now consider the effect on the deformation and thence on the angular momentum. For this purpose it is necessary to have the energies $E_a\cdots E_f$ expressed a little more accurately than implied by the matrix (21), with the terms in $\epsilon=(\Delta/2\omega)^2$ explained in reference 7 now included. The diagonal elements of (21) have added to them 4ϵ, 5ϵ, 5ϵ, 2ϵ, 2ϵ, and 2ϵ, respectively for $u_a\cdots u_f$ and the two nondiagonal elements near the corner, those involving u_a, are multiplied by $(1+\epsilon)$. Thus $5D\epsilon$ is added to E_b and E_c and $2D\epsilon$ to E_d and E_e of (26), and

$$\begin{aligned} \lambda^{\pm}&=3(1+\epsilon)+d\pm|[(d+1+\epsilon)^2+8(1+\epsilon)^2]^{\frac{1}{2}}| \\ &\approx 3(1+\epsilon)+d\pm|[d+1+\epsilon+4/d]|, \end{aligned} \tag{32}$$

the latter expansion being valid for $|d|\gg 1$, as we begin to remove the degeneracy. In this case, with the $1s$ and $2p$ shells filled and a number of nucleons in the $1d-2s$

TABLE II. Calculated moments of inertia compared to the rigid-rotation value for various numbers of nucleons in the $d-s$ shell.

No. in shell Neutrons	Protons	Shape	No. in states a	b	c	d	e	f	Ratio to rigid $D=\frac{1}{2}$	$D=1$
10	10	oblate	4	4	4	4	4		0.50	0.41
10	8	oblate	2	4	4	4	4		∞	−16.9
10	10	(prolate)	4	4	4	4	4		0.05	0.00
10	8	(prolate)	4	4	4	3	3		0.33	0.33
10	2	prolate	4	2	2	2	2		1.7	2.11
6		prolate	2	2	2				0.68	0.67
4	2	prolate	4	1	1				1.91	2.33
4		prolate	2	1	1				1.42	1.67
2	2	prolate	4						2.52	3.17
2		prolate	2						2.52	3.17
1		prolate	1						2.52	3.17

TABLE III. Coefficients k_i for the energy and h_i for the inverse deformation expanded for small values of the parameter $(D/\hbar\omega)$ representing the lifting of the degeneracy. The moment of inertia contains a factor $\omega/h_1\Delta_{\min}=1+h_2(D/\hbar\omega)+h_3(D/\hbar\omega)^2$. The values for the prolate (pro.) and oblate (ob.) cases are listed side by side. Concerning the order of filling states (a) and (f) in the oblate cases, see reference 8.

No. of neutrons in shell	Last state (half) filled with neutrons		g_1	$-k_1$		k_2		$-k_3$		h_1		h_2		h_3	
	pro.	ob.	both	pro.	ob.	pro.	ob.	pro.	ob.	pro.	ob.	pro.	ob.	pro.	ob.
10	d,e	b,c	71	0.0008	0.003	0.790	0.228	1.334	0.666	−11.833	5.916	1.141	1.014	840	105
8	(d,e)	(b,c)	64	0.004	0.006	0.692	0.302	0.672	0.533	−5.333	4.266	1.125	0.773	85.3	43.7
6	b,c	f	57	0.011	0.011	0.574	0.300	0.458	0.444	−3.166	3.166	1.105	0.474	20.1	20.1
4	(b,c)	d,e	50	0.010	0.006	0.408	0.482	0.540	0.000	−3.333	4.166	0.810	0.360	26.7	0
2	a	(d,e)	43	0.009	0.002	0.190	0.280	0.668	0.000	−3.583	7.166	0.419	0.209	38.5	0

shell, N_a in state ϕ_a, N_b in ϕ_b, etc., the energy obtained by summing equation (20) over the nucleons takes the form

$$E/\hbar=g_1\omega[1+(\Delta/3\omega)^2]+g_2\Delta+(D/\hbar)[g_3+g_4(\Delta/2\omega)^2]+g_5D^2/\hbar^2\Delta+\cdots, \quad (33)$$

$$g_1=36+(7/2)(N_a+N_b+N_c+N_d+N_e+N_f),$$
$$g_2=(4/3)N_a+\tfrac{1}{3}(N_b+N_c)-\tfrac{2}{3}(N_d+N_e+N_f),$$
$$g_3=4N_a+6(N_b+N_c)+6(N_d+N_e)+2N_f,$$
$$g_4=4N_a+5(N_b+N_c)+2(N_d+N_e+N_f),$$
$$g_5=4N_a-4N_f.$$

The g_i are here expressed for positive d, applying to the prolate shape with D negative. Minimizing with respect to Δ, we find, to order $(D/\hbar\omega)^2$,

$$E_{\min}=\hbar\omega g_1[1+k_1+k_2(D/\hbar\omega)+k_3(D/\hbar\omega)^2], \quad (34)$$
$$k_1=-(3g_2/2g_1)^2, \quad g_1k_2=g_3+g_4(9g_2/4g_1)^2,$$
$$g_1k_3=-[g_5(2g_1/9g_2)+(g_2g_4)^2(9/4g_1)^3],$$
$$\omega/\Delta_{\min}=h_1[1+h_2(D/\hbar\omega)+h_3(D/\hbar\omega)^2], \quad (35)$$
$$h_1=-2g_1/9g_2, \quad h_2=g_4/2, \quad h_3=(g_5/g_2)h_1^2.$$

Besides confining our attention to moderate deformations, $\Delta\ll\omega$, but still definitely greater than zero, we have specialized for $D\ll\hbar\Delta$ in expanding (32), thus looking at only the initial effect of beginning to lift the degeneracy, in order to avoid the necessity of a separate numerical minimization for each nucleus considered. We thus seek an energy minimum near the one known from the $D=0$ case, but there is unfortunately no guarantee that another minimum instead could not be the one analogous to the situation in real heavy nuclei. (This question might bear further investigation.)

In order both to evaluate this influence of lifting the degeneracy on the calculated moment of inertia, and to compare the stability of the prolate and oblate deformations, the coefficients h_i and k_i of Eqs. (34) and (35) are listed in Table III, for a sequence of ways of populating the $d-s$ shell with neutrons.[8]

The moments of inertia as formulated in Eqs. (25) and displayed in Fig. 2 are calculated with inclusion of the effect of D (the lifting of the degeneracy) entering through the secular problem (which is the main effect), but with the influence of D on the deformation being neglected. That is to say, only the first term h_1 of Eq. (35) was taken into account. Since $\Delta_{\min}$ appears in that calculation of $\mathcal{I}$ in the denominator (as an energy denominator), the terms in $h_2D/\hbar\omega$ and $h_3(D/\hbar\omega)^2$ modify the calculated moment of inertia as fractional corrections. The immediate effect of introducing a small negative D is to reduce $\mathcal{I}_{\text{calc}}$ slightly, through the positive h_2, but at least in the prolate case it takes only a rather small value of $D/\hbar\omega$, of the order of 1/20 or even considerably less for the nearly-filled shell, to enable the h_3 term to compensate this and increase $\mathcal{I}_{\text{calc}}$ above the values such as given in Table II. The behavior of this simple expansion is suggestive of a rather involved result from a more extensive calculation, but it is perhaps likely that it would increase $\mathcal{I}_{\text{calc}}$ in the region just beyond the validity of this expansion, and it shows little promise of decreasing $\mathcal{I}_{\text{calc}}$ by the substantial factor needed in most cases to agree with experiment.

GREATER STABILITY OF PROLATE DEFORMATIONS

The values of $(-k_1)$ listed in Table III contribute to stability and as in Table I show that with $D=0$ the prolate deformation is favored in the first half of the shell and the oblate with the shell more than half-filled. If D is negative, as we assume it must be for heavier nuclei, the larger values of k_2 listed favor stability, and one notes that they slightly favor the oblate shape near the beginning of the shell and strongly favor the prolate shape with the shell half-filled or more. On the other hand, the values of $(-k_3)$ as listed favor the prolate shape throughout the shell, rather strongly toward the beginning and end if D be large enough to make this term matter. For the cases of two and four neutrons near the beginning of the shell, the contribution k_2 begins to challenge the favoring of the prolate shape by k_1 when $D/\hbar\omega$ becomes as large as about 1/20 or $\frac{1}{10}$,

[8] In the oblate case, d is negative (for negative D), and in this expansion for large $|d|$ the absolute value sign in the second line of (32) becomes relevant to the distinction between states (a) and (f). In evaluating the g_i in (34), we have made algebraic combinations ignoring the absolute value sign, which means that we have interchanged the labels (a) and (f). Rather than to rewrite (34), the simplest way to compensate is to invert the order of filling states (a) and (f) in this special calculation, and this has been done in Table III.

but with these values the k_3 contribution is already large enough to tip the balance in favor of the prolate shape. With the shell half-filled or more, values of $D/\hbar\omega$ of only 1/100 suffice to overwhelm the small values of k_1 and make the prolate shape the more stable.

In this connection it should be recalled that D is assumed to be negative for the sake of investigating the $1d-2s$ shell as a simplified analog of heavier nuclei, in which the lifting of degeneracies is expected to involve more complicated manifestations of these same effects. If experimental progress should ever suggest application of these results to the fairly light nuclei to which the calculation explicitly applies, a positive value of D would presumably be more appropriate, in keeping with the appearance of the $2s$ level only 0.87 Mev above the ${}^2d_{\frac{5}{2}}$ ground state of O^{17}, well below the "center of gravity" of the two levels of the ${}^2d$. With small positive values of $D/\hbar\omega$, the oblate shape is favored through at least the last half of the shell in this approximation. Such a variation in the sign of D would correspond to having a central depression of the potential (contributed perhaps by the $1s$ shell which is still an appreciable part of the nucleus) in the region just above O^{16}, and having heavier nuclei better approximated by a well with a flatter bottom than that of the oscillator potential, in keeping with the approximate constancy of nuclear density.

Thus, with D negative, with $1d$ below $2s$, we obtain the result that the prolate shape is stable in this approximation. The extension of this stability to the more-than-half-filled shell comes from the nonlinear terms, arising from expansion of the square root in Eq. (32), in the energy E_a of the state a which is a mixture of the $2s$ state and the $1d$ state with l_z quantum number $k=0$ (in the representation appropriate for large D). The prolate energy is low because just these two $k=0$ states can mix, and the projection $k=0$ favors the prolate shape. The reason for the prolate deformation is then the same as already pointed out by Moszkowski,[9] explicitly in a discussion of the next higher shell consisting of $1f$ and $2p$. In that case, and with spin-orbit coupling taken into account, he shows that it is the depression of the $1f_{7/2}(k=\frac{1}{2})$ by admixture mainly of the $2p_{\frac{3}{2}}(k=\frac{1}{2})$ that accounts for the stability of the prolate shape near the middle of the shell.

It is gratifying to have this availability of more states of lower projection quantum number k to be mixed with each other as a qualitative explanation of the preponderance of prolate deformations. It is not clear that this is the main reason. According to Mottelson and Nilsson and others, the large stable deformations which start just above neutron number $N=88$ involve crossover between the magic number groups. It is possible, too, that pairwise interactions of particles, here neglected except in the average potential, make an essential contribution to the prolate stability in a calculation carried far enough to include correlations of nucleon positions and spin directions. That the effect should be in that direction is made plausible by the consideration that the bulge on the surface of a sphere associated with spheroidal deformation represents a clustering of particles (opposed to the dissociation represented by a depression in the surface), and that with a prolate deformation the bulge has twice the amplitude and about half the surface area as has an equal oblate deformation, thus representing a more compact clustering. Insofar as the few particles in the cluster can correlate their spins in such a way as to take advantage of the saturation properties of nuclear interactions, this is a favorable situation for stability. This is related to, but not quite the same as, the consideration that the cubic terms in the expansion of the surface-tension energy favor the prolate deformation.[10] There is also a Coulomb influence in the same direction.[10] These are presumably all initial manifestations of the fact that fission takes place by way of a prolate distortion, not by way of a ring.

ACKNOWLEDGMENTS

A closely related topic, a perturbation treatment of nucleon behavior in a rotating spheroidal finite square-well potential, has been discussed at some length with Mr. Marvin Rich of the University of California during the first semester of 1955–1956 in the Physics Department at Berkeley. The procedures are sufficiently different from those followed here that no attempt has been made to combine that treatment with this presentation. Discussions with Professor V. Weisskopf, Professor F. Villars, Professor S. A. Moszkowski, Professor Maria G. Mayer, Professor F. Coester, and Dr. B. Mottelson have been very helpful.

APPENDIX

An expression for the moment of inertia in a rotating spheroidal oscillator potential was derived in reference 1 from the rotational energy, but may be derived more simply in that treatment without appeal to the rotational energy by setting the angular momentum,[11] $\hbar$ times Eq. (14) [equation references in this Appendix are to reference 1], equal to $\mathcal{J}_x\Omega$, which yields for closed shells

$$\mathcal{J}_x=(\hbar/4\omega)(\alpha^3-\alpha^{-3})^2\sum{}'(m+1)(n+1), \qquad (36)$$

just as was obtained from the rotational energy, Eq. (15).

There is, unfortunately, a difficulty in the treatment of rotational energy presented in reference 1, as was very kindly pointed out by Maria Goeppert Mayer. It involves two errors which exactly compensate (and thus

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[9] S. A. Moszkowski, Phys. Rev. 99, 803 (1955).

[10] S. A. Moszkowski and C. H. Townes, Phys. Rev. 93, 306 (1954). Note that in the droplet model one expects an oblate deformation at the very beginning of a shell.

[11] That this is angular momentum in the space-fixed system is seen from Eq. (2), which shows that the linear momentum operator refers to this system.

seem to have hidden) one another and leave the equations used, such as Eq. (15), correct. The equation for $E^{(2)}$, which we might call Eq. (8a) [appearing between Eqs. (8) and (9)], should have the opposite sign to give the familiar result that interacting states repel one another in energy. This compensates an error in the interpretation of the meaning of E, of which $E^{(2)}$ is a part. E is the energy of the motion relative to the rotating system, not relative to the space-fixed system as it was taken to be, and to emphasize this it should have been called E'. We have derived a Hamiltonian in terms of the rotating coordinates x_i' which differs from the usual one in the x_i only by the perturbation term $\mathcal{H}^{(1)}$, and we have made the transition to the Schrödinger equation just as we would in the x_i system with this additional term. When we do it in the x_i' system,[12] the operator $\partial/\partial t$ appearing in the Schrödinger equation means differentiation with respect to the time with the x_i' held fixed (or considered as independent coordinates), and when we seek a stationary solution $\psi(x_i')\exp[(i/\hbar)E't]$, the constant E' thus comes to mean energy in the x_i' system.

The Coriolis force on a particle is normal to its velocity (in the rotating system) and thus does no work (in this system) and the first-order energy $E^{(1)\prime}$ vanishes. The fact that $E^{(2)\prime}$ for the ground state is negative indicates that the Coriolis force makes possible a mode of motion with a lowered energy (for example, by pressing toward the center where the potential is lower when the circulation is in a favorable sense).

By altering the sign of Eq. (13), to correct for the error carried through from Eq. (8a), and comparing with Eq. (14), we see that the magnitude of this energy depression may be written

$$E^{(2)\prime}=-\hbar\langle L_x\rangle\Omega/2. \qquad (37)$$

The energy in the space-fixed system, as in the problem of a gyroscope on a merry-go-round, differs from the energy relative to the rotating axes by the term $\hbar\langle L_x\rangle\Omega$ which, when added to this $E^{(2)\prime}$, amounts to the same thing as changing its sign:

$$E^{(2)}=E^{(2)\prime}+\hbar\langle L_x\rangle\Omega=\hbar\langle L_x\rangle\Omega/2. \qquad (38)$$

This represents the rotational energy and agrees in sign and magnitude with Eqs. (13) and (15).

[12] Or when we, alternatively, transform the Schrödinger equation from the space-fixed to the rotating system, which means transforming the time-derivative operator $(\partial/\partial t)_x=(\partial/\partial t)_{x'}+\Sigma(\partial x'/\partial t)_x\partial/\partial x'=(\partial/\partial t)_{x'}+\Omega(z'\partial/\partial y'-y'\partial/\partial z')$.