Evaluation of an Infinite Series

Let

$$f(x)=\pi e^{iax}/\sin\pi a, \tag{A.4}$$

and note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-inx}f(x)dx=\frac{(-1)^n}{a-n}. \tag{A.5}$$

This means that

$$f(x)=\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{a-n}e^{inx}\quad\text{for}\quad -\pi<x<\pi, \tag{A.6}$$

and also

$$\sum_{n=-\infty}^{\infty}\frac{1}{a-n}=\tfrac{1}{2}[f(\pi)+f(-\pi)]=\frac{\pi}{\tan(\pi a)}. \tag{A.7}$$

PHYSICAL REVIEW VOLUME 103, NUMBER 5 SEPTEMBER 1, 1956

Nonlinear Spinor Field*

R. Finkelstein, C. Fronsdal, and P. Kaus†
University of California, Los Angeles, California
(Received April 2, 1956)

A classical spinor field is defined by a variational principle on a Lagrangian with quadratic Dirac and quartic Fermi terms. Localized (particle-like) solutions are found within a class of comparison functions which make the angular momentum stationary for a given charge. It is found that the existence of eigensolutions depends in a radical way on the parameters of the Lagrangian, but that the observable properties of those solutions which do exist depend little on these parameters.

INTRODUCTION

NEW knowledge of the elementary particles is currently recorded by simply adding new terms to the Lagrangian of the total field. Although there is no doubt that this procedure is only provisional, attempts to make inferences about the intrinsic structure of the total field have, except for some efforts to guess new symmetries,[1] been unrelated to experiment. Nevertheless these theories[2] have attracted wide interest, and in any case the problem remains. By considering a very simple model, we shall attempt to present some results on one of the well-known questions which these theories raise, namely: can all the elementary particles be represented as eigenstates of a single underlying field?

The different fundamental theories appearing in the literature have in common the feature (a) that the equations of motion are *nonlinear* partial differential equations. They may be classified by (b) the group of the theory, e.g., the Lorentz group, or some wider group, like that of general relativity, or the generalized theory of gravitation.[2] They may be further classified by (c) their relation to quantum theory: most are quantized in the conventional Hamiltonian way. On the other hand, as is well known, Einstein expected that it would not be necessary to supplement the complete classical field equations with quantum postulates. The recent literature contains several papers in which similar and other unconventional views of the quantum theory are discussed.[3–7] The model to be discussed here will be characterized by (a) nonlinear equations of motion and (b) Lorentz rather than general covariance. We do not discuss point (c); however, the following analysis will be entirely classical.

Dirac has rather recently proposed a new classical theory of the electron.[8] Schrödinger has shown how this theory may be described as a Klein-Gordon-Schrödinger field coupled to a Maxwell field in the usual way, although with a particular choice of gauge.[9] It had been pointed out earlier that there is a class of classical field theories which may be arrived at in this same way—by coupling different representations of the Lorentz group through Lorentz-invariant interactions—i.e., simply by interpreting as classical and unitary precisely the total fields ordinarily considered only in terms of quantum field theory.[10] As a consequence of Schrödinger's remark, Dirac's new field may be related to this class.

Another example belonging to this same class may be arrived at by coupling the Maxwell field to the spinor field; this procedure leads to the differential equations of quantum electrodynamics, except that now the amplitudes are regarded as unquantized. Then, by eliminating the photon field, one obtains[10–12]

$$\gamma_\mu\partial_\mu\psi+\kappa\psi+e^2\int d^4x'[\bar{\psi}(x')\gamma_\mu\psi(x')]\times D_F(x'-x)\gamma_\mu\psi(x)=0,$$

* This work was supported in part by the National Science Foundation.
† Now at RCA Laboratories, Princeton, New Jersey.
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where D_F is the causal Green's function of the Maxwell field. If the boson field has a finite mass, and different tensor character, D_F is replaced by Δ_F and γ_μ by the appropriate matrix.

Ignoring the derivation of this equation, one may regard it as the condition fixing a spinor field which is both nonlocal and nonlinear; the range of the nonlocalization is evidently the Compton wavelength of the associated boson, and vanishes in the limit of infinitely heavy mass. Whenever a spinor field and a boson field are coupled, it is possible to eliminate either the one or the other. If the spinor field is eliminated, the resulting equations of motion, which contain boson amplitudes only, may be characterized as having the "unitary field form." On the other hand, if the boson fields are eliminated the resulting equations may be referred to as having the "action-at-a-distance" form, since only the matter variables, the spinor amplitudes, now appear.[13] It is now possible in principle to quantize either form, i.e., the quantization may be carried through *after* the elimination of either of the coupled fields.

DIRAC-FERMI FIELD

Passing now to the limit of infinitely heavy boson, one obtains a differential equation in which the nonlinear term is a beta-type interaction

$$\gamma_\mu\partial_\mu\psi+\kappa\psi+g^2[\bar{\psi}\gamma_\mu\psi]\gamma_\mu\psi=0.$$

An equation of this type has already been studied from both the classical and quantum viewpoints.[14,15] In this paper we shall continue the investigation of the Lagrangian studied in A. This Lagrangian (which may be referred to as the Dirac-Fermi case) is known to have regular and quadratically integrable (particle-like) solutions. If the charge or spin of such a solution is assigned, the rest mass has a discrete spectrum; these masses are discrete for the same reason as the allowed masses of the hydrogen atom, although the eigenproblem is not of the Schrödinger type but is nonlinear.

Let the Lagrangian density be

$$L=D+\gamma F, \tag{1}$$

where D is the "classical" Dirac part,

$$D=-\tfrac{1}{2}[\bar{\psi}\gamma_\lambda\partial_\lambda\psi-\partial_\lambda\bar{\psi}\gamma_\lambda\psi]-\mu\bar{\psi}\psi, \tag{1a}$$

in which $\bar{\psi}\psi$ has the dimensions of (charge^2/volume); and F is the most general combination of Fermi-type invariants:

$$F=\sum_{\sigma=1}^{5} c_\sigma(\bar{\psi}\Gamma_\rho{}^\sigma\psi)(\bar{\psi}\Gamma_\rho{}^\sigma\psi). \tag{1b}$$

We denote these invariants by S, V, T, A, P as in the theory of beta decay. Since the four spinors are identical, the following identities hold:

$$\begin{aligned} S-T+P&=0,\\ V-A&=0,\\ S-A-P&=0. \end{aligned} \tag{2}$$

[To prove these identities, we recall that these forms are solutions of the equation,

$$F(a,b,c,d)=-F(a,d,c,b),$$

where a, b, c, d may be any four spinors.[16]] The result of combining (1b) and (2) is

$$F=aS+bP, \tag{3}$$

$$a=c_S+c_V+c_T+c_A, \tag{3a}$$

$$b=c_P-c_V+c_T-c_A. \tag{3b}$$

Since the most general form of F is simply a linear combination of scalar and pseudoscalar,[17] one is led to attempt a survey of the entire class of theories characterized by (1). Unfortunately this does not prove possible. The class of solutions which it has been possible to obtain is described in the next paragraph.

SPECIAL SOLUTIONS

One asks for solutions of the variational equation

$$\delta\mathfrak{L}(\psi,\partial_\mu\psi)=0, \tag{A}$$

within a class ($\mathfrak{A}$) of comparison functions, ψ, with the following properties: (a) quadratic integrability, (b) harmonic time dependence, and (c) the angular dependence:

$$\psi=\begin{bmatrix} F\Omega_1{}^\pm \\ F\Omega_2{}^\pm \\ iG\Omega_3{}^\pm \\ iG\Omega_4{}^\pm \end{bmatrix},\quad \text{where}\quad \Omega^+=\begin{bmatrix} aY_{j-\frac{1}{2}}{}^{m-\frac{1}{2}} \\ bY_{j-\frac{1}{2}}{}^{m+\frac{1}{2}} \\ cY_{j+\frac{1}{2}}{}^{m-\frac{1}{2}} \\ dY_{j+\frac{1}{2}}{}^{m+\frac{1}{2}} \end{bmatrix}.$$

Functions of this particular form lead to time independent densities. They also have the property of minimizing the angular momentum for an assigned value of the charge. The functions $\Omega^\pm$ are eigenfunctions of the operator

$$k=\beta[-i\boldsymbol{\sigma}(\mathbf{r}\times\boldsymbol{\nabla})+1].$$

[13] The equations studied in this paper are nonlinear in the spinor amplitudes, and thus correspond, according to the distinction just made, to an action-at-a-distance representation. We hope to clarify this point of view more fully elsewhere. However, in this paper we shall adhere to the more conventional interpretation and language and shall regard these equations as describing nonlinear spinor fields.

[14] Finkelstein, LeLevier, and Ruderman, Phys. Rev. **83**, 326 (1951). This paper will be referred to here as A.

[15] W. Heisenberg, Z. Naturforsch. **9a**, 292 (1954).

[16] See, for example, R. Finkelstein and P. Kaus, Phys. Rev. **92**, 1316 (1953).

[17] Note that $(\bar{\psi}\psi)^2$ and $\Sigma_1{}^5(\bar{\psi}\sigma_\mu\psi)^2$ (Rosenfeld-Møller combination) are the same: $a=1$, $b=0$ for both. The remarks in A about Eq. (6) were based on an error in the reduction of (5b) and make no sense. However, it will appear that the equations actually investigated there, (16a) and (16b), are correct [see Eqs. (5a) and (5b)].

Thus

$$k\Omega^{\pm}=\pm(j+\tfrac{1}{2})\Omega^{\pm}.$$

When a ψ belonging to $\mathcal{A}$ is substituted in the Lagrangian density and the angular dependence is integrated out, the variational problem reduces to the determination of the best radial functions $F(r)$ and $G(r)$. The radial Lagrangian may be written as the sum of two parts

$$\mathfrak{D}=\int\{GF'-FG'-2kr^{-1}FG-\omega(F^2+G^2)+\mu(G^2-F^2)\}r^2dr,$$

$$\mathfrak{F}=\int\{ac_1(k)(G^2-F^2)^2-4bc_2(k)G^2F^2\}r^2dr, \tag{4}$$

and the corresponding differential equations obtained by making independent variations of F and G are

$$G'+(k+1)r^{-1}G+(\mu+\omega)F+2\gamma F[ac_1(G^2-F^2)+2bc_2G^2]=0, \tag{5a}$$

$$F'+(1-k)r^{-1}F+(\mu-\omega)G+2\gamma G[ac_1(G^2-F^2)-2bc_2F^2]=0. \tag{5b}$$

Here

$$c_1(k)=\int_0^{2\pi}\int_0^{\pi}|Y_{|k|-1}{}^{|k|-1}|^4\sin\theta d\theta d\varphi,$$

$$c_2(k)=\int_0^{2\pi}\int_0^{\pi}|Y_{|k|-1}{}^{|k|-1}|^4\cos^2\theta\sin\theta d\theta d\varphi.$$

If one attempts to satisfy conditions (A) over an unrestricted class of functions, then one is led to a set of partial differential equations—say (P); but if the comparison functions are limited to $\mathcal{A}$, then it is only necessary to deal with total differential equations. In general, functions with the special time and angle dependence of $\mathcal{A}$ will not satisfy (P); however, in the case $b=0$, $(c_P+c_T=c_V+c_A)$ functions belonging to $\mathcal{A}$ do satisfy the partial differential equations rigorously. The constant a appearing in (5) will now be set equal to unity, without loss of generality.

EIGENSOLUTIONS WITH MINIMUM SPIN

In Eq. (5) μ, γ, and b are constants which characterize the original Lagrangian, while k, which takes on the values $\pm1, \pm2, \cdots$, fixes the total angular momentum. In a given theory then, the Eqs. (5) depend on the discrete parameter k and the continuous one, ω. Here only the solutions with minimum spin, for which $|k|=1$, are to be considered. For $k=+1$, Eqs. (5) become

$$\begin{aligned}&\frac{df}{dx}+(1+\beta)g+(g^2+\tfrac{1}{2}\lambda f^2)g=0,\\ &\frac{dg}{dx}+\frac{2}{x}g+(1-\beta)f-(f^2+\tfrac{1}{2}\lambda g^2)f=0,\end{aligned} \tag{6}$$

where

$$\lambda=-2(1+\tfrac{2}{3}b),$$
$$x=\mu r,$$
$$\beta=-\omega/\mu,$$
$$f(x)=(\gamma/2\pi\mu)^{\frac{1}{2}}F(r),$$
$$g(x)=(\gamma/2\pi\mu)^{\frac{1}{2}}G(r).$$

We now look for eigensolutions, defined to be those solutions of (6) which are finite at the origin and quadratically integrable. Let $f(0)$ and $g(0)$ be values of f and g at the origin. Any solution is completely specified by $(f(0), g(0), \beta)$; and an eigensolution, since it must be finite at the origin, is completely specified by $(f(0), 0, \beta)$. We recall from the discussion in (A) the following fact: given β, only particular values of $f(0)$ lead to eigensolutions; or given $f(0)$, only particular values of β are allowed. There is thus a one-parameter family of eigensolutions, and this single parameter is fixed as soon as either the angular momentum or the charge integral is specified.

MASS, CHARGE, AND SPIN

The charge-current density is $\epsilon(\bar{\psi}\gamma_\mu\psi)$ as usual, where ϵ is a coupling constant. The eigenfield of a particle with charge e then satisfies the condition

$$\int\psi^*\psi d\mathbf{x}=e/\epsilon\equiv q^2, \tag{7}$$

where q^2 has the dimensions of a square of a charge, and equals $\hbar c$ in the usual theory. Any field satisfying (7) carries the spin

$$\begin{aligned}\int\psi^*\left(\frac{1}{i}\frac{\partial}{\partial\varphi}+\frac{1}{2}\sigma_z\right)\psi d\mathbf{x}&=q^2/2c\\ &=\hbar/2\quad\text{if}\quad q^2=\hbar c.\end{aligned} \tag{8}$$

A condition like (7) or (8) is needed to pass from a continuous to a discrete spectrum, and represents an essential difference between the present nonlinear situation and the usual linear one, in which the spectrum is independent of the normalization. In the present example conditions (7) and (8) are equivalent; usually they are not, and in the case of neutral particles only the angular momentum normalization could be used.

In general there may be several solutions corresponding to the same charge and spin; and when there are, these may be expected to have different mass integrals:

$$Mc^2=-\int T_{44}d\mathbf{x}. \tag{9}$$

For fields, like those of $\mathcal{A}$, with vanishing momentum, this integral expresses the rest mass, and for the given Lagrangian (1) the mass is

$$Mc^2=\mu q^2(\beta+I_C/I_Q), \tag{10}$$

where

$$I_Q=\int_0^\infty (f^2+g^2)x^2dx, \tag{10a}$$

$$I_C=\int_0^\infty (f^4+g^4+\lambda f^2g^2)x^2dx. \tag{10b}$$

In this notation the nonlinear coupling constant, γ, is

$$\gamma=2\pi(I_Q/\mu q^2). \tag{11}$$

From these equations it follows for a given Lagrangian (fixed γ, λ, and μ) and a given spin or charge (assigned q), that I_Q is fixed [Eq. (11)]. As a consequence, eigensolutions of (6) must be found only for a preassigned value of the integral (10a). That is possible only for particular values of β, and therefore of M.

ANALYSIS IN PHASE PLANE[18]

The equations of motion (6) may be written in the following form:

$$\frac{df}{dx}=-\frac{\partial H}{\partial g}, \tag{12a}$$

$$\frac{dg}{dx}=\frac{\partial H}{\partial f}-\frac{2}{x}g, \tag{12b}$$

where

$$H=\tfrac{1}{2}\{(1+\beta)g^2-(1-\beta)f^2+\tfrac{1}{2}(f^4+g^4+\lambda f^2g^2)\}. \tag{12c}$$

It follows that

$$\frac{dH}{dx}=-\frac{2}{x}g\frac{\partial H}{\partial g}=\frac{2}{x}g\frac{df}{dx}. \tag{13}$$

If g remains bounded, as it does for an eigensolution, then as x becomes infinite, the dissipative term vanishes, and the motion becomes "conservative." The asymptotic motion in the finite part of the plane is therefore described by the Hamiltonian equations

$$\frac{df}{dx}=-\frac{\partial H}{\partial g}, \tag{14a}$$

$$\frac{dg}{dx}=\frac{\partial H}{\partial f}. \tag{14b}$$

The equilibrium or singular points of the motion are defined by

$$\frac{dg}{dx}=\frac{\partial H}{\partial f}=f[\beta-1+f^2+\tfrac{1}{2}\lambda g^2]=0, \tag{15a}$$

$$-\frac{df}{dx}=\frac{\partial H}{\partial g}=g[\beta+1+g^2+\tfrac{1}{2}\lambda f^2]=0, \tag{15b}$$

with the following solutions:

(a) $f=g=0$,

(b) $f=0,\quad g^2=-(1+\beta)$,

(c) $g=0,\quad f^2=1-\beta$,

(d) $g^2+\frac{1}{2}\lambda f^2=-(1+\beta),\quad \frac{1}{2}\lambda g^2+f^2=1-\beta.$ (16)

In the neighborhood of an equilibrium point, we have

$$\Delta H=\tfrac{1}{2}[H_{ff}(\Delta f)^2+2H_{fg}(\Delta f)(\Delta g)+H_{gg}(\Delta g)^2], \tag{17}$$

where the nature of the level lines is determined by the discriminant

$$D=[H_{ff}H_{gg}-(H_{fg})^2] \tag{18a}$$

$$=(1+\beta+3g_0^2+\tfrac{1}{2}\lambda f_0^2)(\beta-1+3f_0^2+\tfrac{1}{2}\lambda g_0^2)-\lambda^2f_0^2g_0^2. \tag{18b}$$

At the four singular points (a), (b), (c), and (d),

$$D_a=\beta^2-1, \tag{19a}$$

$$D_b=-2(1+\beta)[\beta-1-\tfrac{1}{2}\lambda(1+\beta)], \tag{19b}$$

$$D_c=2(1-\beta)[\beta+1+\tfrac{1}{2}\lambda(1-\beta)], \tag{19c}$$

$$D_d=4[(1+\tfrac{1}{2}\lambda)^2-\beta^2(1-\tfrac{1}{2}\lambda)^2][\tfrac{1}{4}\lambda^2-1]^{-1}. \tag{19d}$$

The equations of motion in the neighborhood of these points are

$$\frac{d}{dx}\begin{pmatrix}\Delta f\\ \Delta g\end{pmatrix}=\begin{pmatrix}-H_{fg} & -H_{gg}\\ H_{ff} & H_{fg}\end{pmatrix}\begin{pmatrix}\Delta f\\ \Delta g\end{pmatrix}. \tag{20a}$$

The characteristic values of $||H||$ are $\pm(-D)^{\frac{1}{2}}$ so that the singularities are either centers or saddle points; and

$$\frac{d^2}{dx^2}\begin{pmatrix}\Delta f\\ \Delta g\end{pmatrix}=-\begin{pmatrix}D & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}\Delta f\\ \Delta g\end{pmatrix}. \tag{20b}$$

A necessary condition for the existence of an eigensolution is that Δf and Δg approach zero exponentially as x becomes infinite. That is, by (20b), $D<0$, or the origin must be a saddle point. Therefore, by (19a) $|\beta|<1$. Hence the solutions of (b) are never real, and those of (c) are always real. By direct calculation, one finds that the real solutions of (d) lie only in the shaded region of Fig. 1 where the curved boundary is

$$\lambda^*(\beta)=-2(1+\beta)/(1-\beta). \tag{21}$$

The signs of the discriminants are also shown, i.e.,

$$(\lambda-\lambda^*)D_c>0,$$
$$(\lambda-\lambda^*)D_d<0. \tag{22}$$

We designate the different areas of the figure as follows:

right of shaded area, (I)
right shaded area, (II)
left shaded area, (III)
left of shaded area. (IV)

[18] This paragraph is a simplification and extension of the analysis of A. Some of the results given here are also contained in the thesis of R. LeLevier, University of California, Los Angeles, California, 1952 (unpublished).

In the unshaded region, the only singularities besides (a) are at (c) and these are centers or saddle points, as indicated. In (III) there are four d-saddle points and two c centers. In (II) there are four d centers and two c-saddle points. In addition the origin is always a saddle point. Hence in the interval $-2<\lambda<2$ the number of centers always exceeds the number of saddle points by one. This is an illustration of the fact that the Poincaré index of a closed path is $+1$.[19]

The complete classification of the singular points of the conservative motion is shown in Fig. 1. To discuss the actual (nonconservative) solutions of (12), we need to consider (13). In particular, it is useful to study the curve $dH/dx=0$, or

$$g\partial H/\partial g=0. \tag{23}$$

Equation (23) describes (in addition to the f axis) a hyperbola dividing the phase plane into two regions, in each of which the sign of dH/dx is definite. It is important to know under what conditions $dH/dx=0$ intersects $H=0$; direct calculation leads to the following conditions for this intersection:

$$\lambda>-2(1+\beta^2)/(1-\beta^2)\equiv\lambda^{**}, \tag{24a}$$

$$|\lambda|<2:\lambda>-2(1-\beta)/(1+\beta), \tag{24b}$$

$$|\lambda|>2:\lambda<-2(1-\beta)/(1+\beta). \tag{24c}$$

It is not possible to satisfy these conditions in IIIb or IV.

All information is now summarized in Figs. 2(I)–2(IV). These figures are all symmetric in both f and g, but only Fig. 2(I) is shown completed. The figures show the paths passing through the saddle points (the separatrices) and the hyperbola (23).

Examination of IIIb and IV reveals that there is no possibility of connecting a point on the f axis with the origin by a solution curve, i.e., *there are no eigensolutions in regions* IIIb *and* IV. In addition, one may show in IIIa that there exists no eigensolution with an initial f to the right of line d, i.e., the condition there is

$$f_0^2<\{1-\beta+[(1-\beta)^2+h]^{\frac{1}{2}}\}, \tag{25a}$$

where h is the value of H at point d, namely,

$$h=[2+\lambda+(2-\lambda)\beta^2][\lambda^2-4]^{-1}. \tag{25b}$$

CASE $k=-1$

According to Eq. (5), we now have

$$\frac{df}{dx}+\frac{2}{x}f=-\frac{\partial H(f^2,g^2)}{\partial g},\qquad \frac{dg}{dx}=\frac{\partial H(f^2,g^2)}{\partial f}. \tag{26}$$

[19] According to (12c) if f and $g\gg1$, $H\cong\frac{1}{4}(f^4+g^4+\lambda f^2g^2)$. This line, which is a *path*, is closed if $|\lambda|<2$. Hence, if $-2<\lambda<2$, it is possible to find a path which encloses all singularities. The index of this path is the number of centers minus the number of saddle points.

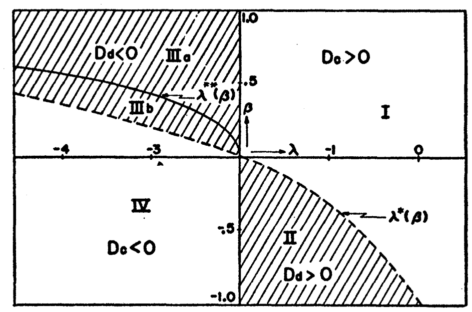

FIG. 1. Singular points (c) and (d) as functions of λ and β. These points are defined in Eq. (16). $D<0$ ($D>0$) indicates saddle point (center). The origin (a) is not shown and is always chosen to be a saddle point. The curve $\lambda^{**}(\beta)$ is defined by Eq. (24).

We next make the substitution $f\to g$ and $g\to f$. Then

$$\frac{dg}{dx}+\frac{2}{x}g=\frac{\partial H(g^2,f^2)}{\partial f},\qquad \frac{df}{dx}=-\frac{\partial H(g^2,f^2)}{\partial g}. \tag{27}$$

These equations are of precisely the form (12) except that f and g are interchanged in H. All the preceding analysis may now be applied. Figure 1 remains valid and Fig. 2 need only be rotated through $\pi/2$. The line $H'=0$ will, however, be different. In addition to the f axis, it consists of

$$2g^2+\lambda f^2=1-\beta.$$

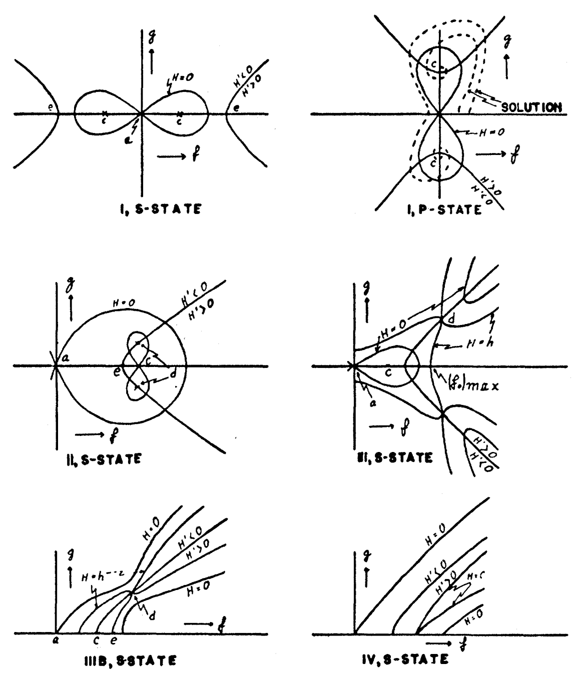

FIG. 2. Phase portraits. The various cases are defined in Fig. 1. Thus IIIb means that the parameters (λ,β) lie in the domain IIIb of Fig. 1. All cases correspond to $k=+1$ except for that labeled p state, for which $k=-1$.

In the previous case, $k=+1$, the line $H'=0$ did not exist for positive λ, but now it is seen to be a closed curve. When λ is negative, $H'=0$ has approximately the same relation to $H=0$ for the $k=+1$ and $k=-1$ cases; but for $k=-1$ it always passes through the points (c) and (d). As before, there are no eigensolutions in the parameter regions III*b* and IV. Again, in III*a* the eigensolution must start to the left of line *d*. Typical solutions are illustrated in Fig. 2 (I, *P* state).

ASYMPTOTIC SOLUTIONS

We shall finally give the exact solution of the asymptotic nonlinear equations in the special case $\lambda=-2$. Then the differential equations are

$$f'=-(1+\beta)g+(f^2-g^2)g, \tag{28a}$$

$$g'=-(1-\beta)f+(f^2-g^2)f, \tag{28b}$$

and

$$\frac{1}{2}\frac{d}{dx}(f^2-g^2)=-2\beta fg. \tag{28c}$$

We introduce the polar variables (J,θ) such that

$$f=J^{\frac{1}{2}}\cosh\theta, \tag{29a}$$

$$g=J^{\frac{1}{2}}\sinh\theta. \tag{29b}$$

Then

$$f^2-g^2=J,$$
$$2fg=J\sinh 2\theta.$$

J is the scalar invariant, and 2θ has the significance of a Lorentz angle since $\tanh\theta=g/f$. Equation (28c) becomes

$$\tfrac{1}{2}dJ/dx=-J\sinh 2\theta. \tag{30}$$

The "energy" integral in the new variables is

$$H=\tfrac{1}{2}\{(1+\beta)J\sinh^2\theta-(1-\beta)J\cosh^2\theta+\tfrac{1}{2}J^2\}. \tag{31a}$$

This may be solved for $\cosh\theta$:

$$\cosh\theta=\frac{2}{\beta}\frac{1}{J}[2H+J-\tfrac{1}{2}J^2] \tag{31b}$$

After substitution in (30), we find

$$x=-\frac{1}{2}\int_{f_0^2}^{J}[(4H+2J-J^2)^2-\beta^2J^2]^{-\frac{1}{2}}dJ. \tag{32}$$

The inversion of this elliptic integral is given in the appendix.

NUMERICAL RESULTS

In (A) the case $\lambda=-2$ was investigated: it was found that for a given value of the coupling constant, γ, there exists only a small number of eigenfields—in general qualitative agreement with the experimental fact that there are only a few elementary particles with spin $\hbar/2$. The mass ratio between the lightest and the heaviest turned out to be of the order of 3 rather than 10^3; but on the other hand some preliminary numerical results in the neighborhood of $\lambda=-2$ suggested that this ratio was greater near $\lambda=0$ (more pseudoscalar). The reason for extending the numerical work was in the first place to decide whether the mass ratio does become large for a particular choice of λ, and more generally to accumulate a certain amount of empirical information about equations of this general type.

The following method was found useful in analyzing the results.[20] A solution will cross the $g=0$ axis several times. Let the crossings be $f_1, f_2\cdots f_n\cdots$, and let the one which is closest to the origin, be $f_{\min}$. When $f_{\min}=0$, the solution is an eigensolution. The $f_{\min}$ are plotted against the starting values f_0. The zeroes of that graph correspond to the eigensolutions. Figure 3 gives such a graph for $\lambda=-1$ and $\beta=0.1$, 0.5 and 0.9. Values of f_0 as high as 1000 have been tried. On the basis of Fig. 3 it was decided that no solutions exist for $\lambda=-1$ when $\beta=0.1$ or 0.5, while solutions do exist for $\beta=0.9$. A similar analysis was performed for other

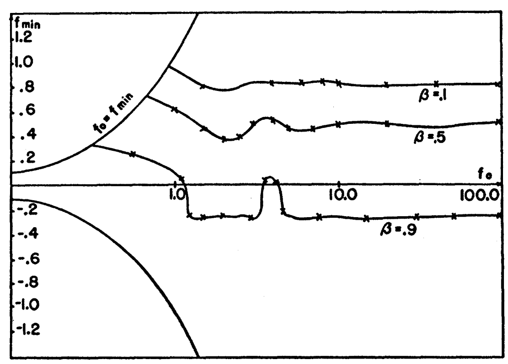

FIG. 3. Plot of f_0 *vs* $f_{\min}$ for $\lambda=-1$.

λ values to decide which λ and β combinations lead to eigensolutions and which are "forbidden."

The results are summarized in Tables I and II and in Fig. 4. In this figure, a solid circle indicates a solution and a cross means that the corresponding point was tested but that no solution was found. *BC* is an approximate boundary drawn through the gates which are shown in the figure. The "elliptic" interval ($|\lambda|<2$) has been covered for $k=+1$. In Table II some results for $k=-1$ are also shown.

In addition, the solutions have the following properties:

(a) When there is a particle-like solution at all, its radius is always of the order of unity (between 0.5 and 3.0) and the field intensity at the origin also does not vary much ($f_0\cong 1$). However, the maximum field intensity ($f_{\max}$) may be much larger than f_0 (when $\lambda=-1.998$ and $\beta=0.1$, for example, there is a solution

[20] These results were obtained with the UCLA high-speed digital computer (SWAC).

for which $f_{max}=30$). This has the result that the charge may be concentrated on a shell near f_{max}. For example, at $\lambda=-2$, the radius of this shell is approximately unity.

(b) The mass depends on λ and β but very little on the number of nodes.

(c) As λ is increased from -2 towards the right (by adding more pseudoscalar), the interval (β_{min}, 1.0). into which all the solutions are crowded, contracts As a result, one finds that masses at $\lambda=+2$ are nearly equal (since they depend mainly on β). It is conjectured that when λ is slightly larger than $+2$, the masses are still closer.

(d) At the boundary between allowed and forbidden regions, a qualitatively new feature was noticed; this is illustrated by the bracketed solutions in the table. On the basis of results reported in (A), it was believed that

TABLE I. S state, $f_{max}(\lambda, \beta$, number of nodes). The numbers in this table represent the maximum value of f. None means that no solution was found for $f_0<1000$. Several numbers in bracket indicate families of 0-node solutions.

λ	β	0-node	1-node	2-node
	0.9	1.05	1.57	2.3
-2	0.5	1.85	4.00	7.5
	0.1	3.60	11.0	16.8
	0.9	1.05	1.57	2.3
-1.9996	0.5	1.85	4.0	7.5
	0.1	5.0	12.0	24.0
	0.9	1.05	1.57	2.3
-1.998	0.5	1.85	4.0	7.5
	0.1	[8, 16, 22, 24, 27, 30$\cdots$]	none	none
	0.9	[1.15, 4.0, 5.0]	none	none
-1	0.5	none	none	none
	0.1	none	none	none
	0.9	[1.37, 3.25]	none	none
0	0.5	none	none	none
	0.1	none	none	none

there was never more than one solution with n nodes for given λ and β. According to the table, however, there may be many nodeless solutions corresponding to a single choice of these parameters, and it is further conjectured that there may also be many solutions with n nodes. Such sets also appear for example in the familiar linear Schrödinger eigenproblem of the hydrogen atom, where the number of nodes depends on $n-l-1$.

(e) At no λ greater than -2 are there large mass ratios. The ratios become largest at $\lambda=-2$ where functions belonging to $\mathcal{A}$ satisfy the partial differential equations (P) based on (A) exactly. The results, illustrated in Fig. 5, are at $\lambda=-2$ very close to those found before.

In Fig. 5, the p solutions are also shown. The degeneracy between $s_{\frac{1}{2}}$ and $p_{\frac{1}{2}}$ is here removed by the nonlinear term. As we saw in the phase analysis, the s and

TABLE II. P state, $f_{max}(\lambda, \beta$, number of nodes). The numbers in parentheses are radii.

λ	β	0-node	1-node	2-node
	0.9	0.53 (3.25)	1.37 (2.25)	2.50 (2.00)
	0.775	1.03 (2.90)	2.50 (2.25)	4.48 (2.00)
-2	0.65	1.52 (3.00)	3.63 (2.35)	6.50 (2.00)
	0.5	2.0	4.5	8.50
	0.1	overflow	overflow	overflow
-1.5	0.9	0.73	spirals	spirals
	0.75	spirals	spirals	spirals
-1.35	0.9	1.0	spirals	spirals
-1.0	0.9	spirals	spirals	spirals

p solutions are not at all alike. However, it turns out that their masses are not very different.

The region IIIa of Fig. 1 has not been explored[21] and larger mass ratios have therefore not been altogether excluded. However, if this region conforms to the pattern of Fig. 5 (of course it may not), then large mass ratios seem to correspond to large values of the coupling constant, if they appear at all. This, according to Eq. (11), implies large I_Q, and according to the empirical results (Table I) corresponds to small β.[22] But according to Fig. 1 there is very little room for such solutions in region IIIa.

DISCUSSION

According to Fig. 4 we have a rather complete picture, except for the region IIIa in which, however, it is possible that there are no solutions at all. Measured by the spread in mass, the results are negative. However, the whole analysis is severely restricted by the initial limitation of the comparison functions to the

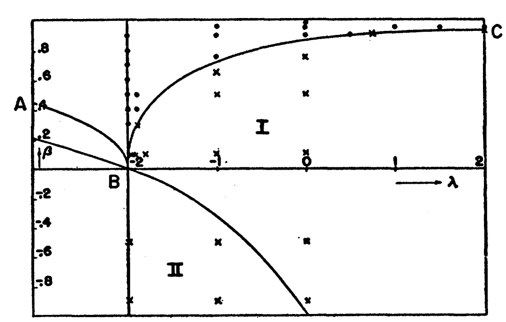

FIG. 4. Summary of SWAC results. Solid circle indicates a solution. Cross means that corresponding point was tested but that no solution was found.

[21] A quite different code would have been required for this region.

[22] In region III, as in region I, small β values can be reached only in the neighborhood of $\lambda=-2$, as indicated by the curve AB in Fig. 4. In region I small β is a necessary condition for large mass ratios, since according to Eq. (10), β is the lower limit of the mass for $\lambda \geq -2$. However, in region III this is not the case, because the nonlinear contribution to the mass, I_C, is no longer positive definite. Therefore small masses are theoretically possible in III even with β values near unity.

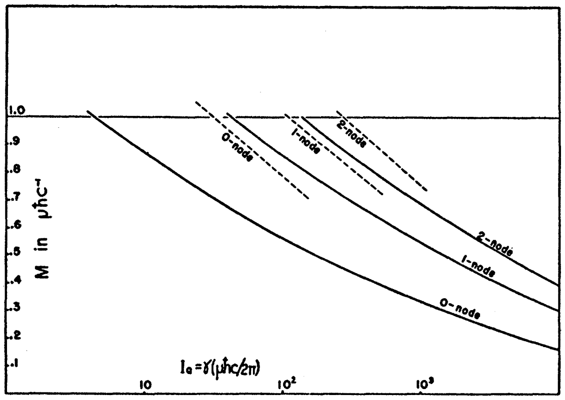

FIG. 5. The mass curves for the $\lambda=-2$ solutions as function of the coupling constant. The continuous (dashed) lines refer to the $s(p)$ solutions.

class $\mathfrak{A}$. It is perhaps worth repeating that the functions belonging to $\mathfrak{A}$ do not satisfy the partial differential equations P except when $\lambda=-2$. It has not been possible to get away from this initial restriction. We are even further from knowing the effect of a nonlocality in the nonlinear interaction. Therefore, although the results presented here represent a considerable increase in empirical (numerical) information over the results of (A), they must still be regarded as fragmentary.

The most striking results are (a) the existence of a small number of discrete masses and (b) a radical dependence of the theory on the nature of the interaction; this second point is brought out most clearly in the discontinuous changes in the appearance of the phase portraits. An unpredictable incorrect feature is the result that the allowed masses all have the same order of magnitude. Since mass density depends on the fourth power of the field intensity, ratios of maximum field intensities of the order of 40 might appear to be sufficient to produce mass ratios of the order of 1600 for particles of the same charge (with given $\int(f^2+g^2)d\mathbf{x}$); although such intensity ratios do appear, the structure of the solution is always such that the mass never becomes very large. Thus large mass ratios are qualitatively possible, but apparently are not realized for the postulated Lagrangian. However, as we have said, the results on which these remarks are based, are quite incomplete. The main deterrent to investigations of the type being considered is the apparent difficulty of connecting with quantum mechanical descriptions of the elementary particles. A particle theory such as we have been considering raises statistical problems which have been much hinted at but not really understood and which may or may not correspond to quantum mechanical uncertainties. Without going into these problems, we wish to suggest a tentative connection between the type of eigensolution found here and the form factors of nonlocal quantum field theories.[23]

[23] R. Finkelstein, Nuovo cimento 1, 1113 (1955).

Finally it may be worth mentioning the possible relevance of our results to quantum field theories of the more usual type. In the work of Heisenberg,[15] referred to earlier, the initial q-number equations lead to c-number equations of a type very similar to those considered here.

ACKNOWLEDGMENTS

We should like to thank Dr. M. Melkanoff for his skillful and most generous assistance with SWAC.

We should also like to thank the Institute for Numerical Analysis of the National Bureau of Standards and Numerical Analysis Research, Department of Mathematics, University of California, Los Angeles for making available computing machine time under the sponsorship of the Office of Naval Research and the Office of Ordnance Research.

One of the authors (C.F.) would like to express his gratitude to Chr. Michelsen's Fund and to the Norwegian Government for financial support.

APPENDIX

The relation between x and J:

$$x=-\frac{1}{2}\int[(4H+2J-J^2)^2-\beta^2]^{-\frac{1}{2}}dJ$$

$$=-\frac{1}{2}\int S_1^{-\frac{1}{2}}S_2^{-\frac{1}{2}}dJ$$

may be expressed in terms of a Jacobian elliptic function.[24] To do this, we must choose the bilinear forms S_1 and S_2 in such a manner that the roots, if they are all real, do not interlace. This is accomplished by setting

$$\begin{aligned}S_1&=(J'-a)(J'-b),\\ S_2&=(J'+a)(J'+b),\\ a&=\{1+4H+\beta\}^{\frac{1}{2}},\\ b&=\{1+4H-\beta\}^{\frac{1}{2}},\\ J'&=J-1.\end{aligned}$$

We next look for linear combinations of S_1 and S_2 that are perfect squares:

$$S_1-\lambda S_2=(1-\lambda)(J'-\alpha)^2=(1-\lambda)J'^2-(a+b)J'(1+\lambda)+ab(1-\lambda).$$

This problem has for solution the 2 values λ_1 and λ_2 of λ that satisfy

$$4(1-\lambda_i)^2ab=(a+b)^2(1+\lambda_i)^2.$$

We note that $\lambda_1\lambda_2=1$, and that they are both negative. We choose $\lambda_1-\lambda_2>0$. For α, we find

$$\alpha_1=(ab)^{\frac{1}{2}},\quad \alpha_2=-(ab)^{\frac{1}{2}}.$$

[24] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1935), fourth edition, p. 514.

Solving for S_1 and S_2, we find

$$S_1 = A_1[J' - (ab)^{\frac{1}{2}}]^2 - B_1[J' + (ab)^{\frac{1}{2}}]^2,$$
$$S_2 = -A_2[J' - (ab)^{\frac{1}{2}}]^2 + B_2[J' + (ab)^{\frac{1}{2}}]^2.$$

Using the relation $\lambda_1\lambda_2 = 1$, we have

$$A_1 = B_2 = (1-\lambda_2)/(\lambda_1-\lambda_2) > 0,$$
$$A_2 = B_1 = (1-\lambda_1)/(\lambda_1-\lambda_2) > 0.$$

Now if we take

$$t = \{A_2/A_1\}^{\frac{1}{2}}[J' - (ab)^{\frac{1}{2}}]/[J' + (ab)^{\frac{1}{2}}],$$

we find

$$x = \pm[5/4A_2(ab)^{\frac{1}{2}}] \int [(1-k^2t^2)(1-t^2)]^{-\frac{1}{2}} dt, \quad k = A_1/A_2,$$

or

$$t = \mathrm{sn}(\pm 4A_2(ab)^{\frac{1}{2}}x, k); \quad J = (ab)^{\frac{1}{2}}(1+tk^{\frac{1}{2}})/(1-tk^{\frac{1}{2}}) + 1.$$

PHYSICAL REVIEW VOLUME 103, NUMBER 5 SEPTEMBER 1, 1956

Nonclassical Transformation in Special Relativity

FRANK B. ESTABROOK
Western Regional Office, Office of Ordnance Research, Los Angeles Ordnance District, United States Army, Pasadena, California

(Received May 14, 1956)

The "nonclassical" complex Lorentz transformation, recently introduced as a mathematical convenience by Reulos, is related to the unique factoring of any 4×4 orthogonal transformation. This factoring also shows, in 4×4 form, the well known possibility of introducing a 2×2 complex spin space, in which unimodular transformations are isomorphic to the proper future-preserving Lorentz group.

REULOS has proposed a new transformation as convenient for calculations in special relativity.[1] We may write its matrix in Minkowski space as

$$P = \begin{bmatrix} P_4 & -P_3 & P_2 & -P_1 \\ P_3 & P_4 & -P_1 & -P_2 \\ -P_2 & P_1 & P_4 & -P_3 \\ P_1 & P_2 & P_3 & P_4 \end{bmatrix}, \tag{1}$$

where

$$P_1^2 + P_2^2 + P_3^2 + P_4^2 = 1. \tag{2}$$

Another such transformation, of opposite chirality, also exists:

$$Q = \begin{bmatrix} Q_4 & -Q_3 & Q_2 & -Q_1 \\ Q_3 & Q_4 & Q_1 & Q_2 \\ -Q_2 & -Q_1 & Q_4 & Q_3 \\ Q_1 & -Q_2 & -Q_3 & Q_4 \end{bmatrix}, \tag{3}$$

where

$$Q_1^2 + Q_2^2 + Q_3^2 + Q_4^2 = 1. \tag{4}$$

These two matrices are each orthogonal by inspection; they commute; they are unique in form (up to changes of sign of row or column); all P-type matrices form a group under matrix multiplication, and similarly for all Q type. Their multiplication rules are those of quaternions: for the Q type, however, the order of the quaternion factors must be reversed. Thus if we define the unimodular matrices

$$\mathbf{Q} = \begin{pmatrix} Q_4 - iQ_3 & -Q_2 + iQ_1 \\ Q_2 + iQ_1 & Q_4 + iQ_3 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} P_4 + iP_3 & P_2 + iP_1 \\ -P_2 + iP_1 & P_4 - iP_3 \end{pmatrix}, \tag{5}$$

the multiplication rules of the 4×4 orthogonal matrices can now be written in 2×2 form:

$$\mathbf{Q}'' = \mathbf{Q}'\mathbf{Q}, \quad \mathbf{P}'' = \mathbf{P}\mathbf{P}'. \tag{6}$$

Now the remarkable fact is that *any* proper orthogonal transformation in Euclidean 4-space can be uniquely factored into a product of these two types:

$$A = QP. \tag{7}$$

Finally, if to the 4-vector (x_1, x_2, x_3, x_4) we associate a 2×2 matrix

$$\mathbf{x} = \begin{pmatrix} x_3 + ix_4 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 + ix_4 \end{pmatrix}, \tag{8}$$

it is immediately found that the determinant of $\mathbf{x}$ is the square of the length of x, and that the orthogonal transformation

$$x' = Ax \tag{9}$$

can also be written in 2×2 matrix form:

$$\mathbf{x}' = \mathbf{Q}\mathbf{x}\mathbf{P}. \tag{10}$$

[1] René Reulos, Phys. Rev. **102**, 535 (1956).