Moments of Inertia of Freely Rotating Systems

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A formula is derived for the moment of inertia of a freely rotating system. Although this formula is similar in form to the one derived by Inglis for a system of particles in an externally rotated potential, there are certain significant differences. The meaning of these differences is discussed and an attempt is made to clarify the question of the validity of the Inglis formula in the case of real nuclei.

THE existence of bands of rotational states in nuclei has been verified recently in an appreciable number of cases. In each rotational band several states are found having energies given by

$$E=E_0+\frac{\hbar^2}{2\mathcal{J}}J(J+1),$$

where E_0 is a constant for the particular band, J is the angular momentum of the state, and $\mathcal{J}$ is a constant, characteristic of each band, known as the "effective moment of inertia." All the states in a given band have apparently a common intrinsic structure; they differ from one another only in the amount of collective rotation present. The experimental evidence indicating this type of structure includes a number of independent observations in addition to the energy values: static quadrupole moments, $E2$ transition probabilities, and beta-decay branching ratios.[1,2] Some of these observations, such as branching ratios in beta decays leading to different members of the same rotational multiplet, require for their interpretation only the assumption of the separability of the internal motion from a collective rotation. Other observations require the evaluation of some intrinsic parameters from a more detailed model. In particular the separation in energy between states of the same rotational band depends upon the effective moment of inertia, whose evaluation depends upon further details of the nuclear model.

One of the more successful attempts to account for the experimentally observed moments of inertia is due to Inglis,[2,3] and is now known as the "cranking model." The nucleus is represented by a system of nucleons moving in a deformed potential which is rotating with a given angular velocity. The energy of the system is calculated by perturbation theory to give a power series in the angular velocity; the coefficient of the quadratic term is interpreted as $\frac{1}{2}\mathcal{J}$. The following expression is obtained for the effective moment of inertia of a two-dimensional system in the state $|0\rangle$:

$$\mathcal{J}_0=2\sum_{k\neq 0}\frac{|\langle 0|L|k\rangle|^2}{E_k-E_0}, \qquad (1)$$

where L is total angular momentum operator, $|0\rangle$ and $|k\rangle$ are states of the system of particles in the deformed potential, unperturbed by rotation, and E_k and E_0 are the corresponding unperturbed energies.

The validity of this approach may be questioned, since the rotation of the nucleus is an externally forced rotation imposed by "cranking" the external potential, while in actual nuclei collective rotations are free rotations resulting from the mutual interaction of the particles themselves. Also, the limitations under which the use of "deformed shell-model" wave functions is justified are not clear. This point is of particular significance since the Inglis formula (1) as it stands requires the use of such functions and is not valid if the exact wave functions for the system are introduced. The latter must be eigenfunctions of L, since the total angular momentum of an isolated system is always a good quantum number; hence all nondiagonal matrix elements of L vanish, and the formula (1) gives a meaningless result.

The use of the external rotation also introduces implicitly an extra degree of freedom into the system which is not present in the real nucleus. This problem has been discussed[4] and has certain implications which will be considered in a subsequent paper.

In this note, a formula is derived for the moment of inertia of a freely rotating system, without the use of externally imposed rotation or additional degrees of freedom. This formula is similar in form to (1) and is valid when the exact wave functions and energy values are introduced. The treatment is applicable to freely rotating systems and also to systems oscillating freely in a potential well binding the system to a fixed orientation. In the latter case the new formula reduces to the Inglis formula (1) in first approximation. Certain similarities between these oscillating wave functions and deformed shell-model wave functions are discussed, but no rigorous justification for the use of the latter is given.

Consider a dynamical system for which bands of rotational states are known to exist. For simplicity, a two-dimensional system is considered; the generalization to three dimensions does not involve any fundamental difficulties. Let the system be specified by an angle ϕ, describing the orientation of a set of moving

[1] A. Bohr and B. R. Mottelson, in *Beta- and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (North Holland Publishing Company, Amsterdam, 1955), Chap. 17.
[2] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, 1 (1955).
[3] D. R. Inglis, Phys. Rev. **96**, 1059 (1954).
[4] Lipkin, de-Shalit, and Talmi, Nuovo cimento **2**, 773 (1955).

coordinate axes fixed in the system, and by a set of independent "intrinsic" coordinates and momenta q_i and p_i, describing the internal degrees of freedom of the system with respect to the moving axes. All the q_i and p_i commute with ϕ and with its canonically conjugate momentum L, the total angular momentum of the system. Assume that the Hamiltonian of the system can be written in the form proposed by Bohr and Mottelson,[5] consisting of a rotational energy, an intrinsic energy, and a coupling term:

$$H=\frac{L^2}{2\mathscr{I}}+H_{\text{int}}(q_i,p_i)+H_{\text{coupl}}(L,q_i,p_i). \tag{2}$$

The existence of rotational bands in the system implies a separation to a good approximation of the intrinsic motion from the collective rotation.[2,4] One can therefore assume that by a proper choice of the system of moving axes, the Hamiltonian (2) can be made approximately separable; that is, (1) H_{coupl} can be neglected; (2) the off-diagonal elements of $1/\mathscr{I}$ can be neglected in the representation in which H is diagonal. The effective moment of inertia $\mathscr{I}$ is therefore an operator whose expectation value depends upon the state of the system, but which does not mix different states. Let us assume further that the operator $\mathscr{I}$ depends only upon the intrinsic variables p_i and q_i.[6] These assumptions necessary for the following treatment can be summarized in mathematical form as follows:

$$\begin{aligned} &\text{(a)} \quad H_{\text{coupl}}=0,\\ &\text{(b)} \quad [H,\mathscr{I}]=0, \\ &\text{(c)} \quad [L,\mathscr{I}]=[\phi,\mathscr{I}]=0. \end{aligned} \tag{3}$$

The experimental value of the moment of inertia associated with a given rotational band is given by the expectation value of the operator $\mathscr{I}$ in any state of the rotational band. An expression for $\langle\mathscr{I}\rangle$ can be obtained directly from the relations (3) and the identities

$$[L,f(\phi)]=-i\hbar f'(\phi), \tag{4a}$$

$$\langle 0|A|k\rangle=\frac{\langle 0|[H,A]|k\rangle}{E_0-E_k}, \tag{4b}$$

where $f(\phi)$ is any periodic function of ϕ with period 2π (i.e., single valued), A is any operator, $|0\rangle$ and $|k\rangle$ two distinct eigenstates of the Hamiltonian (2), and E_0 and E_k the corresponding eigenvalues of H, assumed to be discrete and distinct.

Using Eq. (4b) and the rules for matrix multiplication, one obtains

$$\langle 0|[[H,A],A]|0\rangle=2\sum_{k\neq 0}\frac{|\langle 0|[H,A]|k\rangle|^2}{E_k-E_0}. \tag{5}$$

To get an expression for $\langle 0|\mathscr{I}|0\rangle$, choose[7] $A=\mathscr{I}\sin\phi$. Then, using relations (3) and (4)

$$[H,A]=-i\hbar[L\cos\phi+(\cos\phi)L]/2, \tag{6}$$

and

$$\langle 0|\mathscr{I}|0\rangle=\tfrac{1}{2}\sum_{k\neq 0}\frac{|\langle 0|(\cos\phi)L+L\cos\phi|k\rangle|^2}{(E_k-E_0)\langle 0|\cos^2\phi|0\rangle}. \tag{7}$$ [8]

The expression (7) is by no means unique. Different expressions for the moment of inertia can be obtained by using any other periodic function of ϕ instead of $\sin\phi$ in the expression for the operator A. The choice of $\sin\phi$ is quite arbitrary and is mainly for simplicity. It should be noted that the expression (7) differs formally from the formula (1) only by the presence of the factors $\cos\phi$. It is just these factors which make the relation (7) valid for eigenfunctions of a freely rotating system which are eigenfunctions of L. This can easily be verified by inserting the appropriate wave functions and energy differences in Eq. (7). The result, however, is trivial since the moment of inertia can be computed directly from the energy difference E_k-E_0.

The above derivation can be extended to apply to certain systems which are not in free rotation. For such systems the moment of inertia is not directly obtainable from the energy eigenvalues and the formula (7) is no longer trivial.

Let a term depending only upon ϕ be added to the Hamiltonian (2) to obtain

$$H'=\frac{L^2}{2\mathscr{I}}+H_{\text{int}}(q_i,p_i)+H_{\text{coupl}}(L,q_i,p_i)+V(\phi). \tag{8}$$

If the conditions (3) are still satisfied, then Eq. (6) is valid for the Hamiltonian (8) which differs from (2) only by a term commuting with $\mathscr{I}$ and ϕ. Thus, the above derivation and Eq. (7) are valid for the system described by the new Hamiltonian (8). The states of the system are no longer those describing free rotation but are modified by the potential $V(\phi)$. If this potential has the form of a well centered about $\phi=0$, the possibility exists of bound states in which the system oscillates about the equilibrium position $\phi=0$. If this binding is sufficiently strong, the wave function describing the system is appreciable only in a small region about $\phi=0$, where $\cos\phi$ can be taken as equal to unity. For

[5] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, 16 (1953).

[6] It must be emphasized that these assumptions are never exactly satisfied in any real dynamical system, except that of the rigid rotator, because of the action of centrifugal and Coriolis forces which couple the intrinsic and rotational motions and which change the moment of inertia as a function of the speed of rotation. The treatment given here merely assumes that these effects are negligible to a first approximation.

[7] Special care should be taken in choosing A. Since explicit use is made of matrix multiplication, it must be guaranteed that the state $A|k\rangle$ remains in the space of the set of eigenfunctions $|k\rangle$. As our functions are assumed to be single valued, they should be periodic in ϕ with period 2π. Thus only operators which are periodic in ϕ with period 2π can be used for A. The simpler choice $A=\phi$, which leads to the Inglis formula (1) exactly, is thus erroneous and leads to inconsistent results.

[8] This derivation of the expression for the moment of inertia resulted from a remark of A. Bohr (private communication) regarding a possible derivation of the Inglis formula based upon the relation $[H,\phi]=(i/\hbar)(L/\mathscr{I})$. The difficulties encountered in the interpretation of the relations obtained in this way are avoided in the present derivation, as discussed in the note above. The authors would like to take this opportunity to thank Dr. Bohr for helpful discussions on this subject.

such a strongly bound system, the formula (7) reduces to the same form as the formula (1).

It should be noted that the moment of inertia calculated for the bound system (8) is exactly equal to that of the freely rotating system (2) which differs from it only by the absence of the potential $V(\phi)$. It is therefore possible to calculate the moment of inertia of freely rotating systems by use of the Inglis formula (1) with eigenfunctions of a strongly bound oscillating system having the same intrinsic structure as the original system.

It must be emphasized that there is another important difference between the new formula (7) and the Inglis formula (1), in addition to the purely formal difference involving the factors $\cos\phi$. The expression (7) refers directly to a freely rotating system, or to one having that certain type of bound rotation described by the Hamiltonian (8). The eigenfunctions and energy values of this system appear in the formula. The eigenfunctions and energies used in the Inglis formula (1) are those for a system of particles moving in a deformed potential well which is fixed in space. Two questions are suggested by this difference: (1) Can the derivation of Eq. (7) be extended to apply to systems described by shell-model wave functions; (2) What is the relation between the moment of inertia of the shell-model system and that of the real freely-rotating nucleus?

Although the question of the validity of the use of shell-model wave functions is not rigorously answered in this paper, it can be somewhat clarified by examining the simpler analogous case of collective translation of the entire nucleus (motion of the center of mass of the system).

The close analogy between the problems of collective translation and collective rotation has been discussed.[4] The treatment of rotation given above can be applied to collective translation simply by replacing the operators L and ϕ, respectively by the total linear momentum and the coordinate of the center of mass. The inertial parameter $\mathcal{I}$ given by the formula analogous to Eq. (7) is just the total mass of the system. This formula should be valid for systems moving in free translation and also for those oscillating in a potential binding the system to a fixed position in space. Since center-of-mass motion is well known, the analysis of this problem in the case of nuclei should shed some light on the validity of the use of shell-model wave functions in (7).

The problem of center-of-mass motion in the system described by a shell model with a harmonic oscillator potential has been treated by a number of investigators.[9] It is shown that the states of this system describe a nucleus whose center of mass is bound to the origin of the coordinate system by harmonic oscillator potential, and which therefore undergoes harmonic oscillations. These collective oscillations are not coupled to the internal degrees of freedom of the system ($H_{\mathrm{coupl}}=0$). It is therefore possible to use this model to study the intrinsic motion of a free unbound system which has the same intrinsic motion as the shell model; it is merely necessary to beware of the spurious effects of the center-of-mass oscillations which have no counterpart in the free system.

The treatment of these collective translations is directly analogous to that of the rotational oscillations expressed by the Hamiltonian (8). The translational formula analogous to the Inglis formula (1) therefore gives exactly the total mass of the system when the shell-model wave functions and energies are introduced. The total mass of the shell-model system is of course exactly equal to that of the unbound system.

However, difficulties are encountered if any potential other than the harmonic oscillator is used for the shell model. Exact separation between intrinsic and center-of-mass motions occurs only in the case of the harmonic oscillator potential[10]; in all other cases, H_{coupl} does not vanish and states of different intrinsic properties are mixed by center-of-mass motion. If the shell-model potential depends upon the particle momenta as well as upon the coordinates (e.g., a term of spin-orbit coupling), the equation analogous to Eq. (6) is no longer valid, and the Inglis formula no longer gives the total mass of the system.

Much of the treatment of the center-of-mass problem can be carried over to the rotational case. The shell-model potential used here is not spherically symmetric; a typical example is the anisotropic harmonic oscillator.[3,11] The number of degrees of freedom of the shell-model system is equal to that of the original system; hence degrees of freedom corresponding to the rotational degree of freedom of the original nucleus must exist also in the shell model. The shell-model system does not rotate freely, nor is it held to a fixed orientation; rather, it oscillates with small amplitude about an equilibrium orientation (these fluctuations in nuclear orientation can be shown by choosing some system of axes to specify the orientation of the nucleus, e.g., the principal axes of inertia, and by calculating the mean and mean square values of the Euler angles defining the orientation of these axes).

By analogy with the center-of-mass case, one would expect the Inglis formula to be approximately valid for the rotational case, using shell-model wave functions and energies, if the following two conditions are satisfied: (1) The collective oscillation described by the shell model must not be coupled too strongly to the intrinsic motion; (2) the effective "collective potential" binding the system to a given orientation must commute approximately with ϕ. Methods for checking whether or not these conditions are satisfied in particular cases are being developed and will be reported in a subsequent paper.

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