conditions and related defects could affect the energy levels of the magnetic system of an appreciable fraction of the substance. If, as a first approximation, one wishes to assume that a magnetically dilute system has the same magnetic properties under the same *effective* field, it will be necessary to apply the appropriate filling and demagnetization factors.[6] A check on the combined factor can be obtained by an investigation of the magnetic properties of both macro- and microscopic forms in the temperature range in which they can be kept in thermal contact with liquid helium. Magnetically dilute systems with minimal cooperative effects would seem most likely to avoid possible effects of particle size on the magnetic properties. The investigation of this subject is an interesting problem in itself.

There is another point which seems most difficult of all. One must know at some time before adiabatic demagnetization that the known temperature is uniform throughout the magnetized sample. We see no alternative to allowing it to stand for a very long time with just enough helium gas to bring equilibrium eventually. Also this should not be at a very low initial temperature or the amount of adsorption will spoil the experiment. It will be very difficult to decide when equilibrium has been reached, since the only kind of magnetic susceptibility which can be measured in the field is the adiabatic differential susceptibility, $(\partial\mathbf{I}/\partial\mathbf{H})_s$, and this is very insensitive to temperature under the necessary experimental conditions. A sensitive carbon thermometer used only to indicate equilibrium just before demagnetization is one possibility. Another possible method of determining the time required for equilibrium would be a study at zero field so that the full sensitivity of a magnetic thermometer would be available.

If the measurements are confined to the region above 1°K, the problem becomes somewhat simpler. The obvious method would seem to be the determination of initial susceptibility as a function of temperature while the sample is in thermal contact with helium and the later introduction of heat to the isolated sample by magnetic relaxation. There is, however, another problem which arises at temperatures much above 1°K, namely the heat capacity of the container which would have some contact with outer particles. A single crystal can be suspended without a container. It would appear that the best that could be done with a powder is to enclose it in an extremely thin blown glass bubble which could be suspended and left open to the space within its enclosing vessel. This would avoid the necessity of strength to support pressure changes. We believe that glass is superior to plastic for this purpose.

[6] H. B. G. Casimir, *Magnetism and Very Low Temperatures* (Cambridge University Press, Cambridge, 1940), pp. 9–12.

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Theoretical Analysis of Buildup of Current in Transient Townsend Discharge

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The buildup of current in the transient Townsend discharge has been calculated fully, taking the positive-ion and photon mechanisms at the cathode into account. The two cases of continuous and instantaneous electron supply have been treated. The calculated results are useful for the explanation of the buildup of current during the formative time lag of sparks under a sudden application of overvoltage, and for the transient state of a Townsend discharge in the case of undervoltage application.

I. INTRODUCTION

THE growth of current during the formative time of spark breakdown before the development of space charge effects has been studied by several authors.[1–6] Recently Bandel[5] carried out experiments and a theoretical analysis of the current buildup, but his solution is approximately accurate only for times longer than the positive-ion transit time, and not for shorter times. Following Bandel, Auer[6] made a further analysis of this subject; his analysis is also limited to the special case of γ_p action alone, and it seems to be much too complicated for numerical computation.

In the present paper, a theoretical treatment of the same problem is carried out, based upon the fundamental equations of continuity for the electron and positive-ion streams in a parallel plane gap, from a mathematically more rigorous standpoint than those of the previous investigations.[7] Our analysis shows that the current-time characteristics cannot be given by a single form, but require different forms appropriate to the following three time ranges: (a) times shorter than the electron transit time (t_-); (b) times longer than the

[1] R. Schade, Z. Physik 104, 487 (1937).
[2] W. Bartholomeyczyk, Z. Physik 116, 355 (1940).
[3] A. von Engel and M. Steenbeck, *Elektrische Gasentladungen* (Verlag Julius Springer, Berlin, 1934), Vol. 2, p. 178.
[4] Dutton, Haydon, Jones, and Davidson, Brit. J. Appl. Phys. 4, 170 (1953).
[5] H. W. Bandel, Phys. Rev. 95, 1117 (1954).
[6] P. L. Auer, Phys. Rev. 98, 320 (1955).

[7] More recently P. M. Davidson [Phys. Rev. 99, 1072 (1955)] has discussed this problem from another standpoint.

electron transit time but shorter than the resultant transit time ($\bar{t}=t_-+t_+$, where t_+ is the positive-ion transit time), and (c) times longer than the resultant transit time.

The calculations are carried out for the two cases of externally generated cathode current: (i) a constant photoelectron release (Schade,[1] Jones and collaborators, Bandel,[5] Auer[6]); (ii) an instantaneous photoelectron release at the initial time (Steenbeck, Bartholomeyczyk[2]).

II. GENERAL CONSIDERATION

We consider the system of a uniform field gap with the cathode at $x=0$ and the anode at $x=l$, and with the distribution of charges independent of y and z. Let a constant voltage V be suddenly applied to the gap at the time origin $t=0$. Since the applied field $E=0$ for $t<0$, there are few charges in the gap because of diffusion, notwithstanding a steady external irradiation. Then the fundamental equations which show the conditions of continuity for charges at any time t and at any point x in the gap are

$$\frac{1}{v_-}\frac{\partial N_-}{\partial t}=-\frac{\partial N_-}{\partial x}+\alpha N_-, \tag{G.1}$$

$$\frac{1}{v_+}\frac{\partial N_+}{\partial t}=\frac{\partial N_+}{\partial x}+\alpha N_-. \tag{G.2}$$

The boundary conditions at the cathode and at the anode or an avalanche head are

$$N_-(0,t)=N_0+\gamma_i N_+(0,t)+\gamma_p N_p(t), \tag{G.3}$$

$$N_+(x_i,t)=0, \tag{G.4}$$

where the symbols have the following meanings: $N_-(x,t)$ and $N_+(x,t)$ are the numbers of electrons and positive ions passing through x at time t per cm^2 sec, respectively, and shall be called the "electron stream" and the "positive ion stream," respectively. $N_p(t)$ is the number of chances of ionization by electron collision in the whole gap, and is assumed to be proportional to the number of photons impinging on the cathode when their absorption in gas may be neglected.

$$N_p(t)=\int_0^{x_p}\alpha N_-(x,t)dx. \tag{G.5}$$

N_0 is the number of photoelectrons per cm^2 sec emitted by external agents, e.g., ultraviolet irradiation. v_- and v_+ are the drift velocities of the electrons and positive ions, respectively. $\bar{v}$ is the resultant velocity. α is the first Townsend coefficient. γ_i and γ_p are the second Townsend coefficients representing the action of positive ions and photons at the cathode, respectively, in the same way as in Bandel's definition.[8] $\gamma=\gamma_i+\gamma_p$ is the generalized second Townsend coefficient. x_p is the maximum distance swept by the electrons that leave the cathode at $t=0$:

$$x_p=\begin{cases} v_-t \\ l \end{cases} \quad \text{for} \quad v_-t\lesseqgtr l. \tag{G.6}$$

x_i is the maximum allowable distance where the positive ions to reach x at time t are produced[9]:

$$x_i=\begin{cases} \bar{v}(t+x/v_+) \\ l \end{cases} \quad \text{for} \quad \bar{v}(t+x/v_+)\lesseqgtr l. \tag{G.7}$$

Solving Eq. (G.1) by separation of variables and superposition, we get the following expression for N_-:

$$N_-(x,t)=N_0\sum_k \nu_k \exp[(\alpha-\lambda_k/v_-)x]e^{\lambda_k t}. \tag{G.8}$$

Substituting this in Eq. (G.2), we obtain the expression for N_+:

$$N_+(x,t)=N_0\sum_k\left\{\left(-\frac{\alpha}{\alpha-\lambda_k/\bar{v}}\nu_k\right)\exp[(\alpha-\lambda_k/v_-)x] +\nu_k'\exp(\lambda_k x/v_+)\right\}e^{\lambda_k t}. \tag{G.9}$$

The constants of integration (λ_k, ν_k, and ν_k') and the limits of summation (k) can be determined by the boundary conditions as shown in the following.

First, substituting (G.9) into (G.4), we can determine ν_k' in terms of λ_k and ν_k; it has different values corresponding to the two forms of x_i as follows:

(1) For $\bar{v}(t+x/v_+)\leqq l$ or $t\leqq\bar{t}-x/v_+$,

$$N_+(x_i,t)\equiv N_0\left[\left\{\sum_k\left(-\frac{\alpha}{\phi_k}\nu_k\right)\right\}e^{\alpha v_- t}+\sum_k \nu_k'\exp(\lambda_k v_- t/\bar{v})\right]=0.$$

Therefore

$$\sum_k \frac{\alpha}{\phi_k}\nu_k=0, \tag{G.10}$$

and

$$\nu_k'=0 \text{ for all of } k. \tag{G.11}$$

(2) For $\bar{v}(t+x/v_+)\geqq l$ or $t\geqq\bar{t}-x/v_+$,

$$N_+(x_i,t)\equiv N_0\sum_k\left\{\left(-\frac{\alpha}{\phi_k}\nu_k\right)e^{\phi_{-k}l}+\nu_k'\exp(\lambda_k l/v_+)\right\}e^{\lambda_k t}=0.$$

Therefore

$$\nu_k'=\left(\frac{\alpha}{\phi_k}e^{\phi_k l}\right)\nu_k. \tag{G.12}$$

In these equations,

$$\phi_{-k}=\alpha-\lambda_k/v_-, \quad \phi_k=\alpha-\lambda_k/\bar{v}. \tag{G.13}$$

[8] We will take γ_i action and γ_p action into consideration for the secondary mechanism, which seem to be necessary and sufficient to explain the experimental results on spark time lags by Schade, Fisher and his collaborators, Bandel and Mori (Proc. Fac. Eng. Keiogijuku Univ. 4, 101 (1953)].

[9] The initial conditions,

$$N_-(x,0)=\begin{cases} N_0, & x=0 \\ 0, & 0<x\leqq l, \end{cases} \quad N_+(x,0)=0, \quad 0\leqq x\leqq l$$

are implicitly expressed by Eqs. (G.3)–(G.7).

Consequently,

$$N_+(x,t)=\begin{cases} N_0\sum_k\left(-\frac{\alpha}{\phi_k}\nu_k\right)e^{\phi_{-k}x}e^{\lambda_k t}, & t\leqq \bar{t}-x/v_+ \\ N_0\sum_k\left(\frac{\alpha}{\phi_k}\nu_k\right)\{-e^{\phi_{-k}x}+e^{\phi_k l}\exp(\lambda_k x/v_+)\}e^{\lambda_k t}, & t\geqq \bar{t}-x/v_+. \end{cases} \quad \text{(G.14)}$$

Using the values thus obtained for N_- and N_+, we can express the current densities as follows[10]:

$$J_-(t)=\frac{e}{l}\int_0^{x_p}N_-(x,t)dx=J_{0\alpha}\frac{N_p(t)}{N_0},$$
$$J_+(t)=\frac{e}{l}\int_0^{x_p}N_+(x,t)dx, \quad \text{(G.15)}$$
$$J(t)=J_-(t)+J_+(t).$$

Here $J_0=eN_0$ is the externally generated cathode current density; $J_{0\alpha}=J_0/\alpha l$; $J_-(t)$ and $J_+(t)$ are the current densities at time t due to the motion of all the electrons and positive ions in the gap, respectively; and $J(t)$ is the total current density. Also from Eq. (G.5),

$$N_p(t)=\begin{cases} N_0\left[\left\{\sum_k\left(\frac{\alpha}{\phi_{-k}}\nu_k\right)\right\}e^{\alpha v_- t}+\sum_k\left(-\frac{\alpha}{\phi_{-k}}\nu_k\right)e^{\lambda_k t}\right], & t\leqq t_- \\ N_0\sum_k\left(\frac{\alpha}{\phi_{-k}}\nu_k\right)(e^{\phi_{-k}l}-1)e^{\lambda_k t}, & t\geqq t_-. \end{cases} \quad \text{(G.16)}$$

At the cathode,

$$N_-(0,t)=N_0\sum_k\nu_k e^{\lambda_k t}. \quad \text{(G.17)}$$

$$N_+(0,t)=\begin{cases} N_0\sum_k\left(-\frac{\alpha}{\phi_k}\nu_k\right)e^{\lambda_k t}, & t\leqq \bar{t} \\ N_0\sum_k\left(\frac{\alpha}{\phi_k}\nu_k\right)(e^{\phi_k l}-1)e^{\lambda_k t}, & t\geqq \bar{t}. \end{cases} \quad \text{(G.18)}$$

Thus $N_p(t)$ has different forms for the ranges $t<t_-$ and $t>t_-$, while $N_+(0,t)$ has different forms for $t<\bar{t}$ and $t>\bar{t}$. Hence the boundary conditions are represented by three different equations corresponding to the following three time ranges:

(a) $0\leqq t\leqq t_-$,

(b) $t_-\leqq t\leqq \bar{t}$,

(c) $\bar{t}\leqq t$.

[10] These current densities represent the current intensities in the external circuit divided by the electrode area.

Therefore we must determine the unknown constants λ_k's and ν_k's separately for ranges (a), (b), and (c).

Case (a).—Range:

$$0\leqq t\leqq t_- \quad \text{or} \quad 0\leqq v_-t\leqq l.$$

We add the suffix a to the characteristic constants λ_k and the coefficients ν_k, etc., as λ_{ak}, ν_{ak}, etc. From the boundary condition (G.3),

$$\sum_k \nu_{ak}e^{\lambda_{ak}t}=1+\gamma_i\left\{-\sum_k\left(\frac{\alpha}{\phi_{ak}}\nu_{ak}\right)e^{\lambda_{ak}t}\right\}$$
$$+\gamma_p\left\{\left(\sum_k\frac{\alpha}{\phi_{-ak}}\nu_{ak}\right)e^{\alpha v_- t}-\sum_k\left(\frac{\alpha}{\phi_{-ak}}\nu_{ak}\right)e^{\lambda_{ak}t}\right\}. \quad \text{(a.1)}$$

One of the λ_{ak} must be zero because Eq. (a.1) holds identically for any values of $t(<t_-)$. We call it λ_{a0}; then

$$\lambda_{a0}=0;\ \ \phi_{-a0}=\phi_{a0}=\alpha. \quad \text{(a.2)}$$

For λ_{ak}'s with other values of k, collecting the coefficients of $e^{\lambda_{ak}t}$ in Eq. (a.1) gives

$$\lambda_{ak}^2-\alpha\{(1+\gamma_p)v_-+(1+\gamma_i)\bar{v}\}\lambda_{ak}+\alpha^2(1+\gamma_i+\gamma_p)v_-\bar{v}=0. \quad \text{(a.3)}$$

The λ_{ak}'s are the roots of the above quadratic equation, which has the same form irrespective of k. Hence only two values are to be determined as λ_{ak}'s; we call these λ_{a1} and λ_{a2}, where

$$0<\lambda_{a1}<\lambda_{a2}. \quad \text{(a.4)}$$

Consequently we have only to take into account three terms ($k=0, 1, 2$) in the expressions for N_-, N_+, etc. With the aid of $\lambda_{ak}(k=0, 1, 2)$ determined as above, the coefficients $\nu_{ak}(k=0, 1, 2)$ can be determined from the following simultaneous equations:

Constant term of Eq. (a.1):

$$\nu_{a0}=1-\gamma_i\nu_{a0}-\gamma_p\nu_{a0},$$

$e^{\alpha v_- t}$ term of Eq. (a.1):

$$\nu_{a0}+\frac{\alpha}{\phi_{-a1}}\nu_{a1}+\frac{\alpha}{\phi_{-a2}}\nu_{a2}=0, \quad \text{(a.5)}$$

Eq. (G.10):

$$\nu_{a0}+\frac{\alpha}{\phi_{a1}}\nu_{a1}+\frac{\alpha}{\phi_{a2}}\nu_{a2}=0.$$

In this manner, we have obtained all of the necessary constants. Using these values of λ_{ak} and ν_{ak}, the quantities referring to charge and current are listed as follows:

$$\frac{N_-(x,t)}{N_0}=\begin{cases} \sum_{k=0}^{2}\nu_{ak}e^{\phi_{-ak}x}e^{\lambda_{ak}t}, & x<v_-t \\ 0, & x>v_-t. \end{cases} \quad \text{(a.6)}$$

$$\frac{N_+(x,t)}{N_0}=\begin{cases} \sum_{k=0}^{2}\left(-\frac{\alpha}{\phi_{ak}}\nu_{ak}\right)e^{\phi_{-ak}x}e^{\lambda_{ak}t}, & x<v_-t \\ 0, & x>v_-t. \end{cases} \quad \text{(a.7)}$$

$$\frac{J_-(t)}{J_{0\alpha}}=\sum_{k=0}^{2}\left(-\frac{\alpha}{\phi_{-ak}}\nu_{ak}\right)e^{\lambda_{ak}t}=\frac{N_p(t)}{N_0}, \quad \text{(a.8)}$$

$$\frac{J_+(t)}{J_{0\alpha}}=\sum_{k=0}^{2}\left(\frac{\alpha^2}{\phi_{-ak}\phi_{ak}}\nu_{ak}\right)e^{\lambda_{ak}t}, \quad \text{(a.9)}$$

$$J(t)=J_0\frac{1}{(\lambda_{a2}-\lambda_{a1})t_-}\{e^{\lambda_{a2}t}-e^{\lambda_{a1}t}\}. \quad \text{(a.10)}$$

Case (*b*).—Range:

$$t_-\leqq t\leqq \bar{t} \quad \text{or} \quad \bar{v}t\leqq l\leqq v_-t.$$

Let the characteristic constants and coefficients be marked with suffix b in this time range, as λ_{bk}, ν_{bk}, etc. To determine the unknown quantities λ_{bk} and ν_{bk}, we use the initial condition from Eq. (a.6) and the boundary condition from Eq. (G.3) with the aid of Eqs. (G.16), (G.17), and (G.19) to obtain the following:

$$\frac{N_-(x,t_-)}{N_0}:\ \sum_k(\nu_{bk}e^{\lambda_{bk}t_-})e^{\phi_{-bk}x}=\sum_{k=0}^{2}(\nu_{ak}e^{\lambda_{ak}t_-})e^{\phi_{-ak}x}, \quad \text{(b.1)}$$

and

$$\sum_k \nu_{bk}e^{\lambda_{bk}t}=1+\gamma_i\left\{-\sum_k\left(\frac{\alpha}{\phi_{bk}}\nu_{bk}\right)e^{\lambda_{bk}t}\right\}+\gamma_p\left\{\sum_k\left(\frac{\alpha}{\phi_{-bk}}\nu_{bk}\right)(e^{\phi_{-bk}l}-1)e^{\lambda_{bk}t}\right\}. \quad \text{(b.2)}$$

Since Eq. (b.2) is identical to Eq. (a.1) with respect to t, one of the λ_{bk} must be zero; it is called λ_{b0}.

$$\lambda_{b0}=0;\quad \phi_{-b0}=\phi_{b0}=\alpha. \quad \text{(b.3)}$$

For the other λ_{bk}, comparing $e^{\lambda_{bk}t}$-terms in both sides of Eq. (b.2) gives

$$1=-\gamma_i\frac{1}{1-\lambda_{bk}/\alpha\bar{v}}+\gamma_p\frac{\exp[\alpha l(1-\lambda_{bk}/\alpha v_-)]-1}{1-\lambda_{bk}/\alpha v_-}. \quad \text{(b.4)}$$

This equation has two real roots

$$\lambda_{b1},\quad \lambda_{b2},$$

and an infinite number of complex roots

$$\lambda_{bk}=\lambda_{bk}'+\lambda_{bk}'',\quad k=3,4,5,\cdots.$$

They are in the relation

$$\alpha v_->\lambda_{b2}>\alpha\bar{v}>\lambda_{b1}>\lambda_{b3}'>\lambda_{b4}'>\cdots, \quad \text{(b.5)}$$

and

$$\lambda_{b1}\begin{cases}<0\\=0\\>0\end{cases}\ \text{for}\ \ 1+\gamma_i-\gamma_p(e^{\alpha l}-1)\begin{cases}>0\\=0.\\<0\end{cases} \quad \text{(b.6)}$$

$N_-(0,t)$, $N_-(x,t)$ and other quantities are infinite series,[11] but the predominant terms for current buildup are those involving the real roots λ_{b1} and λ_{b2}. See Appendix. So we may take the three terms involving λ_{b0}, λ_{b1} and λ_{b2} into account for our approximate solution. Moreover, in the following, another approximation will be made by setting the continuity of the electron stream N_- at $t=t_-$ "at the cathode" instead of "in the whole gap."

$$\frac{N_-(0,t_-)}{N_0}:\ \sum_{k=0}^{2}\nu_{bk}e^{\lambda_{bk}t_-}=\sum_{k=0}^{2}\nu_{ak}e^{\lambda_{ak}t_-}. \quad \text{(b.1')}$$

Then the simultaneous equations for the ν_{bk} are as follows:

Eq. (b.1′): $\nu_{b0}+\nu_{b1}e^{\lambda_{b1}t_-}+\nu_{b2}e^{\lambda_{b2}t_-}=\nu_{\mathrm{I}}$,

Constant term of Eq. (b.2):

$$\nu_{b0}=1-\gamma_i\nu_{b0}+\gamma_p(e^{\alpha l}-1)\nu_{b0}, \quad \text{(b.7)}$$

Eq. (G.10): $\nu_{b0}+\dfrac{\alpha}{\phi_{b1}}\nu_{b1}+\dfrac{\alpha}{\phi_{b2}}\nu_{b2}=0,$

where

$$\nu_{\mathrm{I}}=N_-(0,t_-)/N_0. \quad \text{(b.8)}$$

Using the coefficients ν_{bk} determined from the above equations, we get the expressions[12] for the charge streams and currents:

$$\frac{N_-(x,t)}{N_0}=\sum_{k=0}^{2}\nu_{bk}e^{\phi_{-bk}x}e^{\lambda_{bk}t}, \quad \text{(b.9)}$$

$$\frac{N_+(x,t)}{N_0}=\begin{cases}\displaystyle\sum_{k=0}^{2}\left(-\frac{\alpha}{\phi_{bk}}\nu_{bk}\right)e^{\phi_{-bk}x}e^{\lambda_{bk}t}, & x\leqq v_+(\bar{t}-t),\quad \text{(b.10A)}\\ \displaystyle\sum_{k=0}^{2}\left(\frac{\alpha}{\phi_{bk}}\nu_{bk}\right)\times\{-e^{\phi_{-bk}x}+e^{\phi_{bk}l}\exp(\lambda_{bk}x/v_+)\}e^{\lambda_{bk}t}, & x\geqq v_+(\bar{t}-t);\quad \text{(b.10B)}\end{cases}$$

$$\frac{J_-(t)}{J_{0\alpha}}=\sum_{k=0}^{2}\left(\frac{\alpha}{\phi_{-bk}}\nu_{bk}\right)(e^{\phi_{-bk}l}-1)e^{\lambda_{bk}t}=\frac{N_p(t)}{N_0}, \quad \text{(b.11)}$$

$$\frac{J_+(t)}{J_{0\alpha}}=\sum_{k=0}^{2}\left(\frac{\alpha^2}{\phi_{-bk}\phi_{bk}}\nu_{bk}\right)\left\{\left(1+\frac{v_+}{\lambda_{bk}}\phi_{bk}e^{\phi_{-bk}l}\right)e^{\lambda_{bk}t}-\frac{v_+}{\lambda_{bk}}\phi_{-bk}e^{\alpha l}\right\}, \quad \text{(b.12)}$$

$$J(t)=J_{0\alpha}\frac{e^{\alpha l}}{1+\gamma_i-\gamma_p(e^{\alpha l}-1)}\left\{\frac{t-t_-}{t_+}\right\}+J_{0\alpha}\sum_{k=1}^{2}\left(\frac{\alpha^2}{\phi_{-bk}\phi_{bk}}\nu_{bk}\right)\times\left[\left\{\frac{\lambda_{bk}}{\alpha\bar{v}}+\left(\frac{1}{\alpha}+\frac{v_+}{\lambda_{bk}}\right)\phi_{bk}e^{\phi_{-bk}l}\right\}e^{\lambda_{bk}t}-\frac{v_+}{\lambda_{bk}}\phi_{-bk}e^{\alpha l}\right]. \quad \text{(b.13)}$$

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[11] If the initial charge distribution $N_-(x,0)$ be a function of x, $f(x)$, then also the λ_{ak} must be of infinite number and the solution $N_-(x,t)$ must be an infinite series.

[12] Since Bandel, Auer, and even Davidson[7] in his latest paper treated the integral equation for $N_-(0,t)$, the approximate solution shown here is what they have been looking for as an exact solution.

Case (*c*).—Range:

$$\bar{t} \leq t \quad \text{or} \quad l \leq \bar{v}t.$$

Let the constants of integration be characterized with the suffix c in this time range, as λ_{ck} and ν_{ck}, etc.; these constants can be determined as in the preceding range by virtue of the initial and boundary conditions as follows.

$$\frac{N_-(x,\bar{t})}{N_0}: \sum_k [\nu_{ck}\exp(\lambda_{ck}\bar{t})]e^{\phi_{-ck}x} = \sum_k [\nu_{bk}\exp(\lambda_{bk}\bar{t})]e^{\phi_{-bk}x}, \quad \text{(c.1)}$$

$$\sum_k \nu_{ck}e^{\lambda_{ck}t} = 1+\gamma_i\left\{\sum_k\left(\frac{\alpha}{\phi_{ck}}\nu_{ck}\right)(e^{\phi_{ck}l}-1)e^{\lambda_{ck}t}\right\} +\gamma_p\left\{\sum_k\left(\frac{\alpha}{\phi_{-ck}}\nu_{ck}\right)(e^{\phi_{-ck}l}-1)e^{\lambda_{ck}t}\right\}. \quad \text{(c.2)}$$

This is an identity with respect to t. Hence one of the λ_{ck} must be zero; we call it λ_{c0}.

$$\lambda_{c0}=0; \quad \phi_{-c0}=\phi_{c0}=\alpha. \quad \text{(c.3)}$$

Moreover, for λ_{ck} with other values of k,

$$1=\gamma_i\frac{\exp[\alpha l(1-\lambda_{ck}/\alpha\bar{v})]-1}{1-\lambda_{ck}/\alpha\bar{v}} +\gamma_p\frac{\exp[\alpha l(1-\lambda_{ck}/\alpha v_-)]-1}{1-\lambda_{ck}/\alpha v_-}. \quad \text{(c.4)}$$

This equation has one real root λ_{c1} and an infinite number of complex roots.

$$\lambda_{c1}\begin{cases}<0\\=0\\>0\end{cases} \text{for} \quad 1-(\gamma_i+\gamma_p)(e^{\alpha l}-1)\begin{cases}>0\\=0.\\<0\end{cases} \quad \text{(c.5)}$$

For the approximate solution, the following condition may be substituted for Eq. (c.1) as in the range (b).

$$\frac{N_-(0,\bar{t})}{N_0}: \sum_{k=0}^{1}\nu_{ck}\exp(\lambda_{ck}\bar{t})=\sum_{k=0}^{2}\nu_{bk}\exp(\lambda_{bk}\bar{t}). \quad \text{(c.1}')$$

Then the simultaneous equations for the coefficients ν_{ck} are

Eq. (c.1′): $\nu_{c0}+\nu_{c1}\exp(\lambda_{c1}\bar{t})=\nu_{\text{II}}$;

constant term of Eq. (c.2): (c.6)

$$\nu_{c0}=1+\gamma_i(e^{\alpha l}-1)\nu_{c0}+\gamma_p(e^{\alpha l}-1)\nu_{c0},$$

where

$$\nu_{\text{II}}=N_-(0,\bar{t})/N_0. \quad \text{(c.7)}$$

With ν_{c0} and ν_{c1} determined, the solutions are summarized as follows:

$$\frac{N_-(x,t)}{N_0}=\nu_{c0}+\nu_{c1}\exp\{\alpha x+\lambda_{c1}(t-x/v_-)\}, \quad \text{(c.8)}$$

$$\frac{N_+(x,t)}{N_0}=\nu_{c0}(e^{\alpha l}-e^{\alpha x})-\frac{\alpha}{\phi_{c1}}\nu_{c1} \times[\exp\{\alpha l+\lambda_{c1}(t-\bar{t}+x/v_+)\} -\exp\{\alpha x+\lambda_{c1}(t-x/v_-)\}], \quad \text{(c.9)}$$

$$\frac{J_-(t)}{J_{0\alpha}}=\nu_{c0}(e^{\alpha l}-1) +\frac{\alpha}{\phi_{-c1}}\nu_{c1}(e^{\phi_{-c1}l}-1)e^{\lambda_{c1}t}=\frac{N_p(t)}{N_0}, \quad \text{(c.10)}$$

$$\frac{J_+(t)}{J_{0\alpha}}=\nu_{c0}\{\alpha le^{\alpha l}-(e^{\alpha l}-1)\}+\frac{\alpha^2}{\phi_{-c1}\phi_{c1}}\nu_{c1} \times\left\{1+\frac{v_+}{\lambda_{c1}}(\phi_{c1}e^{\phi_{-c1}l}-\phi_{-c1}e^{\phi_{c1}l})\right\}e^{\lambda_{c1}t}, \quad \text{(c.11)}$$

$$J(t)=J_0\frac{e^{\alpha l}}{1-(\gamma_i+\gamma_p)(e^{\alpha l}-1)}+J_{0\alpha}\frac{\alpha^2}{\phi_{-c1}\phi_{c1}}\nu_{c1} \times\left\{\frac{\lambda_{c1}}{\alpha\bar{v}}+\left(\frac{1}{\alpha}+\frac{v_+}{\lambda_{c1}}\right)\phi_{c1}e^{\phi_{-c1}l} -\frac{v_+}{\lambda_{c1}}\phi_{-c1}e^{\phi_{c1}l}\right\}e^{\lambda_{c1}t}. \quad \text{(c.12)}$$

III. SOLUTIONS FOR CRITICAL CASES AND SPECIAL CASES

1. Critical Cases

Let us consider two critical cases in which the magnitude of the applied field is critical according as $1+\gamma_i-\gamma_p(e^{\alpha l}-1)=0$ or $1-(\gamma_i+\gamma_p)(e^{\alpha l}-1)=0$; for these cases, λ_{b1} or λ_{c1} becomes zero, and therefore the results in the general consideration must be somewhat modified.

1.1 *Case of* $1+\gamma_i-\gamma_p(e^{\alpha l}-1)=0$.—The solutions for this case are derived from Eqs. (b.9)–(b.13) by taking the limit $1+\gamma_i-\gamma_p(e^{\alpha l}-1)\to 0$. For example,

$$N_-(0,t)/N_0=\nu_{b1}'+\mu_b t+\nu_{b2}e^{\lambda_{b2}t}, \quad \text{(C.1)}$$

where

$$\mu_b=\left\{\frac{\gamma_i}{\alpha\bar{v}}+\frac{\gamma_p}{\alpha v_-}[\alpha le^{\alpha l}-(e^{\alpha l}-1)]\right\}^{-1}, \quad \text{(C.2)}$$

$$\nu_{b1}'=-\frac{\alpha}{\phi_{b2}e^{\lambda_{b2}t_-}-\alpha}\left\{\nu_1+\mu_b\left(\frac{\bar{t}}{\alpha l}\frac{\phi_{b2}}{\alpha}e^{\lambda_{b2}t_-}-t_-\right)\right\}. \quad \text{(C.3)}$$

1.2. *Case of* $1-(\gamma_i+\gamma_p)(e^{\alpha l}-1)=0$ (application of static breakdown voltage).—

$$N_-(0,t)/N_0=\nu_{\text{II}}+\mu_c(t-\bar{t}), \quad \text{(C.4)}$$

$$\frac{J(t)}{J_0}=\left\{\nu_{\text{II}}+\mu_c\left(t+\frac{\alpha l-1}{\alpha^2 l^2}\bar{t}+\frac{t_+}{2}\right)\right\}e^{\alpha l} +\mu_c\frac{\bar{t}}{\alpha^2 l^2}, \quad \text{(C.5)}$$

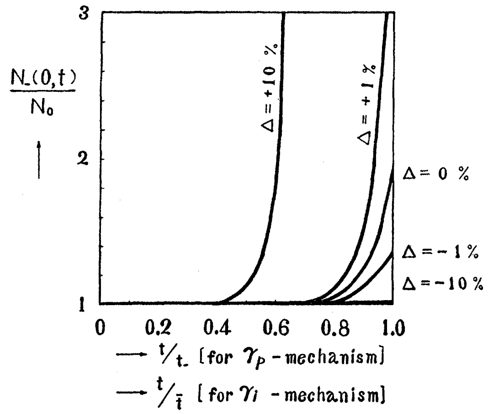

FIG. 1. Cathode electron stream $N_-(0,t)$ *vs* time in electron transit time for γ_p action and in resultant transit time for γ_i action.

where

$$\mu_c=\left\{\left(\frac{\gamma_i}{\alpha\bar{v}}+\frac{\gamma_p}{\alpha v_-}\right)[\alpha l e^{\alpha l}-(e^{\alpha l}-1)]\right\}^{-1}. \qquad \text{(C.6)}$$

With the aid of Eq. (c.5), Eqs. (c.12) and (C.5) give the well-known formula which is derived for a steady-state condition as follows:

$$J(\infty)=\begin{cases} J_0 e^{\alpha l}/1-\gamma(e^{\alpha l}-1), & \lambda_{c1}<0,\ 1-\gamma(e^{\alpha l}-1)>0 \\ \infty, & \lambda_{c1}\geqq 0,\ 1-\gamma(e^{\alpha l}-1)\geqq 0. \end{cases} \qquad \text{(I)}$$

2. Special Cases

The solutions (a.6)–(a.10), (b.9)–(b.13), and (c.8)–(c.12) are those which Bandel and others would want to derive. From these solutions the current-time characteristics can be calculated. Since the general case containing γ_i and γ_p is much complicated for numerical computation, the two special cases in which $\gamma=\gamma_p$ and $\gamma=\gamma_i$ will be dealt with.

2.1. *Case in which* $\gamma=\gamma_p$; $\gamma_i=0$.—

2.1.1. (a)-range: $0\leqq t\leqq t_-$.—

$$\frac{N_-(0,t)}{N_0}=\frac{1}{1+\gamma_p}\{1+\gamma_p\exp[\alpha v_-(1+\gamma_p)t]\}. \qquad \text{(S.1)[13]}$$

2.1.2. (b–c)-*range*: $t_-\leqq t$.—

$$\frac{N_-(0,t)}{N_0}=\frac{1}{1-\gamma_p(e^{\alpha l}-1)}+\frac{\gamma_p e^{\alpha l}}{1+\gamma_p}\times\left\{e^{\gamma_p\alpha l}-\frac{1}{1-\gamma_p(e^{\alpha l}-1)}\right\}\exp\lambda_{bc1}(t-t_-), \qquad \text{(S.2)[14]}$$

where λ_{bc1} is the root of the equation

$$\gamma_p\frac{\exp[\alpha l(1-\lambda_{bc1}/\alpha v_-)]-1}{1-\lambda_{bc1}/\alpha v_-}=1. \qquad \text{(S.3)}$$

2.2. *Case in which* $\gamma=\gamma_i$; $\gamma_p=0$.—

2.2.1. (a–b)-*range*: $0\leqq t\leqq \bar{t}$.—

$$\frac{N_-(0,t)}{N_0}=\frac{1}{1+\gamma_i}\{1+\gamma_i\exp[\alpha\bar{v}(1+\gamma_i)t]\}. \qquad \text{(S.4)}$$

2.2.2. (c)-*range*: $\bar{t}\leqq t$.—

$$\frac{N_-(0,t)}{N_0}=\frac{1}{1-\gamma_i(e^{\alpha l}-1)}+\frac{\gamma_i e^{\alpha l}}{1+\gamma_i}\left[e^{\gamma_i\alpha l}-\frac{1}{1-\gamma_i(e^{\alpha l}-1)}\right]\times\exp[\lambda_{c1}(t-\bar{t})], \qquad \text{(S.5)}$$

where λ_{c1} is the root of the equation

$$\gamma_i\frac{\exp[\alpha l(1-\lambda_{c1}/\alpha\bar{v})]-1}{1-\lambda_{c1}/\alpha\bar{v}}=1. \qquad \text{(S.6)}$$

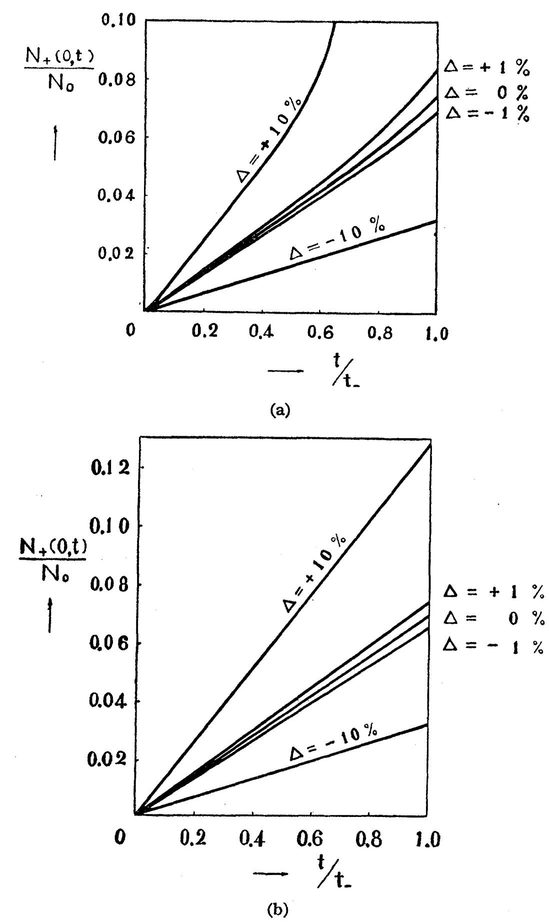

FIG. 2. Cathode ion stream $N_+(0,t)$ *vs* time in electron transit time (a) for γ_p action; (b) for γ_i action.

[13] This is the same as Eq. (3.8) in Auer's paper.

[14] This equation is substantially the same as Eq. (3.14) in Auer's paper.

IV. SOLUTIONS FOR THE CASE OF INSTANTANEOUS ELECTRON SUPPLY

The solutions in the preceding chapters are derived under the application of a step-function-like impulse voltage to a uniform field gap with a cathode photocurrent externally supplied; obviously, they can also be the solutions for the case of continuous voltage but an external supply of a step-function-like cathode photocurrent. Therefore, the solution for the case of an instantaneous cathode electron supply $N_0{}^0$ (per cm^2) at $t=0$ can be obtained by superposing two solutions for

$$N_0(\text{per cm}^2\text{ sec}) \quad \text{for} \quad t\geqq 0,$$

and

$$-N_0(\text{per cm}^2\text{ sec}) \quad \text{for} \quad t\geqq t_0,$$

and taking its limit

$$\lim_{\substack{t_0\to 0\\ N_0\to\infty}} (N_0t_0)=N_0{}^0. \tag{I.1}$$

With the index 0, as in $\nu_k{}^0$, we distinguish the quantities for an instantaneous electron supply ($N_0{}^0$) from those for a continuous electron supply (N_0) with which we have been concerned hitherto.

For the present problem, the fundamental equations (G.1), (G.2) and the boundary conditions (G.3), (G.4) hold (except for N_0). So the solutions for this case

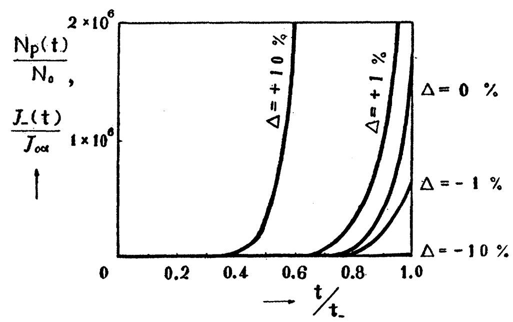

FIG. 3. Electron current density $J_-(t)$, or total number of ionization chances per unit time $N_p(t)$, *vs* time in electron transit time. The values are nearly equal for the three cases: $\gamma=\gamma_p$, $\gamma=\gamma_i$, and $\gamma=0$.

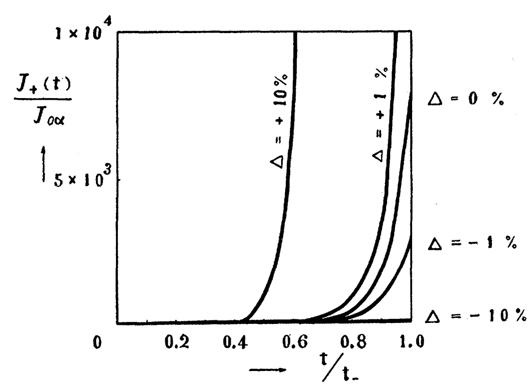

FIG. 4. Ion current density $J_+(t)$ *vs* time in electron transit time. The values of $J_+(t)$ are nearly equal for the three cases: $\gamma=\gamma_p$, $\gamma=\gamma_i$, and $\gamma=0$.

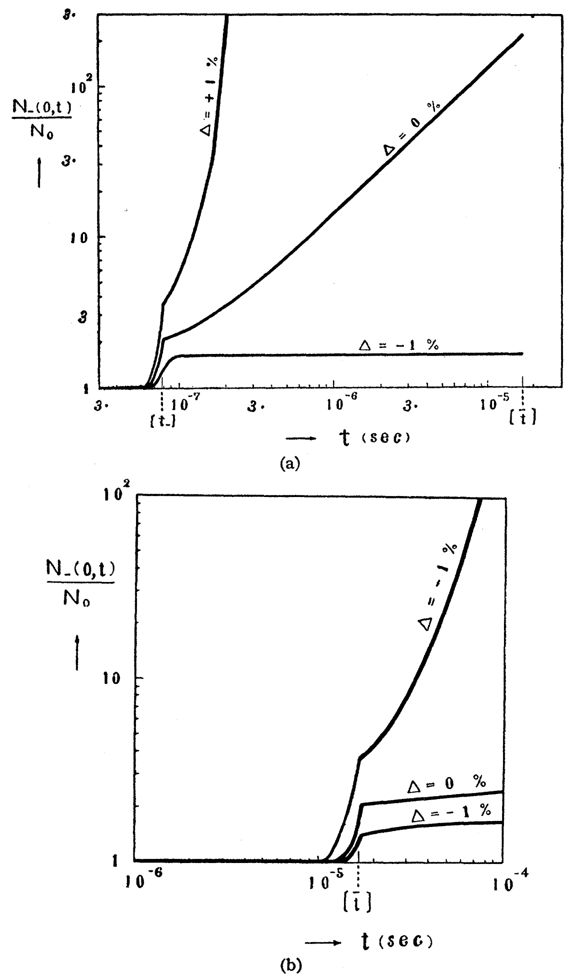

FIG. 5. Cathode electron stream $N_-(0,t)$ *vs* time in the entire time range (a) for γ_p action; (b) for γ_i action.

have the same form as (G.8) and (G.14), as follows:

$$N_-{}^0(x,t)=N_0{}^0\sum_k \nu_k{}^0 e^{\phi-kx}e^{\lambda_k t}, \tag{I.2}$$

$$N_+{}^0(x,t)=\begin{cases} N_0{}^0\sum_k\left(-\dfrac{\alpha}{\phi_k}\nu_k{}^0\right)e^{\phi-kx}e^{\lambda_k t}, & x\leqq v_+(\bar{t}-t)\\ N_0{}^0\sum_k\left(\dfrac{\alpha}{\phi_k}\nu_k{}^0\right)\times[-e^{\phi-kx}+e^{\phi kl}\exp(\lambda_k x/v_+)]e^{\lambda_k t}, & x\geqq v_+(\bar{t}-t). \end{cases} \tag{I.3}$$

However, there may be not any constant term corresponding to $k=0$, because the boundary condition at the cathode has no constant term. Each of the λ_k must be equal to that for a continuous electron supply for all values of k. The determination of the constants of integration $\nu_k{}^0$ is difficult in the same manner as in the previous sections. For this determination, it is advisable

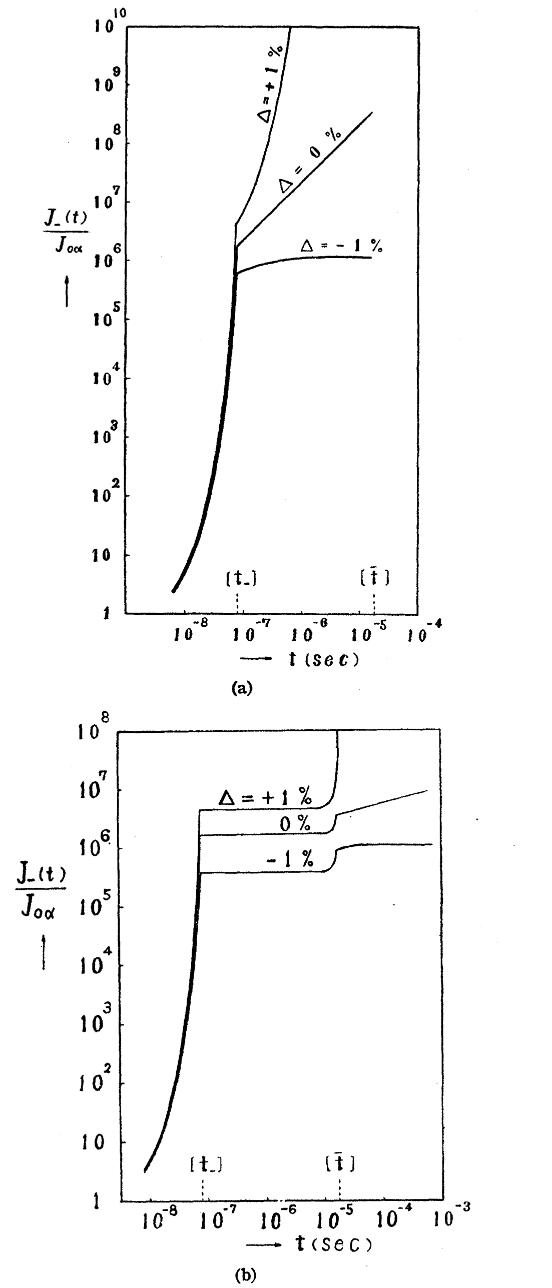

FIG. 6. Electron current density $J_-(t)$ or number of ionization chances $N_p(t)$ *vs* time in the entire time range (a) for γ_p action; (b) for γ_i action.

to superpose the results for continuous $+N_0$ for $t\geqq 0$ and $-N_0$ for $t\geqq t_0$ and to take the limit as mentioned above. The solutions thus obtained are summarized separately for three time ranges.

1. (a)-*range.*—

$$N_-^0(x,t)=\lim_{\substack{t_0\to 0\\ N_0\to\infty}}\{N_-(x,t)-N_-(x,t-t_0)\}$$

$$=N_0^0\sum_k(\lambda_{ak}\nu_{ak})e^{\phi_{-ak}x}e^{\lambda_{ak}t}. \quad \text{(I.4)}$$

Therefore

$$\nu_{ak}^0=\lambda_{ak}\nu_{ak},\quad k=1,2,\cdots. \quad \text{(I.5)}$$

Hence we can get the expressions for N_+, J_-, J_+, etc.

2. (b)-*range.*—The coefficients are derived from the relations:

$$\sum_{k=1}^{2}\nu_{bk}^0e^{\lambda_{bk}t_-}=\nu_{\rm I}^0,\qquad \sum_{k=1}^{2}\frac{\nu_{bk}^0}{\phi_{bk}}=0, \quad \text{(I.6)}$$

where

$$\nu_{\rm I}^0=N_-^0(0,t_-)/N_0^0. \quad \text{(I.7)}$$

The second line of Eq. (I.6) corresponds to Eq. (G.10). Then

$$N_-^0(0,t)=N_0^0\sum_{k=1}^{2}\nu_{bk}^0e^{\lambda_{bk}t}. \quad \text{(I.8)}$$

The other quantities are derived similarly.

3. (c)-*range.*—For the determination of ν_{ck}^0, we have

$$\nu_{c1}^0\exp(\lambda_{c1}\bar{t})=\nu_{\rm II}^0, \quad \text{(I.9)}$$

where

$$\nu_{\rm II}^0=N_-(0,\bar{t})/N_0^0. \quad \text{(I.10)}$$

Using this, we have

$$N_-(0,t)=N_0^0\nu_{c1}^0e^{\lambda_{c1}t}=N_0^0\nu_{\rm II}^0\exp[\lambda_{c1}(t-\bar{t})]. \quad \text{(I.11)}$$

$$J(t)=Q_{0\alpha}\left(\frac{\alpha^2}{\phi_{-c1}\phi_{c1}}\nu_{c1}^0\right)\times\left\{\frac{\lambda_{c1}}{\alpha\bar{v}}+\left(\frac{1}{\alpha}+\frac{v_+}{\lambda_{c1}}\right)\phi_{c1}e^{\phi_{-c1}l}-\frac{v_+}{\lambda_{c1}}\phi_{-c1}e^{\phi_{c1}l}\right\}e^{\lambda_{c1}t}, \quad \text{(I.12)}$$

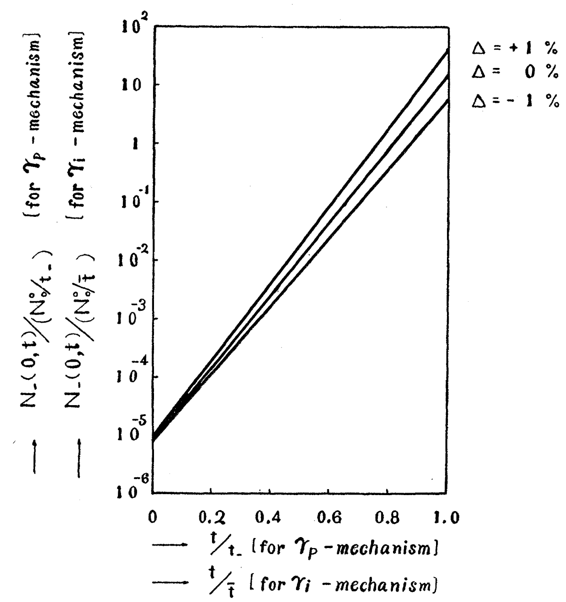

FIG. 7. Cathode electron stream $N_-(0,t)$ *vs* time in electron transit time for γ_p action and in resultant transit time for γ_i action. Instantaneous electron supply.

where

$$Q_0 = eN_0^0, \quad Q_{0\alpha} = Q_0/\alpha l.$$

In particular, for the case in which $1-\gamma(e^{\alpha l}-1)=0$, all of the quantities relating to charge and current are constant in this time range; that is, a steady state has been reached just at $t=\bar{t}$. From Eq. (I.12),

$$J(\infty)=\begin{cases}0\\ Q_0\nu_{\mathrm{II}}{}^0 e^{\alpha l},\ \lambda_{c1}\begin{cases}<0\\=0\\>0\end{cases} \text{for } 1-\gamma(e^{\alpha l}-1)\begin{cases}>0\\=0.\\<0\end{cases}\\ \infty\end{cases} \quad \text{(II)}$$

V. NUMERICAL ILLUSTRATIONS

As an illustrative example, we treat the following case. Gas: air. Pressure: $p=760$ mm Hg. Gap length: $l=1$ cm. Static breakdown voltage[15]: $V_s=31$ kv.

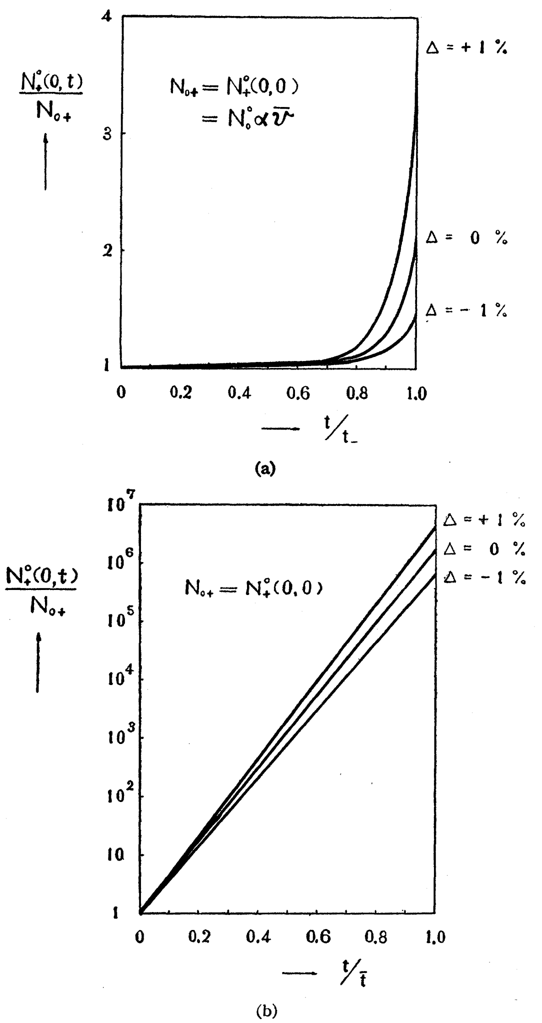

FIG. 8. Cathode ion stream $N_+(0,t)$ *vs* time in electron transit time. Instantaneous electron supply. (a) For γ_p action; (b) for γ_i action.

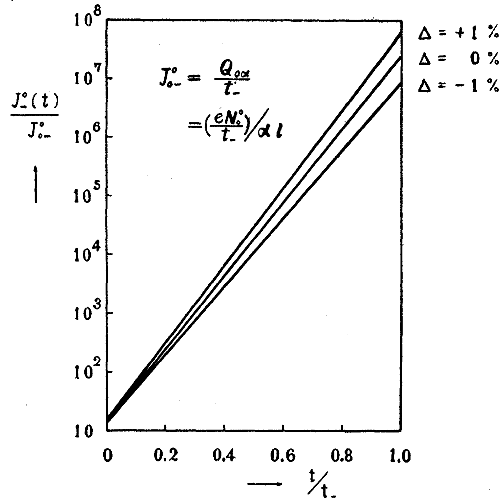

FIG. 9. Electron current density $J_-(t)$ *vs* time in electron transit time. Instantaneous electron supply. The values of $J_-(t)$ are nearly equal for the three cases: $\gamma=\gamma_p$, $\gamma=\gamma_i$, and $\gamma=0$.

The ionization coefficients are

$$\frac{\alpha}{p}=A\left(\frac{E}{p}-B\right)^2,$$

$$A=1.048\times10^{-4}\ (\text{cm}\cdot\text{mm Hg/v}),$$

$$B=27.38\ (\text{v/cm}\cdot\text{mm Hg});$$

these are the same as obtained by Bandel. Also

$$\alpha_s=14.3 \quad \text{for} \quad V=V_s(\Delta=0),$$

$$\gamma=\gamma_s=(e^{\alpha_s l}-1)^{-1}=e^{-\alpha_s l}=6\times10^{-7}.$$

The drift velocities are

$$v_-=1.26\times10^7\ (\text{cm/sec}) \text{ for } \Delta=0,$$

$$v_+=6\times10^4\ (\text{cm/sec}) \text{ for } \Delta=0,$$

$$\bar{v}=5.97\times10^4\ (\text{cm/sec}) \text{ for } \Delta=0,$$

With these data, $N_-(0,t)$, $N_+(0,t)$, and $N_p(t)$ or $J_-(t)$ and $J_+(t)$ are plotted as functions of t for several values of the percent overvoltage Δ in Fig. 1–Fig. 10.

VI. CONCLUSION

In the present paper, the transient Townsend discharge current or current buildup in the formative time of spark breakdown has been calculated, taking two secondary cathode mechanisms γ_i and γ_p into account, through the whole range of time, in more precise and practical form than by previous investigators. The solutions obtained are exact for the time range $t<t_-$ in the general case containing γ_i and γ_p, and for the time range $t<\bar{t}$ in the special case containing γ_i

[15] V_s is taken from *Standard Handbook for Electrical Engineers*, edited by A. E. Knowlton (McGraw-Hill Book Company, Inc., New York, 1949), eighth edition, being the value for a cathode fouled by repeated sparking. Using Bandel's experimental data, the author intends to calculate the current buildup, space-charge formation, etc.

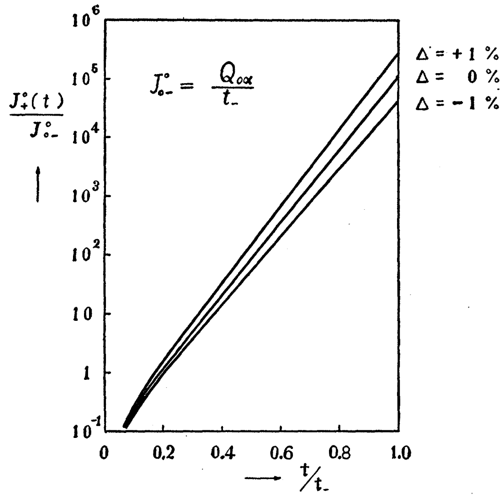

FIG. 10. Ion current density $J_+(t)$ *vs* time in electron transit time. Instantaneous electron supply. The values of $J_+(t)$ are nearly equal for the three cases: $\gamma=\gamma_p$, $\gamma=\gamma_i$, and $\gamma=0$.

only; but they are approximately accurate for other time ranges.

ACKNOWLEDGMENTS

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APPENDIX

We shall discuss the character of the complete solution referred to in Chapter II. The general case containing γ_i and γ_p is too complicated to be treated; so we shall consider the cases of γ_p alone and γ_i alone in the following. In such cases, the characteristic constants λ_k (λ_{bk} or λ_{ck}) can be determined from the single equation,

$$(\gamma\alpha l)\frac{\exp(\alpha l-\Lambda_k)-1}{\alpha l-\Lambda_k}=1, \qquad \text{(A.1)}$$

where

$$\Lambda_k=\Lambda_k'+i\Lambda_k'' = \begin{cases} (\lambda_k' t_-)+i(\lambda_k'' t_-) & \text{for the } \gamma_p \text{ mechanism,} \\ (\lambda_k' \bar{t})+i(\lambda_k'' \bar{t}) & \text{for the } \gamma_i \text{ mechansim.} \end{cases} \qquad \text{(A.2)}$$

From Eq. (A.1),

$$(\gamma\alpha l)e^{-\gamma\alpha l}\exp(\Lambda_k''\cot\Lambda_k'')\cdot\sin\Lambda_k''/\Lambda_k''=1, \qquad \text{(A.3)}$$

and

$$\Lambda_k'=(1+\gamma)\alpha l-(\Lambda_k''\cot\Lambda_k''). \qquad \text{(A.4)}$$

Λ_k'' can be determined graphically from Eq. (A.3), Λ_k' being known from Eq. (A.4) with Λ_k'' already determined. Table I shows such values of $\Lambda_{k'}$ and $\Lambda_{k'}''$ for $k'=1$—8.[16] From these calculations, it can be said that every real part of the complex root Λ_k' is algebraically smaller than the corresponding real root Λ_k and the terms $e^{\lambda_k t}$ of real λ_k become predominant in the series of N or J for large values of t. Consequently the final value of J could be reduced to the correct formula (I) or (II).

TABLE I. Complex roots of Eq. (A.1).

k'	$\Delta(\%)$	$\Lambda_{k'}'$	$\Lambda_{k'}''$
1	−1	−1.09	$2\pi+\frac{1}{2}\pi\times0.2752$
	0	−0.162	$2\pi+\frac{1}{2}\pi\times0.2764$
	+1	0.787	$2\pi+\frac{1}{2}\pi\times0.2775$
2	−1	−1.31	$4\pi+\frac{1}{2}\pi\times0.4677$
	0	−0.377	$4\pi+\frac{1}{2}\pi\times0.4682$
	+1	0.577	$4\pi+\frac{1}{2}\pi\times0.4696$
3	−1	−1.53	$6\pi+\frac{1}{2}\pi\times0.5871$
	0	−0.600	$6\pi+\frac{1}{2}\pi\times0.5884$
	+1	0.352	$6\pi+\frac{1}{2}\pi\times0.5897$
4	−1	−1.66	$8\pi+\frac{1}{2}\pi\times0.6667$
	0	−0.737	$8\pi+\frac{1}{2}\pi\times0.6677$
	+1	0.112	$8\pi+\frac{1}{2}\pi\times0.6686$
5	−1	−1.78	$10\pi+\frac{1}{2}\pi\times0.7212$
	0	−0.857	$10\pi+\frac{1}{2}\pi\times0.7222$
	+1	0.092	$10\pi+\frac{1}{2}\pi\times0.7232$
6	−1	−1.94	$12\pi+\frac{1}{2}\pi\times0.7600$
	0	−1.017	$12\pi+\frac{1}{2}\pi\times0.7609$
	+1	−0.063	$12\pi+\frac{1}{2}\pi\times0.7618$
7	−1	−2.07	$14\pi+\frac{1}{2}\pi\times0.7894$
	0	−1.147	$14\pi+\frac{1}{2}\pi\times0.7902$
	+1	−0.198	$14\pi+\frac{1}{2}\pi\times0.7910$
8	−1	−2.19	$16\pi+\frac{1}{2}\pi\times0.8124$
	0	−1.322	$16\pi+\frac{1}{2}\pi\times0.8131$
	+1	−0.313	$16\pi+\frac{1}{2}\pi\times0.8138$

The neglect of complex roots give N_+ and J_+ a small discontinuity at $t=t_-$ for the γ_p mechanism and at $t=\bar{t}$ for the γ_i mechanism. The author will discuss this subject more extensively in the future.

[16] The suffix k' is the ordinal number for complex roots.