Nuclear Many-Body Problem*

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A self-contained and largely new description is given of Brueckner's method for studying the nucleus as a system of strongly interacting particles. The aim is to develop a method which is applicable to a nucleus of finite size and to present the theory in sufficient detail that there are no ambiguities of interpretation and the nature of the approximations required for actual computation is clear.

It is shown how to construct a model of the nucleus in which each nucleon moves in a self-consistent potential matrix of the form $(\mathbf{r}'|V|\mathbf{r})$ (Sec. II). The potential is obtained by calculating the reaction matrix for two nucleons in the nucleus from scattering theory. Some complications arise in the definition of the energy levels of excited nucleons (Sec. III). The actual wave function is obtained from the model wave function by an operator which takes into account multiple scattering of the nucleons by each other (Sec. IV).

The method of Brueckner is a vast improvement over the normal Hartree-Fock method since, in calculating the self-consistent potential acting on an individual particle in the model, account is already taken of the correlations between pairs of nucleons which arise from the strong internucleon forces (Sec. V). Although the actual wave function is *derivable* from a wave function which corresponds essentially to the shell model, the probability of finding a large nucleus of mass number A "actually" in its shell model state is small (of order $e^{-\alpha A}$, where α is a constant) (Sec. VI). The influence of spin is investigated (Sec. VIII). In the case of an infinite nucleus, an integral equation is obtained for the reaction matrix, just as in the theory of Brueckner and Levinson (Sec. IX).

The exclusion principle must be applied in intermediate states in solving the integral equation for the reaction matrix. This makes an enormous difference for the solution. When the exclusion principle is used, the scattering matrix is very nearly given by the Born approximation, for any well-behaved potential (Sec. X). Numerical results are given for the case when nucleons interact only in S states, an assumption which leads to saturation without a repulsive core. The agreement with observation is fair to poor, owing to the poor assumption for the interaction (Sec. XI). Brueckner's result that three-particle clusters give a small contribution to the energy is confirmed, although the numerical value is many times his result; the calculation is then extended to the case of a repulsive core (Sec. XII). The dependence of the binding energy on the mass number A is investigated for saturating and nonsaturating interactions (Sec. XIII). Terms of relative order $1/A$ are calculated, and it is shown that these terms are much smaller than Brueckner and Levinson found, making the method also applicable to relatively small nuclei (Sec. XIV). Some aspects of the problem of the finite nucleus are discussed, including that of degeneracy (Sec. XVI).

I. INTRODUCTION

NEARLY everybody in nuclear physics has marvelled at the success of the shell model. We shall use the expression "shell model" in its most general sense, namely as a scheme in which each nucleon is given its individual quantum state, and the nucleus as a whole is described by a "configuration," i.e., by a set of quantum numbers for the individual nucleons. For instance, the collective model would be included in this definition of the shell model, the only differences being that the potential in which the individual nucleons move is not spherically symmetric but ellipsoidal, and that the emphasis is on different phenomena.

The shell model, defined in this wider sense, has had many triumphs in explaining the positions and properties of states of the nucleus.[1] In addition, Feshbach, Porter, and Weisskopf[2] have shown that the fundamental concept of quantum states of individual nucleons continues to have good meaning even for free neutrons of moderate energy interacting with the nucleus.

While the success of the model has thus been beyond question for many years, a theoretical basis for it has been lacking. Indeed, it is well established that the forces between two nucleons are of short range, and of very great strength, and possess exchange character and probably repulsive cores. It has been very difficult to see how such forces could lead to any over-all potential and thus to well-defined states for the individual nucleons.

In view of this paradox, it has often been suggested to abandon the idea of interaction of nucleons in pairs inside the nucleus, and to assume instead that nucleons in large aggregates act "collectively" by creating an overall, smoothly varying potential. In particular, it has been suggested[3,4] that there be a general "meson potential" in the nucleus, with high meson density and relatively little variation of the potential from point to point. Such an assumption is again very difficult to reconcile with known facts about mesons: From the analysis[5] of the scattering of mesons of moderate energy (a few hundred Mev), it follows that the coupling of mesons with nucleons is rather *weak* at distances of the order of $\hbar/\mu c = 1.4\times10^{-13}$ cm. Indeed, the coupling at these distances is essentially proportional to the pseudo*vector* coupling constant f^2, which, according to the analysis[5] of meson scattering, has a value of only about 0.08. The pseudo*scalar* coupling constant $g^2=(2Mf/\mu)^2\approx14$ becomes important only for phenomena which involve a transfer of momentum of the order of Mc, and hence

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[1] M. G. Mayer and J. H. D. Jensen, *Elementary Theory of the Nuclear Shell Model* (John Wiley and Sons, Inc., New York, 1955).
[2] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).
[3] M. H. Johnson and E. Teller, Phys. Rev. **98**, 783 (1955).
[4] L. I. Schiff, Phys. Rev. **84**, 1, 10 (1951); **86**, 856 (1952).
[5] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

at distances of the order of the proton Compton wavelength $\hbar/Mc$. Only at such very small distances from a nucleon will the meson density become high. It is difficult to see how this should lead to a meson potential smooth in space inside the nucleus, or how nonlinear terms in the meson field could be important at an *average* position in the nucleus—of course, they will be very important in the immediate neighborhood of each nucleon. Thus one is automatically led back to the same difficulties which occur when two-body forces between the nucleons are assumed.

But even apart from the specific idea of the meson potential, there are strong arguments to show that the two-body forces continue to exist inside a complex nucleus and are not replaced by a general smooth potential.[6] These arguments have recently been summarized by Brueckner, Eden, and Francis.[7] The point is that there is abundant evidence that the nuclear wave function contains very strong components of high momentum which manifest themselves in such processes as the capture of π mesons and the photoelectric effect caused by high-energy (>100 Mev) γ rays. In both these cases, a large amount of energy without much momentum is given to the nucleus. In order for a nucleon to absorb this energy, and at the same time to conserve momentum, the nucleon must have had a large momentum before it absorbed the π meson or the γ ray; i.e., the wave function of the nucleus in its normal state must contain components corresponding to large momenta of an individual nucleon. Another process in which these components manifest themselves is the "pickup process" in which an incoming proton of high speed picks up a neutron from inside the nucleus to form a deuteron: in order to be "picked up," the neutron must have had a large momentum in the nucleus. Perhaps the best way of measuring the momentum distribution in the nucleus is to study the energy distribution of protons scattered "quasi-elastically" by a nucleon in the nucleus; this again shows large components at high momentum. All these processes show that the "potential" is fluctuating violently from point to point in the nucleus, which is compatible with the assumption that two-body forces continue to act inside the nucleus without much modification.

Similar difficulties exist in understanding the success of the "cloudy crystal ball" model. It is true that Lane and Wandel[8] could explain that the imaginary part of the potential between nucleon and nucleus is very small, but only by essentially assuming the validity of the shell model. On the other hand, Lane, Thomas, and Wigner,[9] using orthodox methods of quantum mechanics and some seemingly plausible assumptions, obtained values for this imaginary part which were about 20 times too large. This large result had always been assumed correct before the war, leading to the hypothesis of a "black" nucleus.

Brueckner has developed a powerful mathematical method which for the first time promises to resolve this paradox. In a series of papers with his collaborators, he finds that one can calculate the nuclear energy levels using a self-consistent field method, even though the forces are of short range. Furthermore, the nuclear wave function retains the strong high-momentum components which are indicated by experiment.[7]

The general method has been developed by Brueckner and Levinson,[10] on the basis of the theory of multiple scattering of Watson.[11] The theory was evaluated for a square-well interaction between nucleons by Brueckner[12]; one of his main results was that the nucleons appear to have an effective mass of only about one-half their actual mass; this point will be discussed in Secs. X and XI. Brueckner's calculation[12] essentially replaced two earlier papers[13] in which a more special assumption was made about the interaction between two nucleons, namely that it was the particular potential derived from meson theory by Brueckner and Watson.[14] The newer approach,[12] in which the nuclear interaction is required only to be in accord with the known facts on two-nucleon scattering and is otherwise left arbitrary, is more satisfactory than the older one,[13] in which the problem of deriving two-body nuclear interactions from meson theory gets mixed up with the entirely separate problem of deriving nuclear structure from a given two-body interaction.

In a further paper,[15] Brueckner finds that for a very large nucleus, the corrections to the self-consistent field approximation are exceedingly small, of the order of 1 part in 1000 of the energy. These corrections arise only from intermediate states in which three (or more) particles are successively excited. In the same paper, Brueckner develops a greatly improved scheme for perturbation theory, the "linked cluster expansion." Other corrections[10] are of relative order $1/A$, where A is the number of particles in the nucleus (see also Sec. XIV). Further papers have been concerned with applications[16] of the theory and with a different presentation of the method.[17]

In spite of its apparent great accomplishments, the

[6] See, e.g., J. Heidmann, Phys. Rev. **80**, 171 (1950); J. S. Levinger, Phys. Rev. **84**, 43 (1951).
[7] Brueckner, Eden, and Francis, Phys. Rev. **98**, 1445 (1955).
[8] A. M. Lane and C. F. Wandel, Phys. Rev. **98**, 1524 (1955).
[9] Lane, Thomas, and Wigner, Phys. Rev. **98**, 693 (1955).
[10] K. A. Brueckner and C. A. Levinson, Phys. Rev. **97**, 1344 (1955). This paper will be quoted as BL.
[11] K. M. Watson, Phys. Rev. **89**, 575 (1953); also N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).
[12] K. A. Brueckner, Phys. Rev. **97**, 1353 (1955).
[13] Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954); K. A. Brueckner, Phys. Rev. **96**, 508 (1954).
[14] K. A. Brueckner and K. M. Watson, Phys. Rev. **92**, 1023 (1953).
[15] K. A. Brueckner, Phys. Rev. **100**, 36 (1955). This paper will be quoted as BC.
[16] For example, Brueckner, Eden, and Francis, Phys. Rev. **99**, 76 (1955); **100**, 891 (1955).
[17] R. J. Eden and N. C. Francis, Phys. Rev. **97**, 1366 (1955).

theory of Brueckner *et al.* has not been readily accepted by nuclear physicists. This is in large measure the result of the very formal nature of the central proof of the theory (reference 10, Sec. II). In addition, the definitions of the various concepts used in the theory are not always clear. Two important concepts in the theory are the wave functions of the individual particles, and the potential V_c "diagonal" in these states (reference 10, Sec. V). The paper by Brueckner and Levinson defines rather clearly how the potential is to be obtained from the wave functions, but not how the wave functions can be constructed from the potential V_c. Apparently, BL assume tacitly[18] that the nucleon wave functions are plane waves, but in this case, the method is only applicable to an infinite nucleus. For a finite nucleus, no prescription is given for obtaining the wave functions.

It is the purpose of the present paper to show that the theory of Brueckner gives indeed the foundation of the shell model. It will be shown in Sec. II that a self-consistent scheme *can* be developed for a *finite* nucleus which permits determination not only of its energy levels, but also of the wave functions of the nucleons and of the whole nucleus. This extension, like the original BL scheme, already includes in the self-consistent field all two-particle correlations, which the Hartree method would include in the "configuration interaction" (Sec. V). Essentially, the Hartree method replaces the action of all other particles on a given one by an average potential, while the Brueckner method treats the interaction of any *two* particles exactly, and only replaces the action of any further particles on the interacting pair by an average. Brueckner's method would even give an improvement over the Hartree method for the electrons in an *atom*, and will undoubtedly be very important for the theory of electrons in solids.

In order to satisfy all requirements, the self-consistent potential acting on a nucleon in a finite nucleus must be taken to be a potential matrix, $(\mathbf{r}'|V|\mathbf{r})$, or transformed to momentum space, $(\mathbf{k}'|V|\mathbf{k})$. The Hartree method which uses a potential $V(r)$, and the original BL method for an infinite nucleus which gives a potential $V(k)$ depending on momentum only, are then special limits of our self-consistent potential.

An actual solution of the problem is attempted only for an infinite nucleus. In this case, the choice of wave functions is obvious, namely plane waves. Then the problem reduces to the determination of the reaction matrix of scattering theory for a given set of nucleon wave functions. This requires the solution of an integral equation in momentum space (Sec. IX). From the derivation of this equation, it is clear that the intermediate states must satisfy the exclusion principle; this fact was realized by Brueckner, but was not taken into account in his actual solution. In fact, the exclusion principle is found to have a decisive influence on the solution (Sec. X). Fortunately, the exclusion principle makes the integral equation *easier* to solve, by virtue of the rather large radius of the Fermi sphere of occupied states in momentum space. If the potential is well-behaved, in particular if it does not contain a repulsive core, the scattering matrix turns out to be essentially equal to the matrix of the interaction potential between two nucleons. This result, which would not be true without the exclusion principle, means, most surprisingly, that the theory reduces nearly to the Born approximation. The error in the Born approximation is only about 5%.

This result is closely connected with the rapid convergence of the theory itself. Brueckner[15] already realized that the correction terms due to three-particle clusters (and higher order perturbations) are small only by virtue of the Pauli principle. Our own recalculation of the three-particle cluster correction gives a result about 20 times larger than Brueckner's (Sec. XII) but still small enough to guarantee rapid convergence of the theory, *viz.*, about 1% of the main term in the interaction. The successive approximations are then (1) the Born approximation, (2) the correction arising from the solution of the two-particle scattering equation which is about 5% of the first approximation, (3) the three-particle cluster term which is about 1% of the first order. Thus the successive approximations converge by about equal steps, which is very plausible. (In Brueckner's original theory, the second approximation was about equal to the first and the third only about 0.1 percent of it, which seemed rather miraculous.) Indeed, once the importance of the Pauli principle is realized, it would seem possible—for a well-behaved nuclear interaction—to solve the entire nuclear problem by a suitably arranged perturbation theory.

This is not possible, of course, if the interaction between nucleons contains a repulsive core. In this case, the Born approximation would give an infinite result, and the reaction matrix is in no way similar to the Born approximation. To include the Pauli principle in this case, a different treatment is required; this will be discussed in a future paper by Bethe and Goldstone. The exclusion principle can be shown to act like an increase of the effective radius of the core.

In order to obtain a feeling about the orders of magnitude, a special model is calculated in Sec. XI. Since the results on repulsive-core potentials are not yet available, a different potential was postulated which would give nuclear saturation in a simple way. Interaction was assumed to exist only in s states, as Brueckner[12] did in his evaluation. The results are fair to poor; in particular, the equilibrium density comes out about 30 times too high. This is not too surprising in view of the arbitrary nature of the potential assumed.

Indeed, we consider it premature to attempt to

[18] This assumption is only mentioned once in their paper, and this as late as Sec. VI.

obtain quantitative results from the Brueckner method until the method itself is fully established, including a treatment of a repulsive core plus an attractive potential together with the Pauli principle. Once the method is established, there will still remain the question of the proper two-nucleon potential to be used. It is not at all likely that all potentials which give the same nucleon-nucleon scattering at low energy, will also give the same nuclear binding energies; in fact, the example of the repulsive core shows that this cannot be generally the case. It will be very interesting to see whether we can learn additional facts about the nucleon-nucleon interaction from a study of complex nuclei.

In any case, confidence in the Brueckner method cannot be based on numerical agreement (or lack of agreement) between the results derived from it and experiment. Such agreement will show whether the assumptions made about nuclear forces are acceptable or not. The *method* must rest entirely on mathematical proof and internal consistency.

Some contributions are made in this paper towards the mathematical development. In Sec. III, the steps required to obtain the reaction matrix are discussed explicitly. Certain complications appear; in particular, it is not possible to assign a unique energy to a nucleon in an excited state. Fortunately, these problems are of no practical importance for well-behaved potentials. In Sec. IV, the model operator connecting the actual wave function with the model wave function, is written down. In Sec. VII, the definitions are discussed which are necessary to make the theory applicable to finite nuclei. In Sec. VIII, the influence of nucleon spin and charge, and their behavior in intermediate states, are treated in detail.

Brueckner and Levinson[10] have shown that there are certain correction terms in the theory which are of order $1/A$. In Sec. XIV it is shown that these are numerically about 100 times smaller than BL found, i.e., they are of the order of $1/A$ times the three-particle cluster corrections of Sec. XII. Matrix elements, which correspond to the excitation of one nucleon rather than two, and which would vanish for an infinite nucleus, likewise give contributions of order $1/A$ or less (Sec. XV).

Brueckner[15] has obtained the somewhat surprising result that the energy of any large system should be proportional to the number of particles, A. It is pointed out that this holds only if the density is kept fixed. The actual energy is obtained by varying the density until the energy is a minimum; then only saturating forces give an energy proportional to A for the system; for nonsaturating systems for which the density varies with A, the energy per particle also varies (Sec. XIII).

The problems for finite nuclei are discussed in Sec. XVI. They are essentially two, *viz.* (a) the behavior of the self-consistent potential near the surface of the nucleus and (b) the degeneracy of many configurations. Configuration interaction should be taken into account only between degenerate configurations. The theory has enough flexibility to permit a definition of "degeneracy" which is convenient for calculation. Generally, the theory justifies in all essential respects the practice of current shell model calculations.

The reader who is interested in results but not in complicated theoretical developments, is advised to read Sec. II, then Eq. (3.1) and the explanation of the symbols in it but not the rest of Sec. III, then Secs. IX–XII and XVI. The other sections may be left out without essential loss of continuity. A knowledge of the papers by Brueckner and collaborators[10–17] is not required.

II. DEFINITION OF SELF-CONSISTENT FIELD

We consider a finite (but large) nucleus. We shall assume a potential V in which each individual nucleon moves, then calculate the wave functions of the nucleons in this potential, and then construct the potential V again from the wave functions. The resulting potential should of course be equal to the starting potential—this is the condition of self-consistency.

The potential V must be assumed to be a potential matrix,

$$(\mathbf{r}'|\,V\,|\mathbf{r}), \tag{2.1}$$

rather than a simple potential such as is assumed in the Hartree theory for atoms. In that theory, it is assumed that

$$(\mathbf{r}'|\,V\,|\mathbf{r})=V(r)\delta(\mathbf{r}'-\mathbf{r}). \tag{2.2}$$

That the more general assumption (2.1) is necessary, follows from Brueckner's[12] result for the infinite nucleus, *viz.*, that the diagonal elements of V in a momentum representation depend strongly on the momentum $\mathbf{k}$ (see also Secs. X and XVI of this paper). Such velocity-dependent potentials can, in ordinary space, be represented only by a nonlocal potential matrix (2.1). This generalization of the potential concept is one of the reasons why Brueckner's method has so much greater power than the original Hartree method. In effect, a nonlocal potential (but of course one of special type) is already used in the Fock method.

A state of a nucleon in the potential (2.1) must satisfy the Schrödinger equation ($\hbar=1$):

$$(1/2M)\nabla^2\psi_n(\mathbf{r})+E_n\psi_n(\mathbf{r})=\int d^3\mathbf{r}'(\mathbf{r}'|\,V\,|\mathbf{r})\psi_n(\mathbf{r}'). \tag{2.3}$$

The ψ_n form a complete set, and can easily be shown to be orthogonal:

$$\int\psi_{n'}{}^*(\mathbf{r})\psi_n(\mathbf{r})d^3r=0, \quad \text{if } E_n\neq E_{n'}. \tag{2.4}$$

For reasonable assumptions about the potential, i.e., essentially a well of a certain radius, there will be discrete states (the states of the shell model) for $E_n<0$, and continuum states for $E_n>0$.

A state of the nucleus is defined by specifying which of the one-nucleon states ψ_n are occupied by nucleons. The set of occupied one-nucleon quantum states will be called the "configuration." Each configuration will be described by a Slater determinant wave function which contains all the occupied one-nucleon states. This determinant we call the model wave function and we denote it by Φ. Because of the properties of the nucleon wave functions, the model wave functions for different configurations are orthogonal. It is essential for this purpose that all nucleon wave functions are taken in the *same* potential V. For the present we disregard degeneracies (see, however, Sec. XVI).

Among the configurations, we single out one whose properties we wish to calculate. This we call the chosen configuration. This need not necessarily be the lowest state of the nucleus, but we shall generally consider states of relatively low energy (say, below 20-Mev excitation energy). Throughout this paper, the properties of this chosen configuration will be calculated more carefully than those of other configurations.

It is often convenient to assign definite quantum numbers to each nucleon, e.g., by considering the main diagonal of the Slater determinant. Thus we may speak of the state n_j of nucleon j; the states in the chosen configuration we shall denote by superscripts 0, thus: n_j^0.

Now the actual interaction between two nucleons is introduced; following Brueckner, it will be denoted by v. It is a function of the distance between the two nucleons, r_{ij}, their spins and charge, and it may contain exchange operators. It is assumed to be determined from experiments on the two-nucleon system. Three-body forces are assumed to be negligible. Whether this is in fact true for nuclear forces is not known at present; but it is clearly the simplest assumption to make, at least until it is clearly proved that no agreement with observation can be obtained without three-body forces.

The matrix elements of the known operator v_{ij} can be determined between two arbitrary configurations of nucleons. Obviously, the states of all nucleons other than i and j must remain unchanged in the transitions. The states of all these other nucleons will influence the matrix elements of v_{ij} only indirectly, by determining the potential operator $(\mathbf{r}'|V|\mathbf{r})$ of the self-consistent field and thereby the wave functions ψ_i and ψ_j. However, since we now consider the wave functions ψ_n in the potential V as *given*, we need only consider matrix elements such as

$$(n_i'n_j'|v_{ij}|n_in_j)_N, \tag{2.5}$$

where the subscript N indicates that the matrix element is to be taken between the determinant (model) wave functions describing the initial and final configurations of the nucleus, which contain nucleons i and j in the states n_i, n_j and n_i', n_j', respectively. We shall assume that all other nucleons are in the states prescribed by the chosen configuration, n_k^0. The states n_i, n_j, n_i', n_j' need not coincide with n_i^0 and n_j^0, respectively.

The matrix element (2.5) vanishes automatically if any one of the four states n_i, n_j, n_i' or n_j' coincides with one of the states occupied by one of the other nucleons, n_k^0, or if $n_i=n_j$ or $n_i'=n_j'$. If none of these prohibitions due to the Pauli principle exists, we have

$$(n_i'n_j'|v_{ij}|n_in_j)_N = (n_i'n_j'|v_{ij}|n_in_j)_P-(n_j'n_i'|v_{ij}|n_in_j)_P, \tag{2.6}$$

where the matrix elements with the subscripts P are taken without regard to the presence of other nucleons, and are given by

$$(n_i'n_j'|v_{ij}|n_in_j)_P=\int\psi_{n_i'}^*(\mathbf{r}_i)\psi_{n_j'}^*(\mathbf{r}_j)v_{ij}(r_{ij}) \times\psi_{n_i}(\mathbf{r}_i)\psi_{n_j}(\mathbf{r}_j)d^3r_id^3r_j. \tag{2.7}$$

The integrals in (2.7) imply also summations over spin and charge, and the operator v_{ij} may involve spin and charge operators. The second term in (2.6) is the well-known exchange term. There are obvious selection rules between primed and unprimed nucleon states.

It will be useful to consider a very large nucleus. Then the wave functions over most of the nucleus can be approximated by plane waves. This model will be used predominantly from Sec. IX on. In this case, momentum will be conserved between the two nucleons, *viz.*,

$$\mathbf{k}_i'+\mathbf{k}_j'=\mathbf{k}_i+\mathbf{k}_j. \tag{2.8}$$

Then the final spatial wave function of nucleon j is completely specified if that of nucleon i, and the initial states of both, are given.

In the next section, we shall show how at least the most important matrix elements of V, the one-nucleon potential, can be determined from the interaction v_{ij}. These matrix elements are obtained with respect to the wave functions $\psi_n(\mathbf{r})$ (n representation). Once they are known, we may obtain V in the $\mathbf{r}$ representation by the usual formula of transformation theory:

$$(\mathbf{r}'|V|\mathbf{r})=\sum_n\sum_{n'}\psi_n^*(\mathbf{r}')(n'|V|n)\psi_{n'}(\mathbf{r}). \tag{2.9}$$

For self-consistency, the resulting $(\mathbf{r}'|V|\mathbf{r})$ must be identical with the initial V used in Eq. (2.3). Just as in the original Hartree theory, this self-consistency must be achieved by trial and error.

BL developed their theory without first defining a potential V. This was made possible by their tacit assumption[18] that the one-nucleon wave functions are plane waves. This assumption is appropriate for an infinite nucleus and follows in this case from the general argument of invariance of all physical quantities with respect to translation. Thus in this special case a knowledge of the potential V is not required to obtain the wave functions. In the case of a finite nucleus, just as in the case of the Hartree-Fock atom, the first step in the development of the self-consistent

field must be a "guess" of the one-particle potential V. Our method in this paper contains the Hartree-Fock theory as an approximation (Sec. V) and goes over into the Brueckner-Levinson theory in the limit of an infinite nucleus (Sec. IX).

In every case, and thus also in the case of an infinite nucleus, the set of one-nucleon wave functions must be uniquely defined. The fact that the wave functions in an infinite nucleus must be plane waves defines them except for one parameter, *viz.*, the density of the nucleus. To determine the density, the self-consistency requirement must be used. Eden[19,20] has shown that this requirement can be satisfied by formally calculating the energy of the nucleus as a function of the density, and then finding the minimum of this function. Thus the self-consistency in this case is equivalent to the variational problem. This will be discussed in more detail in Sec. IX, and also in E,[20] Sec. IV.

III. REACTION MATRIX

The main feature of the Brueckner theory is the use of the reaction matrix[21] for the scattering of two nucleons while they are moving in the nuclear medium. Consider, for example, the scattering of two nucleons i and j from their states n_i^0, n_j^0 in the chosen configuration to some other states n_i', n_j'. The reaction matrix G for this problem is the solution of the equation:

$$(n_i',n_j'|G_{ij}|n_i^0n_j^0) = (n_i'n_j'|v_{ij}|n_i^0n_j^0)_N+\sum_{n_i''n_j''}(n_i'n_j'|v_{ij}|n_i''n_j'')_N \times\frac{Q}{e}(n_i''n_j''|G_{ij}|n_i^0n_j^0). \quad (3.1)$$

The denominator e, or rather $-e$, means the excitation energy of the two nucleons i and j in the intermediate state $n_i''n_j''$, as compared with their state $n_i^0n_j^0$ in the chosen configuration, both calculated in the nuclear medium, i.e., under the influence of all the other nucleons. It will be calculated presently.

It is clearly necessary to exclude states for which $e=0$. This is done by BL by using the principal value in their Eq. (3) which corresponds to our (3.1). (Since they are dealing with an infinite nucleus their Eq. (3) contains an integral over intermediate states rather than a sum.) Following Eden,[20] we have introduced the operator Q to fulfill the same purpose as the principal value operator: we define Q to be zero for all states for which $e=0$, i.e., for the chosen configuration and for all other configurations which have the same energy. In addition, we are free to set $Q=0$ for all configurations which have *nearly* the same energy as the chosen one, as long as we specify clearly at the beginning of any particular calculation which configurations we shall thus regard as "nearly degenerate." This will be discussed further in Sec. XVI, where it will be explained why this freedom in the definition of Q is useful.

It may be noted in passing that singly excited states cannot be reached from the chosen configuration in the case of an infinite nucleus because of the momentum conservation (2.8); i.e., for an infinite nucleus it is impossible to excite one nucleon without also exciting the other. In a large but finite nucleus, the matrix elements $(n_i^0n_j''|v_{ij}|n_i^0n_j^0)$ are not zero but small, of order Ω^{-1} (where Ω is the volume of the nucleus) compared with those elements in which both nucleons are simultaneously excited with (approximate) momentum conservation (see Sec. XV); leaving them out in (3.1) would therefore be no great change and will be done at various points to simplify the calculation.

As has been pointed out in Sec. II, the matrix elements v_N of the interaction between states of the complete nucleus vanish if any of the states $n_i'n_j'n_i''n_j''$ is occupied by some other nucleon $k\neq i,j$ in the chosen configuration, i.e., if any of them is equal to some n_k^0. Thus either these states n_i', etc. are the states of nucleons i and j themselves in the chosen configuration, i.e., they are n_i^0 and n_j^0, or they must be outside the chosen configuration, i.e., they are normally unoccupied states. For the final state $n_i'n_j'$, both of these possibilities exist. For the intermediate state $n_i''n_j''$, however, the operator Q_{ij} (as defined) excludes the possibility that either n_i'' or n_j'' is equal to the chosen states n_i^0, n_j^0: therefore, in this case, both nucleons must be in normally unoccupied states. Thus the Pauli principle must operate in the intermediate state. We could indicate this by explicitly summing in (3.1) only over unoccupied states n_i'' and n_j''.

The Pauli principle has the consequence that the denominators e are in general not even nearly zero, especially if the ground state of the nucleus is calculated. The e are all negative. Further, the matrix G_{ij} has by (3.1) the same selection rules as v_{ij}: it vanishes if $n_i'=n_j'$, or if either n_i' or n_j' is equal to any of the states occupied by other nucleons, n_k^0. In this way, the matrix G_{ij} depends on the entire chosen configuration and not merely on the states of nucleons i and j.

The excitation energy $-e$ is given by

$$-e=E(B)-E(C), \quad (3.2)$$

where $E(C)$ is the energy of the chosen configuration C, and the intermediate configuration $B(n_in_j)$ differs

[19] R. J. Eden, Phys. Rev. **99**, 1418 (1955).

[20] R. J. Eden, Proc. Roy. Soc. (London) **A235**, 408 (1956). This paper will be quoted as E.

[21] The real solution of the scattering equation (3.1), corresponding to standing waves, is required, not the complex solution corresponding to outgoing waves; therefore we are dealing with a reaction rather than a scattering matrix. Accordingly, the notation t_{ij} is not appropriate since it is commonly used for the complex scattering matrix. We did not wish to use r or R because the distance between nucleons and the nuclear radius occur frequently, and we have therefore adopted the notation G used by M. L. Goldberger, Phys. Rev. **84**, 929 (1951). Avoidance of t also has the advantage of making the notation more different from that for the kinetic energy for which we use T, as Eden does in E. We think that H_0 for the kinetic energy should be avoided since the free-nucleon assumption does not afford even a "zero-order" approximation to the problem.

from $C(n_i^0 n_j^0)$ by having the states n_i^0 and n_j^0 empty and the states n_i and n_j filled. It will be shown in Sec. IV, after Eq. (4.17), that for any configuration with states n_i occupied, one has in good approximation

$$E=\sum_i (n_i|T_i|n_i)+\sum_{ij}(n_i n_j|G_{ij}|n_i n_j)+w_{1C}, \quad (3.3)$$

where w_{1C} is a constant which is nearly the same for all states of low excitation, and which therefore cancels in taking the difference (3.2). Now the interactions G_{kl} between any two nucleons k,l other than i,j are the same in the two configurations B and C and therefore cancel in (3.2). Therefore we get

$$-e_{ij}=\mathcal{E}(n_i,n_j;n_i^0,n_j^0)+\Delta G_{ij}, \quad (3.4)$$

where

$$\begin{aligned}\mathcal{E}(n_i,n_j;n_i^0,n_j^0)&=(n_i|T_i|n_i)+\sum_{k\neq i}(n_i n_k^0|G_{ik}|n_i n_k^0;n_i^0 n_j^0,n_j)\\&-(n_i^0|T_i|n_i^0)-\sum_{k\neq i}(n_i^0 n_k^0|G_{ik}|n_i^0 n_k^0)\\&\qquad+\text{same for nucleon } j \quad (3.5)\end{aligned}$$

may be considered as the excitation energy of nucleons i and j in the field of all the others.

The quantum numbers listed after the semicolon in the first G_{ik} matrix in (3.5) give the empty states and the extra occupied state in configuration B. The last term in (3.4) is

$$\begin{aligned}\Delta G_{ij}&=(n_i n_j|G_{ij}|n_i n_j)-(n_i n_j^0|G_{ij}|n_i n_j^0)\\&-(n_i^0 n_j|G_{ij}|n_i^0 n_j)+(n_i^0 n_j^0|G_{ij}|n_i^0 n_j^0). \quad (3.6)\end{aligned}$$

Since this is only the interaction between one pair of nucleons while (3.5) is the interaction between one nucleon and A others, ΔG_{ij} is of relative order $1/A$. We shall therefore neglect it although it could be taken into account if we desired.

The reaction matrix $(n_i n_k^0|G_{ik}|n_i n_k^0;n_i^0 n_j^0,n_j)$ is calculated by solving an equation similar to (3.1). However, we must remember that in configuration B the states n_i^0 and n_j^0 are empty, whereas there is an extra nucleon in state n_j. Therefore we must count the excitation energy of nucleon i from the empty state n_i^0, not from the "initial" state n_i. Further, we must include in the energy denominator the constant excitation energy of nucleon j in going from state n_j^0 to n_j. Explicitly, we have to solve

$$\begin{aligned}&(n_i' n_k'|G_{ik}|n_i n_k^0;n_i^0 n_j^0,n_j)\\&=(n_i' n_k'|v_{ik}|n_i n_k^0)_N-\sum(n_i' n_k'|v_{ik}|n_i'' n_k'')\\&\times\frac{Q_{ik}}{\mathcal{E}(n_i'',n_j,n_k'';n_i^0,n_j^0,n_k^0)}\\&\qquad\times(n_i'' n_k''|G_{ik}|n_i n_k^0;n_i^0 n_j^0,n_j). \quad (3.7)\end{aligned}$$

The denominator $\mathcal{E}(n_i'',n_j,n_k'';n_i^0,n_j^0,n_k^0)$ is a sum of three terms similar to the two in (3.5). Thus it is ordinarily "larger" than the denominator (3.5) occurring in (3.1). This effect has been mentioned by Brueckner[15] in BC and called "propagation off the energy shell." In a naive application of the theory of the scattering matrix, one might have expected the denominator $\mathcal{E}(n_i',n_k';n_i,n_k^0)$ instead of that found in (3.7); this would be much smaller and might even become zero or negative.

The excitation energy (3.5) can in general not be written as

$$\mathcal{E}(n_i,n_j;n_i^0,n_j^0)=E(n_i)+E(n_j)-E(n_i^0)-E(n_j^0), \quad (3.8)$$

i.e., as the difference between the energies of particles i and j in the now occupied state and in the chosen state. It is true that the second line of (3.5) may be considered as the energy of particle i in the chosen configuration,

$$E(n_i^0)=(n_i^0|T_i|n_i^0)+\sum_{k\neq i}(n_i^0 n_k^0|G_{ik}|n_i^0 n_k^0). \quad (3.9)$$

But the first line of (3.5) depends not only on the state occupied by particle i in configuration B, but also on the two empty states n_i^0, n_j^0 and on the state n_j now occupied by particle j; it can therefore not be written as $E(n_i)$. We cannot define the energy of a normally empty state n_i, but this energy depends on the other states which are empty or occupied.[22] The energy "$E(n_i)+E(n_j)$" depends not just on two labels, $n_i n_j$, but on four, $n_i n_j n_i^0$ and n_j^0.

This is obviously a rather complicated situation. Fortunately, for the two most important types of interaction the dependence on all the additional quantum numbers is not great, so that (3.8) is probably a good approximation. The first type is a simple interaction of the "classical" type, like Yukawa, square well, etc., without a repulsive core. For this type, it will be shown in Sec. X that the sum term on the right-hand side of (3.1) or (3.7) gives a relatively small contribution, so that for instance (3.7) reduces nearly to the first term on the right, the interaction matrix v_{ik}. This then does not depend on the supernumerary quantum numbers n_i^0, n_j^0, n_j, and in this case the first line of (3.5) is a function of n_i alone so that (3.8) is valid.

The second important type of interaction is a repulsive core.[23] In this case the difference between the interaction matrix v and the reaction matrix G is large. However, the main contribution to the sum in (3.1) comes from intermediate states of very high energy. For these states, the kinetic energy term $(n_i|T_i|n_i)$ in (3.5) is far more important than the interaction term G. This kinetic energy, however, depends only on the state n_i, not on n_i^0, n_j^0, and n_j.

We shall therefore assume in most of our calculations

[22] I am indebted to J. Goldstone for pointing this fact out to me.

[23] This interaction will be treated in a forthcoming paper by the author and J. Goldstone, to be published in Proc. Roy. Soc. (London), which will be quoted as BG.

that (3.8) is a good approximation. It is, however, an approximation only; in BL and other papers of the Brueckner school, it was assumed to be exact, and it was assumed to be the only task of the theory to determine the dependence of the eigenvalue $E(n)$ on the momentum k. The difficulty of defining $E(n)$ for unoccupied states n exists whether we treat an infinite nucleus or a finite one.

However, for a finite nucleus the difficulty has a further unpleasant consequence: it makes it impossible to give a unique definition of the one-particle potential V. It would obviously be desirable to define V in such a way that the excitation energy of a particle as defined by (3.5), agrees with the difference of the eigenvalues of the particle calculated by solving (2.3) in the given potential V. But this cannot be done since the excitation energy (3.5) can in general not be written in the form (3.8). Therefore (3.5) cannot be simplified by introducing one-particle eigenvalues.

It is possible, however, to define V in such a way that the eigenvalue $E(n_i^0)$ represents the energy required to remove particle i from the chosen configuration. This energy is given by the second line of (3.5). Thus we wish to choose V in such a way that

$$(n_i^0|V|n_i^0)=\sum_{k\neq i}(n_i^0 n_k^0|G_{ik}|n_i^0 n_k^0) \tag{3.10}$$

for all states n_i^0 in the chosen configuration. The G_{ik} in (3.10) do not depend on any quantum numbers except n_i^0 and n_k^0, in contrast to the G_{ik} occurring in (3.7). The right-hand side of (3.10) therefore depends only on n_i^0, once the chosen configuration is selected. By the way, the sum on the right-hand side of (3.10) may be extended to include $k=i$ because this term is automatically zero owing to the Pauli principle. The eigenvalue of a nucleon in an occupied state of the chosen configuration is then

$$E(n)=(n|T|n)+\sum_j(nn_j^0|G|nn_j^0). \tag{3.11}$$

It is also possible to define uniquely the elements of V leading from an occupied to an empty state, by setting

$$(n_i|V|n_i^0)=\sum_j(n_i n_j^0|G_{ij}|n_i^0 n_j^0), \tag{3.12}$$

where again the term $j=i$ gives zero contribution. In the final state, only the state n_i^0 of the chosen configuration is empty and n_i outside that configuration is occupied; thus the matrix element depends on these two quantum numbers.

However, it is not possible to give any unique definition of the elements of V between two *unoccupied* states, diagonal or nondiagonal. The diagonal elements should be given by the second term on the right of (3.5),

$$(n_i|V|n_i)=\sum_{k\neq i}(n_i n_k^0|G_{ik}|n_i n_k^0; n_i^0 n_j^0, n_j), \tag{3.13}$$

but as we have seen, they depend not only on the state of the nucleon n_i considered, but also on the states n_i^0, n_j^0, and n_j. If $(n_i|V|n_i)$ for an unoccupied state n_i must be defined, one might take a suitable average of (3.13) over n_i^0, n_j^0, and n_j. This obviously would have no physical meaning, but would serve merely to give a definition of the self-consistent potential $(\mathbf{r}'|V|\mathbf{r})$. To obtain this potential from (2.9), *all* matrix elements $(n'|V|n)$ must be known.

As we have seen, the matrix elements starting from occupied states are well defined. The nondiagonal matrix elements $n'\neq n$ are small, of order $\Omega^{-\frac{1}{3}}$ if the nuclear volume Ω is large (Sec. XV). The diagonal elements, $n'=n$ (unoccupied), can only be defined with some arbitrariness. To see what this arbitrariness means for the potential $(\mathbf{r}'|V|\mathbf{r})$, let us assume that we can use plane wave functions, which is legitimate for a large nucleus. Then the contribution of the diagonal elements for unoccupied states to (2.9) is

$$\sum_{n(\text{unocc.})}\psi_n^*(\mathbf{r}')(n|V|n)\psi_n(\mathbf{r}) = (2\pi)^{-3}\int_{k_F}^{\infty}d^3k(k|V|k)e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}. \tag{3.14}$$

This is therefore associated with the behavior of $(\mathbf{r}'|V|\mathbf{r})$ as a function of $\mathbf{r}-\mathbf{r}'$, and particularly with the rapid oscillations of this function. Presumably, these have little influence on the wave functions ψ_n which are solutions of the Schrödinger equation in this potential. For a large nucleus, these wave functions are essentially free-particle functions in a box. Therefore we believe that the arbitrariness in the matrix elements of V between unoccupied states will not have very much influence on the set of wave functions ψ_n for the individual nucleons.

The following may be an acceptable procedure: choose $(\mathbf{r}'|V|\mathbf{r})$ by "guessing," and calculate the ψ_n, the interaction matrix v_{ij}, and the reaction matrix G_{ij}. Then calculate the matrix elements of V between *occupied* states from (3.11) and (3.12); they are well defined. If they do not agree with the corresponding matrix elements of the originally chosen $(\mathbf{r}'|V|\mathbf{r})$, correct the latter suitably. However, do not attempt to correct or even define the matrix elements of V between unoccupied states, but instead rely on the "first guess" of V in this respect. This would presumably simplify the procedure of obtaining a self-consistent solution.

It is essential that V be the same potential for all states ψ_n of single nucleons, in order to guarantee orthogonality.

IV. CONCERNING THE PROOF OF THE METHOD

Following BL,[10] the actual wave function Ψ of the nucleus in any given configuration (not necessarily the ground state) is related to the model wave function by an operator F, thus

$$\Psi=F\Phi. \tag{4.1}$$

F has been called the model operator by Eden,[20] and it is defined implicitly by the two equations

$$F=1+\frac{Q}{e}\sum_{ij} I_{ij}F_{ij}, \tag{4.2}$$

$$F_{ij}=1+\frac{Q}{e}\sum_{lm\neq ij} I_{lm}F_{lm}, \tag{4.3}$$

where the sums go over all pairs of nucleons, each pair being counted only once, and the second sum excludes the term in which both nucleons l and m are identical with the nucleons i and j, but includes terms in which *one* of the two nucleons l, m coincides with one of the pair i, j. The operator I is related to the scattering matrix G defined in Sec. III. We write

$$I_{ij}=G_{ij}-\bar{G}_{ij}, \tag{4.4}$$

and, following Eden, we define $\bar{G}_{ij}$ as the operator which comprises the diagonal elements of G_{ij} and those in which the state of only *one* nucleon changes; for a more precise definition, see Sec. VII and E. Then I_{ij} is essentially the "nondiagonal" part of the operator G_{ij}, more precisely the part which corresponds to the simultaneous change of the quantum states of both nucleons i and j. "Diagonal" is to be understood with respect to the set of model wave functions defined in Sec. II. The denominators $-e$ denote the excitation energy of the nucleus in the model state which is established when the operators to the right of e have operated on the "initial" model wave function Φ of Eq. (4.1).

The excitation energy is to be calculated as in (3.4), by taking the difference of the *model* energies between the excited and the chosen configuration [see Eq. (4.19)]. One must not use the difference between the actual energies, and certainly not the difference between the model (unperturbed) energy of the excited state and the actual (perturbed) energy of the chosen configuration, as is done in the Brillouin-Wigner perturbation theory (see Sec. XIII, especially reference 50). The operator Q is defined as in Sec. III, below Eq. (3.1).

A slight modification, introduced in E,[20] is desirable. The actual nuclear wave function Ψ must of course satisfy the Schrödinger equation

$$E\Psi=H\Psi=\sum_i T_i\Psi+\sum_{ij} v_{ij}\Psi. \tag{4.5}$$

Now instead of requiring that (4.1) be true with Φ the model wave function defined in Sec. II (which would require further modification of F), we shall postulate that (4.2), (4.3) is the correct form of the model operator, but shall define a modified model wave function Φ' by setting

$$\Psi=F\Phi'. \tag{4.6}$$

Following BL[10] and Eden,[20] we can then derive the Schrödinger equation for Φ' as follows. Comparing (4.2) and (4.3), we have

$$F=F_{ij}+\frac{Q}{e}I_{ij}F_{ij} \tag{4.7}$$

(no summation over ij). Therefore

$$v_{ij}F=\left[v_{ij}+v_{ij}\frac{Q}{e}(G_{ij}-\bar{G}_{ij})\right]F_{ij}, \tag{4.8}$$

where (4.4) has been used. The last term, with $\bar{G}_{ij}$, will turn out to be an unimportant correction. The other terms may be combined by using the integral equation (3.7) for G, which has the form[24]

$$G_{ij}=v_{ij}+v_{ij}\frac{Q}{e}G_{ij}. \tag{4.9}$$

Therefore

$$v_{ij}F=G_{ij}F_{ij}-v_{ij}\frac{Q}{e}\bar{G}_{ij}F_{ij} \tag{4.10}$$

$$=\bar{G}_{ij}\left(F-\frac{Q}{e}I_{ij}F_{ij}\right)+I_{ij}F_{ij}-v_{ij}\frac{Q}{e}\bar{G}_{ij}F_{ij}, \tag{4.11}$$

where we have again used (4.4) and (4.7). We now sum over ij and transform the third term of (4.11), using (4.2):

$$\begin{aligned}\sum I_{ij}F_{ij}&=\sum(1-Q)I_{ij}F_{ij}+\sum QI_{ij}F_{ij}\\&=\sum(1-Q)I_{ij}F_{ij}+e(F-1).\end{aligned} \tag{4.12}$$

Collecting all terms in (4.5) and (4.6), we have then

$$EF\Phi'=\left(\sum_i T_i+\sum_{ij}\bar{G}_{ij}+e\right)F\Phi'-e\Phi'+w\Phi', \tag{4.13}$$

where w is an operator consisting of three parts,

$$w_1=\sum_{ij}(1-Q)I_{ij}F_{ij}, \tag{4.14}$$

$$w_2=-\sum_{ij}\bar{G}_{ij}\frac{Q}{e}I_{ij}F_{ij}, \tag{4.15}$$

$$w_3=-\sum_{ij}v_{ij}\frac{Q}{e}\bar{G}_{ij}F_{ij}. \tag{4.16}$$

We shall now show that the terms involving $F\Phi'=\Psi$ in (4.13) can be greatly simplified by proper choice of e. For this purpose, we consider $\Psi=F\Phi'$ expanded in terms of the original model wave functions of Sec. II (Slater determinants). In order for this expansion to be useful, all the model wave functions (for all configurations B) must satisfy the "model wave equation"

$$(E-H_M)\Phi_B=0, \tag{4.17}$$

[24] It should be noted that the denominator here is e just as in (4.2) and (4.3). Thus excitations of nucleons other than i and j have to be included in e. This is in the spirit of Sec. III and different from E, Eq. (2.7). It avoids the last correction term in E, Eq. (2.15), which could give rise to a spurious dependence on higher powers of A.

where H_M is the "model Hamiltonian"

$$H_M=\sum T_i+\sum \bar{G}_{ij}. \tag{4.18}$$

It will be shown in Sec. VII how $\bar{G}_{ij}$ must be defined in order that (4.17) be true. In particular, the nondiagonal elements of $\bar{G}_{ij}$ must be defined by (7.4) and (7.7), regardless of the excitation of nucleons other than i and j. This definition insures also that the Φ_B form an orthogonal system. It should be noted that H_M has only diagonal elements with respect to the model wave functions Φ_B but that $\sum T_i$ and $\sum \bar{G}_{ij}$ alone would also have nondiagonal elements. Further, according to (7.3), the diagonal elements of $\bar{G}_{ij}$ are equal to those of G_{ij} and are to be calculated by solving (4.9), taking the excitations of *all* nucleons into account in the denominators e.

We now define the operator e also as diagonal in the model wave functions, thus

$$e=E_C-E_B, \tag{4.19}$$

where E_C is the model energy of the chosen configuration. Then (4.13) reduces to

$$(E-E_C)F\Phi'+e\Phi'=w\Phi'. \tag{4.20}$$

We can first determine E. Assuming that the chosen configuration is not degenerate, i.e., that the nucleus contains only closed shells (the more general case will be considered below), we project Eq. (4.20) on the model wave function of the chosen configuration, Φ_C. From (4.19), we see that $e=0$ in this case.
Further, from (4.2),

$$(\Phi_C,F\Phi')=(\Phi_C,\Phi'), \tag{4.21}$$

because $Q=0$ in the configuration C. Therefore (4.20) yields[25]

$$E-E_C=(\Phi_C,w\Phi')/(\Phi_C,\Phi')\equiv w_C. \tag{4.22}$$

This, together with (4.17) and (4.18), is the basis of Eq. (3.3). Equation (4.19) justifies our treatment of energy denominators in Sec. III. Further, (4.20), may be rewritten

$$e\Phi'=(w-w_CF)\Phi'. \tag{4.23}$$

The operator $w-w_CF$ is small. Its diagonal matrix elements are zero by (4.22), and we therefore consider its nondiagonal elements

$$(\Phi_B,\ (w-w_CF)\Phi_C) \tag{4.24}$$

for states $E_B\neq E_C$. Then the contribution of w_1 is zero because of the factor $1-Q$, that of w_2 and w_3 is extremely small according to Sec. XIV, and therefore (4.24) is proportional to w_C. This energy correction, however, is also small as will be shown in Sec. XII. Therefore, very nearly,

$$\Phi'\approx\Phi_C, \tag{4.25}$$

and (4.22) may be simplified to

$$w_C\approx(\Phi_C,w\Phi_C). \tag{4.26}$$

Now consider this expression, in particular the part due to w_1 which will be shown in Sec. XII and XIV to be the most important contribution. The operator $1-Q$ equals 1 in our case because the nucleus is ultimately in state C. Hence

$$(\Phi_C,w_1\Phi_C)=(I_{ij}\Phi_C,F_{ij}\Phi_C). \tag{4.27}$$

Now when (4.27) is evaluated in a straightforward manner, certain terms appear which are proportional to A^2 and higher powers of the number of nucleons A in the nucleus. These terms were discovered by Brueckner in his "linked cluster" paper,[15] BC, and shown by him to be spurious, i.e. to be compensated by other terms, *viz.* the energy perturbations in the energy denominators. Brueckner's prescription is to take into account only linked clusters in the expression (4.26) for w_C.

For the purpose of defining linked clusters, we will regard F as the sum of the infinite series of terms which would be obtained by iteration of F_{ij} in (4.2) using (4.3). Each term consists of a sum of products of two-suffix symbols. We will consider one such product acting on a single term in the Slater determinant Φ; it will involve a series of suffixes ij, lm, pq, $\cdots$ and there will be no summation. We will say that the pair ij is "directly linked" to pq if either one of the numbers i, j is equal to either one of p, q. We will say that ij is "linked" to lm if in the term we are considering it is possible to go from ij to lm via a chain of directly linked pairs. The term will be called a "linked cluster" term if all pairs of suffixes in it are linked. Thus a third order term having suffixes ij, jk, kl is a linked cluster.

Brueckner, in BC, has shown that there should not be any contributions from "unlinked clusters" to the energy E of the nucleus. Unlinked clusters are the analog of "disconnected graphs" in field theory, which are known not to give any contribution in that theory. In our case also, they would be entirely unphysical, giving rise to the appearance of terms proportional to A^2 and higher powers of A in the expression for the energy of the nucleus. Indeed, Brueckner shows by direct evaluation that at least up to fourth order of standard perturbation theory, the contributions from unlinked clusters actually cancel. This is a great advance over the Brillouin-Wigner formulation of perturbation theory in which these troublesome terms always occur.

[25] The matrix elements here, and in the foregoing, must be taken with respect to the model wave functions Φ_B because only in these is the operator e diagonal [see the derivation of (4.20)]. They must not be taken with respect to the complete nuclear wave function, and in particular (4.22) must not be replaced by

$$(\Psi_C,w\Phi'), \tag{a}$$

as BL do in their Eq. (25). The use of (4.22) is at the same time a great simplification of the calculation, and it greatly reduces the numerical value of the correction terms w_2 and w_3, as will be shown in Sec. XIV.

Goldstone[26] has shown quite generally that unlinked clusters must be omitted, by direct application of field theoretic methods to the problem of the complex nucleus. His method is exact, and shows that w_C is given by (4.27) with the proviso that only linked clusters must be taken into account. The corrections w_2 and w_3 of (4.15), (4.16) do not appear but there are some minor modifications of the prescription for calculating the matrices G and $\bar{G}$.

We shall now discuss the matrix elements

$$(\Phi_B, w_1\Phi_C) \tag{4.28}$$

in more detail. Because of the factor $1-Q$, these elements are different from zero only if the state B has the same energy as the chosen configuration. We must therefore distinguish the two cases in which (a) the chosen configuration is degenerate with other states, and (b) it is nondegenerate. In case (b), w_1 has only a diagonal element. Thus w_1 will only contribute to the energy of the nucleus but will leave $\Phi'=\Phi$. It will be shown in Sec. XII that the diagonal element of w_1 can arise only from the successive interaction of three or more nucleons (three-particle clusters). As a consequence, this term is rather small *per nucleon*, but it is proportional to the number of nucleons. It will change rather little from one configuration to another because it depends only on the behavior of the wave function when three nucleons come very close together, and not on its over-all behavior. Thus the diagonal term of w_1 for any configuration may be replaced by a constant, *viz.*, its expectation value for the chosen configuration, w_{1C}. This result has been used in (3.5).

In case (a), the diagonal term will also arise and have substantially the same value as in case (b). But in addition, there are matrix elements of w_1 linking the various degenerate configurations. For instance, all the degenerate configurations may have 3 nucleons in the $1g$ shell, and differ by the magnetic quantum numbers assigned to these 3 nucleons. Then the operator I_{ij} may change the magnetic quantum numbers so as to cause a transition from one of the degenerate configurations to another. Thus w_1 will in this case have nonvanishing matrix elements even if F_{ij} in (4.14) is replaced by unity, while in case (b) the matrix elements vanish if this replacement is made, because the diagonal elements of I_{ij} are zero by definition (see Sec. XII). The nondiagonal matrix elements of w_1 between degenerate configurations are therefore apt to be considerably larger than the contribution of *one* pair ij to the diagonal element w_1. On the other hand, only the nucleons in incomplete shells will contribute to the nondiagonal elements, and therefore the nondiagonal elements of w_1 will not be proportional to A but to a lower power of A. The nondiagonal elements can be treated by the usual methods of perturbation theory for degenerate states, as will be described in Sec. XVI.

[26] J. Goldstone, Proc. Roy. Soc. (London), (to be published).

The two perturbation terms w_2 and w_3 have already been treated by BL. They show that the diagonal matrix elements of w_2 and w_3 give a contribution to the energy which is independent of A, and is therefore of order $1/A$ relative to the main binding energy of the nucleus. In Sec. XIV, we shall confirm this result and shall show that the numerical value of this contribution is much smaller than BL estimated, *viz.*, only a few hundred kev for the whole nucleus, rather than 10 Mev: hence the method remains good down to small nuclei.

The nondiagonal elements of w_2 and w_3 give contributions of order $1/A$ to the energy and to the wave function, as will be shown in Sec. XIV. They are therefore entirely negligible.

V. DISCUSSION AND COMPARISON WITH THE HARTREE-FOCK METHOD

The actual wave function defined by (4.1) to (4.3) may be expanded in a series of terms involving no, one, two, . . . interactions. The first approximation to F is obtained if F_{ij} in (4.2) is replaced by unity. Then putting the nondiagonal elements of I_{ij} equal to those of G_{ij} (see Sec. VII), we get for the wave function of a nucleus containing A nucleons:

$$\Psi_C=\Phi(n_1^0,n_2^0,\cdots n_A^0)-\sum_{i<j}\sum_{ni}{}'\sum_{nj}{}' \times\frac{1}{\mathcal{E}(n_i,n_j;n_i^0,n_j^0)}(n_in_j|G_{ij}|n_i^0n_j^0) \times\Phi(n_1^0,n_2^0,\cdots,n_i,\cdots,n_j,\cdots,n_A^0). \tag{5.1}$$

Here the model wave functions have been explicitly designated by the configuration to which they belong, and the primes mean that the sums go only over unoccupied states n_i, n_j. Terms including more than two excited nucleons have been omitted in (5.1). If G_{ij} were further approximated by v_{ij}, then this approximation would correspond to the first-order (not zero order) wave function of standard perturbation theory. Thus the wave function Ψ includes, besides the model wave function Φ, also all states which can be generated from Φ by letting an arbitrary pair of nucleons i and j scatter each other. In higher approximations, i.e., when F_{ij} from (4.3) is inserted into (4.2), multiple scattering is also taken into account (see below). Furthermore, the use of G_{ij} in (5.1) constitutes an improvement over the use of v_{ij} in first-order perturbation theory.

The Hartree-Fock method is obtained from the Brueckner method by the following approximations:

(1) The wave function is assumed to be Φ, rather than Ψ.

(2) The second term in (3.1) is left out, i.e., the G matrix is replaced by the v matrix, $(n_i'n_j'|v_{ij}|n_in_j)_N$. The Hartree approximation itself is obtained by the further approximation of replacing $(n_i'n_j'|v_{ij}|n_in_j)_N$ by $(n_i'n_j'|v_{ij}|n_in_j)_P$, i.e., neglecting the exchange term in (2.6). Clearly, for nucleons which have exchange

forces between them, this would be a very poor approximation.

To see that the Brueckner method is a vast improvement over the Hartree-Fock approximation, we consider merely the first step in solving Eq. (3.1) by iteration, i.e., we replace G_{ij} in the sum in (3.1) by v_{ij}. (The Hartree-Fock approximation would be obtained by neglecting the sum altogether.)

Let $n_i=n_i'$ and $n_j=n_j'=n_j^0$, where n_i^0, n_j^0 denote as usual the states of nucleons i and j in the chosen configuration. Sum over all pairs of nucleons i, j. Then, according to (3.10), the left-hand side of (3.1) will yield the total potential energy of the nucleus, $E_{\text{pot}}^{(2)}$, the superscript 2 indicating that this is the second approximation. We find

$$E_{\text{pot}}^{(2)}=E_{\text{pot}}^{(1)}+\sum_B Q\frac{|(B|W|C)|^2}{E_C-E_B}, \tag{5.2}$$

where $E_{\text{pot}}^{(1)}$ is the first (Hartree-Fock) approximation to the potential energy, C is the "chosen" configuration of the nucleus, B any excited configuration, and

$$W=\sum_{ij} v_{ij} \tag{5.3}$$

is the sum of the interaction operators over all nucleon pairs. Clearly (5.2) is exactly the second-order perturbation theory result for the potential energy, and the second term is the full "configuration interaction," summed over *all* excited configurations which can be reached by the excitation of two nucleons from the chosen configuration. Thus (5.2) is a far better approximation than has ever been attempted in calculating either atomic or nuclear energy levels, and it is only the first step of the Brueckner method in improving the Hartree-Fock approximation.

A great advantage of the use of the scattering matrix G, instead of a perturbation expansion, will be that unpleasant divergences common in perturbation theory may be avoided. The case of the greatest practical importance in nuclear physics is that of a respulsive core, i.e., an infinite repulsive potential for $r<a$. Then the matrix elements of the interaction v are infinite, but the elements of the scattering matrix G are finite.[23] There are probably also advantages in the treatment of Coulomb interactions.

The wave function (5.1) also contains configuration interaction. This is true in spite of the fact that in (5.1) the crude approximation has been made of replacing F_{ij} by unity, which means the neglect of multiple scattering. But even in this crude approximation, the wave function contains strong two-particle correlations between the nucleons, and thus is much superior to the Hartree-Fock wave function. Moreover, it contains the high-momentum components which are required by high-energy experiments, as discussed in Sec. I and by Brueckner, Eden, and Francis.[7]

Thus the Brueckner method gives an improvement, both of the wave function and of the energy, over the Hartree-Fock method. In the Hartree-Fock method, the motion of *one* particle is considered in the average potential produced by all the others. In the Brueckner method, the interaction of *two* particles is calculated accurately, and only the influence of the third and further particles is replaced by an average potential. Thus the Brueckner method is a logical extension of the Hartree method; the price for its increased power and accuracy is of course the increased complication in the determination of the self-consistent field (see Sec. III).

Perhaps the most important advance is that in the Brueckner method the error can in principle be calculated, by evaluating the matrix elements of the perturbations (4.14) to (4.16) between model wave functions. In the Hartree method, the error has always been assumed to be small, and has turned out to be small in practical applications to atoms, but no method was available heretofore to estimate the error quantitatively.

The Brueckner theory is simplified, and made more similar to the Hartree-Fock theory, if it is possible to make the approximation (3.8), i.e., to assign to each nucleon an energy $E(n_i)$ as given by (3.11). One may then identify the second term in (3.11) with the potential energy of nucleon i and set

$$V(n_i)=\sum_j(n_in_j^0|G_{ij}|n_in_j^0). \tag{5.4}$$

For the occupied states $n_i=n_i^0$, as was mentioned in (3.9), the matrix elements of G_{ij} occurring in (5.4) are always well defined and therefore the definitions (5.4), (3.11) are sensible. BL assumed (5.4) to be correct for all states.

The total energy of the nucleus is given by (3.3). Neglecting w_{1C}, we obtain the model energy, but this is still not equal to the sum of the eigenvalues $E(n_i)$ of all the nucleons, because in (3.3) each pair ij must be counted only once while summation of (5.4) over all i would count each pair i,j twice. We have for the total model energy

$$E=\sum(n_i|T_i|n_i)+\tfrac{1}{2}\sum(n_i|V|n_i) \tag{5.5}$$

$$=\sum_i E(n_i)-\tfrac{1}{2}\sum(n_i|V|n_i). \tag{5.6}$$

The factor $\frac{1}{2}$ in (5.5), and the subtracted term in (5.6), are of course very familiar from the ordinary Hartree model, but nevertheless seem always to create some difficulty. Further discussion will be found in Sec. IX.

As has been stated, (5.4) and (3.11), and therefore (5.5) and (5.6), are strictly applicable only to the chosen configuration; for excited nucleons, the G_{ij} in (5.4) depends actually on several additional quantum numbers.

VI. ACTUAL AND MODEL WAVE FUNCTIONS

It is interesting to ask how closely the actual wave function Ψ_C is approximated by the model wave function Φ_C. For this purpose, we consider the actual

wave function expanded in model wave functions,

$$\Psi_C=\sum c_B\Phi_B, \tag{6.1}$$

$$c_B=(\Phi_B,\Psi_C). \tag{6.2}$$

To calculate the expansion coefficients, we use the expressions (4.1)–(4.3) for Ψ_C whose first three terms are

$$\Psi_C=\Phi_C+\sum_e \overset{Q}{I}_{ij}\Phi_C+\sum\sum_{e\ e} \overset{Q}{I}_{ij}\overset{Q}{I}_{lm}\Phi_C+\cdots. \tag{6.3}$$

The second term expresses one, the third term two scatterings.

To obtain c_B, it is convenient to consider a nucleus of given volume Ω and number of particles A. Both are assumed large and proportional to each other. We then wish to calculate the dependence of the various terms on Ω. Since Ω is assumed large, momentum will be almost conserved in the matrix elements of I_{ij} (see Sec. XV). These will of course have the same dependence on Ω as v_{ij}; the latter is given by

$$\begin{aligned}(\mathbf{k}_i'\mathbf{k}_j'|v_{ij}|\mathbf{k}_i\mathbf{k}_j)&\\ =\int\psi^*(\mathbf{k}_i',\mathbf{r}_i)\psi^*(\mathbf{k}_j',\mathbf{r}_j)v_{ij}(r_{ij})&\\ \times\psi(\mathbf{k}_i,\mathbf{r}_i)\psi(\mathbf{k}_j,\mathbf{r}_j)d\tau_i d\tau_j&\\ =\Omega^{-2}\int e^{-i\mathbf{k}_i'\cdot\mathbf{r}_i-i\mathbf{k}_j'\cdot\mathbf{r}_j}e^{i\mathbf{k}_i\cdot\mathbf{r}_i+i\mathbf{k}_j\cdot\mathbf{r}_j}&\\ \times v_{ij}(r_{ij})d\tau_i d\tau_j.&\end{aligned} \tag{6.4}$$

The factor Ω^{-2} comes from the normalization of the wave functions, each of the four wave functions having a normalization factor $\Omega^{-\frac{1}{2}}$. Now in the last form of the integral, the exponential is $\exp[i(\mathbf{k}_i-\mathbf{k}_i')\cdot(\mathbf{r}_i-\mathbf{r}_j)]$, considering that $\mathbf{k}_j'-\mathbf{k}_j=\mathbf{k}_i-\mathbf{k}_i'$ by momentum conservation. Then, holding the relative coordinate $\mathbf{r}_i-\mathbf{r}_j$ fixed, we may integrate over $d\tau_j$ and obtain a factor Ω while the integral over $\mathbf{r}_i-\mathbf{r}_j$ gives a factor independent of Ω. Thus

$$I_{ij}\sim v_{ij}\sim\Omega^{-1}. \tag{6.5}$$

Essentially the same result would be obtained for a finite nucleus, i.e., if the wave functions ψ are not exactly plane waves in the volume Ω but tail off at the surface.

If we consider two definite nucleons $\mathbf{k}_i$, $\mathbf{k}_j$, then the *amplitude* of the model state in which $\mathbf{k}_i$ has been changed to $\mathbf{k}_i'$ by means of the term I_{ij} in (6.1) is proportional to Ω^{-1}, and the "probability" of finding the nucleus in this state is therefore proportional to Ω^{-2}. Now for given $\mathbf{k}_i$ and $\mathbf{k}_j$, the final state $\mathbf{k}_i'$ can be chosen freely, and the number of possible quantum states $\mathbf{k}_i'$ per unit volume in momentum space is proportional to Ω. The other final state $\mathbf{k}_j'$ is determined by momentum conservation. Thus for given i and j, the number of possible "final" states is proportional to Ω and the total probability of exciting any of these is therefore proportional to Ω^{-1}. Since the number of pairs of nucleons ij is proportional to Ω^2, the total probability of finding the nucleus in an excited model state is proportional to Ω, the probability of the chosen state being set equal to 1. Of course this takes into account only the action of the second term in (6.1). Since Ω is proportional to A, we may set the probability of an excited model state equal to αA, where α is a numerical coefficient depending on the actual nuclear forces.

We can easily generalize this result, especially if we assume that there are no linked clusters containing more than two particles each. Then (6.1) is equivalent to a product over all possible pairs of particles,

$$F=\prod_{ij}\left(1+\overset{Q}{I}_{ij}\right). \tag{6.6}$$

Each term gives a probability $2\alpha/A$ for finding just the pair ij excited, with α defined in the last paragraph. Since there are $\frac{1}{2}A^2$ pairs, the total probability of finding any pair excited is $e^{\alpha A}$ if we still set the probability of the chosen model state equal to 1. However, it is more convenient to change the normalization and to set the *total* probability of any model state equal to one. Then the probability of "finding" the nucleus "actually" in the state described by the model wave function is

$$|(\Phi_C,\Psi_C)|^2\sim e^{-\alpha A}. \tag{6.7}$$

This is exponentially small for a large nucleus. Therefore it would be entirely wrong to say that a large nucleus is "actually" in its model state; the probability for this is "infinitesimal," and is smaller the larger the nucleus. The result (6.7) is not changed if clusters of three or more particles are taken into account, except of course for the value of α.

The difficulty of many[27] "old-fashioned" treatments of nuclear structure was that they attempted to obtain the actual nuclear wave function Ψ_C which is complicated when expressed in model functions Φ_C. The great achievement of the Brueckner theory is that it permits the calculation of energy levels and other nuclear properties in terms of the model function of the chosen configuration, Φ_C. For the energy this has already been demonstrated. We shall now discuss other properties.

The most important observable properties of a nucleus (other than the energy) depend on one-nucleon operators, such as the magnetic moment which is the sum of the magnetic moment operators of the individual nucleons, the dipole moment for an optical transition between two nuclear levels, or the momentum distribution of a nucleon which influences some of the experi-

[27] Swiatecki (to be published) has shown, however, that the energy of a nucleus may be obtained by ordinary perturbation theory provided the interaction between nucleons is well-behaved, and that then each term in the perturbation expansion is proportional to A.

ments mentioned in Sec. I. We shall therefore consider the evaluation of one-nucleon operators, in particular of the diagonal matrix elements for the chosen configuration of an operator $M=\sum M_i$, where M_i is a one-nucleon operator such as the magnetic moment; we have

$$\langle M\rangle=\sum_i(\Psi_C,M_i\Psi_C)/(\Psi_C,\Psi_C). \tag{6.8}$$

We consider Ψ_C expanded in terms of model wave functions. Then, because of the orthogonality of the model wave functions,

$$(\Psi_C,M_i\Psi_C)=\sum_B\sum_{n_i}\sum_{n_i'}(\Psi_C,\Phi_{Bn_i'})\times(\Phi_{Bn_i},\Psi_C)(\psi_{n_i'},M_i\psi_{n_i}), \tag{6.9}$$

where B is a configuration of the $A-1$ nucleons other than i, and Bn_i denotes the configuration of our nucleus in which the $A-1$ nucleons are in configuration B and nucleon i in state n_i. The last factor in (6.9) depends only on the states of nucleon i; it is immaterial whether the other nucleons are excited or not. We are therefore only interested in

$$\sum_B(\Psi_C,\Phi_{Bn_i'})(\Phi_{Bn_i},\Psi_C), \tag{6.10}$$

which represents essentially the probability that one particular nucleon i is excited. This probability is of the order of magnitude one, regardless of the size of the nucleus. This follows from (6.7); the desired probability (6.10) will be the ratio of the excitation probability for all nucleons, $e^{\alpha A}$, to that for $A-1$ nucleons, $e^{\alpha(A-1)}$, giving e^α which is independent of A.

This result makes it possible to calculate the expectation value (6.8) with a finite amount of labor. To carry out such a calculation, it will probably be convenient to use the same techniques as have been applied in field theory, especially the method of Feynman diagrams.[26] The chosen configuration corresponds to the vacuum state in field theory; a transition of one particle is considered as the creation of a pair, consisting of a hole in the state n_i^0 of the chosen configuration and of a particle in the state n_i' outside the Fermi sphere. The interaction v_{ij} or G_{ij} connects two particle lines. The operator M is then an additional operator which may be inserted in one of the particle lines; it is similar to the action of an external potential in field theory.

The analogy with field theory is close. The physical vacuum state has only an infinitesimal projection on the bare vacuum state, just as the actual wave function Ψ_C of the nucleus has only a very small projection on the model state Φ_C, Eq. (6.7). Nevertheless, the properties of single electrons (such as their magnetic moment), or the interaction of two particles, can be derived in field theory by considering just these particles themselves and a limited number of photons, electron pairs, etc. with which they interact directly. One may disregard all disconnected diagrams and in the nuclear case all unlinked clusters.

VII. DEFINITION OF THE MATRICES $\bar{G}$ AND I

While the reaction matrix G is defined by Sec. III, particularly by Eqs. (3.1) and (3.7), there is still some arbitrariness in the definition of the matrix $\bar{G}$ and hence of I, Eq. (4.4). There is further arbitrariness in the one-nucleon potential V. This can be used to make the operator in (4.13),

$$\sum T_i+\sum\bar{G}_{ij} \tag{7.1}$$

diagonal in the model wave functions. Then e in (4.3) becomes a number, i.e., e is also diagonal in the model wave functions. Furthermore, it is desirable to choose I_{ij} such that it has no diagonal matrix elements, and no matrix elements leading from the chosen configuration to configurations in which only one nucleon is excited, because this choice will greatly reduce the number of terms which needs to be considered in the perturbations w_1 to w_3, Eqs. (4.14) to (4.16).

We thus set

$$(n_in_j|I_{ij}|n_in_j)=0, \tag{7.2}$$

$$(n_in_j|\bar{G}_{ij}|n_in_j)=(n_in_j|G_{ij}|n_in_j), \tag{7.3}$$

where, if n_i and/or n_j are normally unoccupied, the right-hand side of (7.3) should still depend on the states left empty in the chosen configuration (see Sec. III). The choice (7.3) actually does not affect the above condition that (7.1) should be diagonal; the diagonal elements of (7.1) can be chosen arbitrarily without violating this condition.

Since the one-nucleon wave functions are orthogonal, the only nonvanishing elements of the operator T_i are (a) the diagonal elements and (b) elements in which the state of nucleon i changes and all other nucleons remain in the same state. Therefore $\bar{G}$ must not have any elements in which *both* nucleons i and j change their state,

$$(n_i'n_j'|\bar{G}_{ij}|n_in_j)=0 \quad \text{if } n_i',\, n_j'\neq n_i,\, n_j, \tag{7.4}$$

$$(n_i'n_j'|I_{ij}|n_in_j)=(n_i'n_j'|G_{ij}|n_in_j) \quad \text{if } n_i',\, n_j'\neq n_i,\, n_j. \tag{7.5}$$

If one nucleon (i) changes its state, we must have

$$(n_i'|T_i|n_i)+\sum_j(n_i'n_j|\bar{G}_{ij}|n_in_j)=0, \quad \text{if } n_i'\neq n_i. \tag{7.6}$$

Thus the sum in (7.6) must be chosen to be independent of the states occupied by the other nucleons, and it is most convenient to set

$$(n_i'n_j|\bar{G}_{ij}|n_in_j)=(n_i'n_j^0|G_{ij}|n_in_j^0) \quad \text{if } n_i'\neq n_i, \tag{7.7}$$

whether or not $n_j=n_j^0$, and whether or not all the other nucleons k are in the chosen configuration. However, as we have shown in Sec. III, the right hand side of (7.7) is still not defined if *both* n_i and n_i' differ from the chosen state n_i^0; but if *either* of them is equal to n_i^0, there is no ambiguity; in the other case, we may choose an arbitrary state n^0 to be empty, but this must be taken to be the same state for *all* matrix elements of type

(7.7), i.e., independent of i. Once $\bar{G}$ is chosen, I is determined by (4.4); in particular, (7.7) gives

$$(n_i n_j^0|I_{ij}|n_i^0 n_j^0)=0 \quad \text{if } n_i \neq n_i^0. \tag{7.8}$$

Thus the elements of I which start from the chosen configuration and go to configurations with only one nucleon excited are indeed zero as desired.

Since the model wave functions are orthogonal and the one-nucleon wave functions satisfy the Schrödinger equation (2.3), we have

$$(n'|T|n)+(n'|V|n)=0 \quad \text{if } n' \neq n. \tag{7.9}$$

In order to satisfy (7.6), (7.9), and (7.7), we must choose the one-nucleon potential such that

$$(n'|V|n)=\sum_j (n'n_j|\bar{G}_{ij}|nn_j) \tag{7.10}$$

$$=\sum_j (n'n_j^0|G_{ij}|nn_j^0). \tag{7.11}$$

This result was already used in (3.12). The matrix elements of V must of course be independent of the choice of the particular nucleon i in (7.7); therefore the sum in (7.10) must not exclude any nucleon j. If either n or n' is equal to n_i^0, the term $j=i$ in (7.11) vanishes automatically because of the Pauli principle, so that there is no difficulty in this case. Thus the matrix elements of V between an occupied and an empty state are uniquely and reasonably defined by (7.11) and depend only on the initial and final state of a single nucleon. Those between two empty states cannot be uniquely defined as was shown in (3.13); since G is not a proper two-nucleon operator, V cannot be a proper one-nucleon operator. This ambiguity can only be resolved by arbitrary choice of these elements of V, but this choice has only little influence on the one-nucleon wave functions (end of Sec. III).

It will be shown in Sec. XV that the nondiagonal matrix elements of V and T, and the elements of G_{ij} in which only one nucleon changes its state, are small, of order $\Omega^{-\frac{1}{2}}$ or less compared with the other matrix elements.

VIII. SPIN CONSIDERATIONS

In this section we wish to evaluate the matrix elements of the interaction,

$$(n_i' n_j'|v_{ij}|n_i n_j)_N, \tag{8.1}$$

and in particular to evaluate the sums over spin and charge of the two nucleons. We are particularly interested in the case when n_i, n_j are the states of particles i and j in the chosen configuration, although nearly all our results will also apply to more general cases. If $n_i=n_i^0$ and $n_j=n_j^0$, then (8.1) corresponds to the removal from the chosen configuration of the nucleons in states n_i^0 and n_j^0, and their placing into the states $n_i' n_j'$, so that the final configuration is defined by four quantum numbers, $n_i^0 n_j^0 n_i' n_j'$, which describe the states empty in the chosen configuration and those occupied outside that configuration. The states $n_i^0 n_j^0$ should therefore be considered as having definite spins and charges, these being determined by the spins and charges of the nucleons which are left in the same orbitals (spatial wave functions) after n_i^0 and n_j^0 have been removed.[28]

We shall denote the spatial wave function (orbital) of the nucleon state n_i by m_i, the spin state by α_i and the charge state by γ_i, and similarly for the states n_j, n_i', and n_j'. Whenever we need to specify *definite* values of spin or charge, we call the spin wave functions α and β, and the charge wave functions γ and δ. The interaction will be written for the present as

$$v_{ij}=v_1+v_2\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j, \tag{8.2}$$

where v_1 and v_2 depend on the spatial coordinate $\mathbf{r}_{ij}$ but may still contain a Majorana exchange operator P_M. As is well known, the symmetry of the nuclear wave function in i and j needs to be taken into account only once, and we choose to do this in the final state; thus we take the initial state to be

$$|n_i n_j\rangle=m_i\alpha_i\gamma_i(i)m_j\alpha_j\gamma_j(j), \tag{8.3}$$

and the final state

$$|n_i' n_j'\rangle=m_i'\alpha_i'\gamma_i'(i)m_j'\alpha_j'\gamma_j'(j) - m_j'\alpha_j'\gamma_j'(i)m_i'\alpha_i'\gamma_i'(j). \tag{8.4}$$

With these definitions, no normalizing factor $1/\sqrt{2}$ is required. Finally, in applying (8.2) to (8.3), it is useful to distinguish immediately the cases when the two particles have equal or opposite spin.

A. Initial spin equal, $\alpha_i=\alpha_j=\alpha$ (this is not meant to imply that the spin is necessarily up). Then also in the final state we must have $\alpha_i'=\alpha_j'=\alpha$; otherwise the matrix element will vanish. Then, without loss of generality, we may set $\gamma_i'=\gamma_i$, $\gamma_j'=\gamma_j$ (if $\gamma_i=\gamma_j$, this makes no difference; if $\gamma_i\neq\gamma_j$, this defines which state is called n_i'). The application of (8.2) to (8.3) gives

$$v_{ij}|n_i n_j\rangle=(v_1+v_2)m_i(i)m_j(j)\alpha\gamma_i(i)\alpha\gamma_j(j) \tag{8.5}$$

and the matrix element with (8.4) is

$$(n_i' n_j'|v_{ij}|n_i n_j) =(m_i'(i)m_j'(j),\,(v_1+v_2)m_i(i)m_j(j)) - (\gamma_i\gamma_j)(m_j'(i)m_i'(j),\,(v_1+v_2)m_i(i)m_j(j)). \tag{8.6}$$

The scalar product $(\gamma_i\gamma_j)$ is 1 if $\gamma_i=\gamma_j$ and 0 if $\gamma_i\neq\gamma_j$; thus the second (exchange) term in (8.6) will be present only when the two interacting nucleons have initially the same charge as well as the same spin. The matrix elements remaining unevaluated in (8.6) are purely spatial matrix elements, for which we write in abbreviated notation:

$$(n_i' n_j'|v_{ij}|n_i n_j)=(v_1+v_2)_D-(\gamma_i\gamma_j)(v_1+v_2)_X. \tag{8.7}$$

[28] I am indebted to J. Goldstone for pointing out that the wave function of the two "removed" nucleons, i, j should not be symmetrized by itself in spin, charge, and space, because this is in general incompatible with leaving the "residual nucleus" after removal of i and j in a definite configuration.

D meaning "direct" and X "exchange." It should be noted that in the final state spin and charge of both particles i, j are the same as in the initial state.

B. Initial spins different, $\alpha_i=\alpha$, $\alpha_j=\beta$. Then without loss of generality we set $\alpha_i'=\alpha$, $\alpha_j'=\beta$, thus defining which of the two final states is n_i'. Application of (8.2) on (8.3) gives

$$v_{ij}|n_in_j\rangle=(v_1-v_2)m_i(i)m_j(j)\alpha\gamma_i(i)\beta\gamma_j(j) +2v_2m_i(i)m_j(j)\beta\gamma_i(i)\alpha\gamma_j(j). \quad (8.8)$$

We then have to distinguish two cases for the charges.

1. Initial charges equal, $\gamma_i=\gamma_j=\gamma$. Then also the final charges must be $\gamma_i'=\gamma_j'=\gamma$. Using (8.4), the matrix elements are found to be

$$(n_i'n_j'|v_{ij}|n_in_j)=(v_1-v_2)_D-2v_{2X}. \quad (8.9)$$

2. Initial charges different, $\gamma_i=\gamma$ and $\gamma_j=\delta$. This is the only case where two different final states are possible:

(a) Final charges=initial charges, $\gamma_i'=\gamma$, $\gamma_j'=\delta$. Then

$$(n_i'n_j'|v_{ij}|n_in_j)=(v_1-v_2)_D. \quad (8.10)$$

(b) Charges interchanged in final state, $\gamma_i'=\delta$, $\gamma_j'=\gamma$. Then

$$(n_i'n_j'|v_{ij}|n_in_j)=-2v_{2X}. \quad (8.11)$$

It is convenient now to introduce the interaction in singlet and triplet state, v_s and v_t, instead of v_1 and v_2. We have

$$v_t=v_1+v_2, \quad (8.12)$$

$$v_s=v_1-3v_2, \quad (8.13)$$

and we get from (8.7) to (8.11) the following matrix elements:

1. Charges and spins equal for both nucleons, wave functions $\alpha\gamma$, $\alpha\gamma$ (both initially and finally):

$$v_{ij}=v_{tD}-v_{tX}. \quad (8.14)$$

2. Spins equal, charges different, wave functions $\alpha\gamma$, $\alpha\delta$ (initially and finally):

$$v_{ij}=v_{tD}. \quad (8.15)$$

3. Spins different, charges equal, wave functions $\alpha\gamma$, $\beta\gamma$ (initially and finally):

$$v_{ij}=\tfrac{1}{2}(v_{tD}+v_{sD})+\tfrac{1}{2}(v_{sX}-v_{tX}). \quad (8.16)$$

4. Spins and charges different, initial wave functions $\alpha\gamma$, $\beta\delta$: (a) final wave functions the same $\alpha\gamma$, $\beta\delta$:

$$v_{ij}=\tfrac{1}{2}(v_{tD}+v_{sD}). \quad (8.17)$$

(b) final charges interchanged, wave function $\alpha\delta$, $\beta\gamma$:

$$v_{ij}=\tfrac{1}{2}(v_{sX}-v_{tX}). \quad (8.18)$$

The most important matrix element is the diagonal one, summed over all spins and charges of nucleon j, keeping those of nucleon i fixed. This is obtained by summing Eqs. (8.14) to (8.17) which yields

$$\sum_{\text{spin, charge}} v_{ij}=3v_{tD}+v_{sD}+\tfrac{1}{2}v_{sX}-\tfrac{3}{2}v_{tX}. \quad (8.19)$$

The coefficients, 1 and 3, of v_s and v_t, are the statistical weights. The signs of the exchange terms reflect the symmetry of the spatial wave function. The factors $\frac{1}{2}$ with the exchange terms arise from the fact that these terms exist only between two like nucleons.

If the two nucleons are to be followed from the chosen configuration through a set of intermediate states and back to the chosen configuration, their spins and charges are, in most cases, simply left unchanged in the process. Only in the single case when both spin and charge are initially different, two kinds of intermediate states must be considered: the two nucleons may retain spin as well as charge, or else they may retain their spins but exchange their charges. The latter elements will be absent if $v_{sX}=v_{tX}$, i.e., in particular if the triplet and singlet forces are assumed to be the same. This is not a bad approximation to the known interaction in the S state. The forces in the odd-parity states of two nucleons are not sufficiently well known to decide whether $v_s=v_t$ is a good approximation.

If the equation for the scattering matrix, (3.1), is solved by iteration, as in Sec. X, then in the second approximation the effect of a pair of intermediate orbitals $m_i'm_j'$ will be given by the sum of the squares of the matrix elements (8.14) to (8.18). Although the result is elementary we shall not give it, but shall first simplify our assumptions about the interaction.

We assume now that the spatial dependence of the interaction is that of a Serber force, i.e.,

$$v_{ij}(\mathbf{r}_{ij})=v(r_{ij})\tfrac{1}{2}(1+P_M), \quad (8.20)$$

where P_M is the Majorana exchange operator. As is well known, the Serber force is zero for all two-nucleon states of odd parity (odd orbital momentum), while for even parity it is simply $v(r_{ij})$. Later on (Sec. XI) we shall consider the assumption that there is interaction only in S states; this is a special case of the Serber force for which the following equations, especially (8.28), remain valid, although the analytic form (8.20) is not.

If (8.20) is assumed, then

$$v_D=v_X=\tfrac{1}{2}(m_i'm_j'|v(r_{ij})|m_im_j) +\tfrac{1}{2}(m_j'm_i'|v(r_{ij})|m_im_j), \quad (8.21)$$

where the right-hand terms are ordinary spatial matrix elements,

$$(m'n'|v(r_{ij})|mn)=\int\psi_{m'}^*(\mathbf{r}_i)\psi_{n'}^*(\mathbf{r}_j) \times v(r_{ij})\psi_m(\mathbf{r}_i)\psi_n(\mathbf{r}_j)d\tau_id\tau_j. \quad (8.22)$$

Because of the equality $v_D=v_X$, the matrix elements (8.14) to (8.18) simplify considerably:

1. Spin and charge of the two nucleons equal:

$$v_{ij}=0. \tag{8.23}$$

There is, in this case, no interaction because the Pauli principle requires the spatial wave function to be antisymmetric which makes it odd parity.

2. Spin equal, charge different:

$$v_{ij}=v_t. \tag{8.24}$$

3. Spin different, charge equal:

$$v_{ij}=v_s. \tag{8.25}$$

Pauli principle, and even parity of the spatial wave function (Serber force) together require an antisymmetric spin function, hence a singlet state.

4. Spin and charge different.

(a) Final wave function same as initial:

$$v_{ij}=\tfrac{1}{2}(v_t+v_s). \tag{8.26}$$

(b) Final wave function has charges interchanged:

$$v_{ij}=\tfrac{1}{2}(v_t-v_s). \tag{8.27}$$

The sum over all diagonal elements, (8.23) to (8.26), which was previously given by (8.19), now simplifies to

$$\sum v_{ij}=\tfrac{3}{2}(v_t+v_s). \tag{8.28}$$

This permits a simple interpretation: we consider the interaction of a given nucleon i with all the four nucleons in the orbital m_j. This is found to be effectively equal to the interaction with *three* nucleons, since that with the nucleon of the same spin and charge is exactly zero, Eq. (8.23). The effective interaction with each of the three nucleons is the average of the triplet and the singlet interaction. This is related to the fact that for even parity, there are equally many triplet as singlet states. This may be seen by considering the 16 possible spin-charge states of two nucleons: ten of these have symmetric spin-charge wave function and hence antisymmetric spatial wave functions, i.e., odd parity (9 of these have isotopic spin $T=1$ and spin $S=1$ while one has $T=0$, $S=0$). Six states have antisymmetric spin-charge wave function, of which 3 are $T=1$, $S=0$ and three are $T=0$, $S=1$.

We mentioned previously that the second-order contribution to the scattering matrix from an intermediate orbital m_i' is given by the sum of the squares of the matrix elements, in our case (8.23) to (8.28). This yields

$$\sum(v_{ij})^2=\tfrac{3}{2}(v_s^2+v_t^2). \tag{8.29}$$

Since the observed interaction is not very different in triplet and singlet even states, no great error will be made if $\frac{1}{2}(v_s^2+v_t^2)$ is replaced by the square of the average interaction, $\frac{1}{2}(v_s+v_t)$.

Throughout this paper, we shall therefore adopt the following simplified procedure. We replace the nuclear matrix elements of v by the average of singlet and triplet interaction for states of even parity. With these elements of v, we then calculate the G matrix, using (3.1). From G we obtain the effective one-particle potential V by assuming that each nucleon interacts only with 3 out of 4 of the other nucleons (i.e., with 3 nucleons in each orbital m_i).

Although we shall use the Serber interaction throughout this paper, it is still interesting to write down the average interaction (8.19) for the more general case when there is a Serber interaction v_e in even states and another interaction v_o in states of odd parity. Thus we write

$$v_{tD}=v_{te}+v_{to}, \quad v_{tX}=v_{te}-v_{to}, \tag{8.30}$$

and similarly for the singlet. Then (8.19) becomes

$$\sum v_{ij}=\tfrac{3}{2}v_{te}+\tfrac{3}{2}v_{se}+(9/2)v_{to}+\tfrac{1}{2}v_{so}. \tag{8.31}$$

Thus each type of state contributes according to its statistical weight, the triplet odd states being $T=S=1$ (weight 9) and the singlet odd ones $T=S=0$ (weight 1, see above).

IX. INFINITE NUCLEUS

The simplest case to which the theory can be applied is that of an infinite nucleus, in other words of nuclear matter. Clearly, the Coulomb interaction must be neglected in order to get finite results. In this case, the state of nuclear matter is fully characterized by its density (or more generally, by the neutron and proton density separately).

All positions are equivalent, and the proper wave functions are obviously plane waves. This obviates what is normally a most difficult part of determining the self-consistent field, *viz.*, the search for proper wave functions. The energy of a nucleon in the chosen configuration, which in Eq. (3.11) was shown to depend only on the state n_i^0 of that nucleon, will now depend only on its momentum $\mathbf{k}$, and we may write

$$E(k)=V(k)+k^2/2M. \tag{9.1}$$

In other words, the "self-consistent potential" V is now diagonal in a momentum representation,

$$(\mathbf{k}'|V|\mathbf{k})=V(k)(2\pi)^3\delta(\mathbf{k}-\mathbf{k}'). \tag{9.2}$$

The self-consistent field problem then consists primarily in finding $V(k)$. It may be noted that we have here a situation opposite to that of the conventional Hartree model where V is diagonal in the *position*; the general case discussed in Sec. II where V is neither diagonal in $\mathbf{r}$ nor in $\mathbf{k}$ comprises the Brueckner theory of an infinite nucleus and the Hartree theory as special cases.

The energy of a cubic centimeter[29] of nuclear matter is

$$(2\pi)^{-3}\int 4d^3k[\tfrac{1}{2}V(k)+k^2/2M], \tag{9.3}$$

[29] It is more convenient to calculate immediately energies per unit volume or per particle than to calculate first the total energy and then express the result in terms of the total number of particles.

where the factor $\frac{1}{2}$ in the first term arises again from the need of counting each pair of interacting nucleons only once, while the factor 4 in front comes from the fact that for each momentum state $\mathbf{k}$ there are four nucleons, of two spin directions and two charge values.

The determination of $V(k)$ is the only problem of self-consistency which is treated by Brueckner *et al.* In fact, Brueckner[12] has pointed out that this may be a rather tedious problem because the value of $V(k)$ is required in the second term of (3.1) or (9.10), and this second term may be large and, with Brueckner's procedure, is indeed large. Fortunately, however, this second term is greatly reduced if the exclusion principle is taken into account immediately (see Sec. X) in its evaluation (which Brueckner does not do), and in this case a very rough initial choice for $V(k)$ will suffice for the calculation of the second term in (3.1).

We shall use for the nucleons plane-wave functions normalized to unit density. Then, if the interaction $v_{ij}(r_{ij})$ is an ordinary potential, its matrix elements for one-nucleon functions are[30]

$$(\mathbf{k}_i'\mathbf{k}_j'|v_{ij}(r)|\mathbf{k}_i\mathbf{k}_j)_P = w_{ij}(\mathbf{k}_i-\mathbf{k}_i')(2\pi)^3\delta(\mathbf{k}_i'+\mathbf{k}_j'-\mathbf{k}_i-\mathbf{k}_j), \quad (9.4)$$

where δ is the 3-dimensional δ function and

$$w_{ij}(q)=\int v_{ij}(r_{ij})e^{i\mathbf{q}\cdot\mathbf{r}_{ij}}d^3r_{ij} \quad (9.5)$$

is the Fourier transform of the nucleon interaction. If we use a Serber force (8.20), then the matrix element of v_{ij} for the nucleus is given by (8.21) which becomes in our case

$$(\mathbf{k}_i'\mathbf{k}_j'|v_{ij}|\mathbf{k}_i\mathbf{k}_j)_N=\tfrac{1}{2}[w_{ij}(\mathbf{k}_i-\mathbf{k}_i')+w_{ij}(\mathbf{k}_i-\mathbf{k}_j')] \times(2\pi)^3\delta(\mathbf{k}_i'+\mathbf{k}_j'-\mathbf{k}_i-\mathbf{k}_j). \quad (9.6)$$

In the second (exchange) term, the nucleon i goes into the final state $\mathbf{k}_j'$ and *vice versa*.[31]

The Eq. (3.1) determining the scattering matrix G becomes now an integral equation[32]:

$$(\mathbf{k}_i'\mathbf{k}_j'|G_{ij}|\mathbf{k}_i\mathbf{k}_j) = (\mathbf{k}_i'\mathbf{k}_j'|v_{ij}|\mathbf{k}_i\mathbf{k}_j)_N-(2\pi)^{-6}\int d^3k_i'' \times d^3k_j''(\mathbf{k}_i'\mathbf{k}_j'|v_{ij}|\mathbf{k}_i''\mathbf{k}_j'')_N \times\frac{1}{E(k_i'')+E(k_j'')-E(k_i^0)-E(k_j^0)} \times(\mathbf{k}_i''\mathbf{k}_j''|G_{ij}|\mathbf{k}_i\mathbf{k}_j), \quad (9.7)$$

where k_i^0 and k_j^0 denote the states occupied by nucleons i and j in the chosen configuration. In (9.7) and the following, we are making the very essential simplification of assuming that (3.8) is valid for the energy of the intermediate state. We thus disregard the complications which were extensively discussed in Sec. III, *viz.*, that the scattering matrix G_{ij} and the energy of excited states $E(k_i)$ depend on the empty states and on other excited nucleons which may be present. The influence of this simplification will be briefly discussed in Sec. X. Since each matrix element of v_{ij} contains a δ function expressing conservation of momentum, we write

$$(\mathbf{k}_i'\mathbf{k}_j'|v_{ij}|\mathbf{k}_i\mathbf{k}_j)_N = (\mathbf{k}_i'|\tilde{v}_{ij}|\mathbf{k}_i\mathbf{k}_j)_N(2\pi)^3\delta(\mathbf{k}_i'+\mathbf{k}_j'-\mathbf{k}_i-\mathbf{k}_j), \quad (9.8)$$

$$(\mathbf{k}_i'\mathbf{k}_j'|G_{ij}|\mathbf{k}_i\mathbf{k}_j)_1 = (\mathbf{k}_i'|\tilde{G}_{ij}|\mathbf{k}_i\mathbf{k}_j)(2\pi)^3\delta(\mathbf{k}_i'+\mathbf{k}_j'-\mathbf{k}_i-\mathbf{k}_j), \quad (9.9)$$

and obtain the integral equation

$$(\mathbf{k}_i'\mathbf{k}_j'|\tilde{G}_{ij}|\mathbf{k}_i\mathbf{k}_j) = (\mathbf{k}_i'\mathbf{k}_j'|\tilde{v}_{ij}|\mathbf{k}_i\mathbf{k}_j)_N-(2\pi)^{-3}\int d^3k_i'' \times(\mathbf{k}_i'\mathbf{k}_j'|\tilde{v}_{ij}|\mathbf{k}_i''\mathbf{k}_j'')_N \times\frac{P}{E(\mathbf{k}_i'')+E(\mathbf{k}_i+\mathbf{k}_j-\mathbf{k}_i'')-E(\mathbf{k}_i^0)-E(\mathbf{k}_j^0)} \times(\mathbf{k}_i''\mathbf{k}_j''|\tilde{G}_{ij}|\mathbf{k}_i\mathbf{k}_j), \quad (9.10)$$

where of course $\mathbf{k}_j''$ and $\mathbf{k}_j'$ are defined by momentum conservation, and P means the principal part.

In Sec. III we have shown that the intermediate states $\mathbf{k}_i''\mathbf{k}_j''$ must be different from all the states occupied by other nucleons, $\mathbf{k}_l^0$. Further, since the principal value of the integral is to be taken, they cannot *both* be equal to the chosen states $\mathbf{k}_i^0\mathbf{k}_j^0$ but at at least one of them must represent an unoccupied state. Now momentum conservation requires that if $\mathbf{k}_i''\neq\mathbf{k}_i$, then also $\mathbf{k}_j''\neq\mathbf{k}_j$; so at least if i and j are initially in the chosen states ($\mathbf{k}_i=\mathbf{k}_i^0$ and $\mathbf{k}_j=\mathbf{k}_j^0$), then *both* $\mathbf{k}_i''$ and $\mathbf{k}_j''$ must be unoccupied states. Hence, with plane wave functions, we may simply use the principal value P of the integral, and we do not need to use the more restrictive operator Q of Sec. III.

[30] The factor $(2\pi)^3$ in (9.4) and further equations is most easily understood as follows: If the momentum components of the momenta $\mathbf{k}_i$, $\mathbf{k}_j$, $\mathbf{k}_i'$, $\mathbf{k}_j'$ are considered quantized in a large volume, then momentum conservation permits exactly one quantized momentum state for $\mathbf{k}_j'$ if $\mathbf{k}_i$, $\mathbf{k}_j$, and $\mathbf{k}_i'$ are given, and for this one state the matrix element is w, Eq. (9.5). In the continuum treatment, we have to integrate over k_j' momentum space with the volume element $(2\pi)^{-3}d^3k_j'$; if then the integrand is Eq. (9.4), the integration yields (9.5).

[31] Brueckner, in reference 15, erroneously takes the final state to be $-\mathbf{k}_i'$ which is correct only if the center of mass of the two nucleons is at rest.

[32] The integral in (9.7) goes only *once* over each momentum space k_i'', k_j'' and does not contain factors 4 for spin and charge. This is because the spin-charge function of the two nucleons in the intermediate state $k_i''k_j''$ is completely determined, *viz.*, it is the same as in the initial state k_ik_j, and in the final state $k_i'k_j'$, as was shown in Sec. VIII.

As we have just shown, the integral in (9.10) extends over all states in which *both* $\mathbf{k}_i''$ and $\mathbf{k}_j''=\mathbf{k}_i+\mathbf{k}_j-\mathbf{k}_i''$ are greater than k_F, the radius of the Fermi sphere. The value of k_F is given by

$$(2\pi)^{-3}4(4\pi/3)k_F^3(4\pi/3)r_0^3=1, \tag{9.11}$$

where r_0 is the radius of the sphere containing one nucleon. The most accurate information on r_0 for actual nuclei comes from the Stanford electron scattering experiments[33] and gives (for Au for which experiments and analysis are best)

$$r_0=(1.180\pm0.012)\times10^{-13}\text{ cm}. \tag{9.12}$$

Equation (9.11) yields

$$k_F=3^{2/3}\pi^{1/3}/2r_0=1.524/r_0=1.29\times10^{13}\text{ cm}^{-1}. \tag{9.13}$$

The potential energy V defined in (9.2) is now

$$V(k_i)=3(2\pi)^{-3}\int_0^{k_F}d^3k_j(\mathbf{k}_i\mathbf{k}_j|\tilde{G}_{ij}|\mathbf{k}_i\mathbf{k}_j), \tag{9.14}$$

i.e., the integral over the diagonal elements of the reaction matrix G. As was explained in Sec. VIII, the factor 3 takes into account the fact that nucleon i interacts with the three nucleons of momentum $\mathbf{k}_j$ which differ from it in either spin or charge. It may be noted that $Mv_{12}(r_{12})$ is of dimension (length)$^{-2}$; for the Fourier transform defined in (9.5) we have $Mw_{12}(q)$ $\sim$(length)$^{+1}$, and we shall see later (Sec. XI) that Mw_{12} is related to the effective range of the potential; the $\tilde{G}$ matrix elements are of the same dimensions as w, and $V(k)$ according to (9.14) is again of the dimension of a potential, i.e. $MV\sim$length^{-2}.

The potential energy $V(k)$, and therefore the total energy of one nucleon $E(k)=V(k)+k^2/2M$, will depend on the radius of the Fermi sphere in momentum space, k_F, in addition to its dependence on k. Similarly, the interaction between two nucleons, $(\mathbf{k}_i\mathbf{k}_j|\tilde{G}_{ij}|\mathbf{k}_i\mathbf{k}_j)$, will depend on k_F, because the integral in (9.10) extends over the momentum space outside the Fermi spheres $k_i''=k_F$ and $k_j''=k_F$. If $\tilde{G}$ were replaced by $\tilde{v}$, this quantity would no longer depend on k_F but the potential energy $V(k)$, Eq. (9.14), would of course still do so.

The total energy of a nucleus containing A nucleons is,[34] according to (5.5),

$$W_{\text{tot}}=AW=\Omega\times4(2\pi)^{-3}\int^{k_F}d^3k\left[\tfrac{1}{2}V(k)+\frac{k^2}{2M}\right], \tag{9.15}$$

where Ω is the volume of the nucleus.[35] This clearly depends on the density of nuclear matter,

$$\rho=A/\Omega=(3/4\pi)r_0^{-3}=(2/3\pi^2)k_F^3, \tag{9.16}$$

and may be calculated as a function of ρ. However, as was pointed out by Eden[19,20] (see also Sec. II of this paper), we have not yet taken into account the requirement that the wave functions and the potential V be self-consistent. This requirement cannot easily be formulated for an infinite nucleus but this can be done for one which is extremely large. The proper procedure would of course be to choose a potential matrix $(\mathbf{r}'|V|\mathbf{r})$, calculate the wave functions $\psi_n(\mathbf{r})$, reconstruct V from this, etc., until a self-consistent solution is found. This would clearly be very difficult, and furthermore, since the nucleus is very large, it is clear physically that the exact dependence of $(\mathbf{r}'|V|\mathbf{r})$ on the positions of $\mathbf{r}$ and $\mathbf{r}'$ within the boundary layer will have a negligible influence on the wave functions and the energy of the nucleus, because it will affect only the surface energy. Obviously, the only parameter in V which will seriously influence the nuclear energy is the over-all dimension of the potential well, i.e., the size of the nucleus for a given number A of nucleons. A problem of this type, where self-consistency depends almost entirely on the choice of one parameter, is eminently adapted to the variational method. Indeed, Eden[19,20] has shown that the self-consistent field problem in this case is equivalent to the variational problem of making the total energy a minimum as a function of the density ρ.

Thus, while it is formally possible to calculate the $V(k)$ and the total energy for an arbitrary density, the result would in general not approximate the self-consistent solution for a large but finite nucleus. Only if the total energy is made a minimum as a function of density, will a self-consistent solution be obtained. The "solutions" for other densities may thus be considered as spurious, but they are of course useful for obtaining such quantities as the compressibility of nuclear matter, etc.

The variational condition may be rewritten

$$W=\int d^3k\left[\tfrac{1}{2}V(k)+\frac{k^2}{2M}\right]\Big/\int d^3k, \tag{9.17}$$

$$\frac{dW}{d\rho}=dW/dk_F=0. \tag{9.18}$$

Since W is a function of density only, the extra energy added when another nucleon is added to a large nucleus and the *density* kept constant, is given by W; in other words, the binding energy is $-W=|W|$. Also if the *volume* is kept constant, the extra energy is W by virtue of (9.18); we have

$$\left(\frac{\partial W_{\text{tot}}}{\partial A}\right)_\Omega=W+A\left(\frac{\partial W}{\partial A}\right)_\Omega=W+\rho\frac{dW}{d\rho}=W. \tag{9.19}$$

Another way of calculating the binding energy of an extra nucleon is to consider the energy per unit volume, W_{vol}, and differentiate it with respect to the density,

[33] Hahn, Ravenhall, and Hofstadter, Phys. Rev. **101**, 1131 (1956), where references to the earlier literature are also given.

[34] This time the sum goes over all nucleons regardless of spin, and hence there is simply a factor 4.

[35] One should avoid calling the volume of the nucleus v or V because this would be confused with the interaction or the potential.

i.e., effectively k_F. We have

$$\left(\frac{\partial W_{\text{tot}}}{\partial A}\right)_\Omega=\Omega\left(\frac{\partial W_{\text{vol}}}{\partial A}\right)_\Omega=\frac{dW_{\text{vol}}}{d\rho}$$
$$=\frac{\pi^2}{2k_F^2}\frac{d}{dk_F}\frac{1}{2\pi^3}\int d^3k\left[\tfrac{1}{2}V(k)+\frac{k^2}{2M}\right], \quad (9.20)$$

using (9.16). Inserting (9.14), this becomes

$$\left(\frac{\partial W_{\text{tot}}}{\partial A}\right)_\Omega=\frac{1}{k_F^2}\frac{d}{dk_F}\left[\int k^2dk\frac{k^2}{2M}+\frac{3}{4}\frac{1}{16\pi^4}\right.$$
$$\left.\times\int^{k_F}\int d^3k_1d^3k_2(\mathbf{k}_1\mathbf{k}_2|\tilde{G}_{ij}|\mathbf{k}_1\mathbf{k}_2)\right]. \quad (9.21)$$

The first term, the kinetic energy, gives

$$k_F^2/2M, \quad (9.22)$$

i.e., the kinetic energy of the fastest nucleon. The second term gives three contributions: (a) from differentiation with respect to the upper limit of the k_1 integral, (b) from the same for k_2, and, (c) from the dependence of $\tilde{G}_{ij}$ on k_F arising from the solution of (9.10). We shall disregard contribution (c); this will be shown (Sec. X) to be a good approximation for normal or higher density. The contributions (a) and (b) are equal, and the factor 2 arising from this cancels the factor $\frac{1}{2}$ with V in (9.15). Since the volume element in k_1 space is $4\pi k_1^2dk_1$, twice the contribution of (a) is

$$\frac{3}{4}\frac{2\times4\pi}{16\pi^4}\frac{k_F^2}{k_F^2}\int d^3k_2(k_Fk_2|\tilde{G}_{ij}|k_Fk_2)=V(k_F), \quad (9.23)$$

if we use (9.14). Together with (9.22), we have then

$$\frac{\partial W_{\text{tot}}}{\partial A}=V(k_F)+\frac{k_F^2}{2M}. \quad (9.24)$$

This is the total one-nucleon energy of the most energetic nucleon, so that we obtain

$$W=E(k_F). \quad (9.25)$$

Thus, if (c) can be neglected, one may either use the eigenvalue of the most energetic nucleon $E(k_F)$, *or* the average energy W from (9.17).

It is perhaps somewhat surprising that one calculates in one case an average energy and in the other case a maximum energy, and that these two should be equal. This is made possible by the factor $\frac{1}{2}$ in the potential energy when the average W is calculated in (9.17); this should just compensate for the difference between average and maximum. This argument shows, of course, that the energy of the top of the Fermi distribution, $E(k_F)$, must be negative. The relation (9.25) is only true if the nuclear density is chosen so as to make W a minimum; otherwise, W does not represent the energy of an additional nucleon, added at constant volume, as can be seen from (9.19). On the other hand, this energy *is* correctly given by $E(k_F)$ provided the above-mentioned contribution (c) is negligible.

X. EXCLUSION PRINCIPLE

We shall now try to solve the integral equation (9.10) for the scattering matrix $\tilde{G}_{ij}$. Contrary to the procedure of Brueckner,[12] we shall take the Pauli exclusion principle into account from the beginning. It will be shown that this makes a large difference to the solution, and further, that the solution becomes easier rather than more difficult by taking the Pauli principle into account.

For simplicity, we shall consider the case[36]:

$$\mathbf{k}_j=-\mathbf{k}_i. \quad (10.1)$$

By momentum conservation in (9.10), we then have also

$$\mathbf{k}_i'+\mathbf{k}_j'=\mathbf{k}_i''+\mathbf{k}_j''=0. \quad (10.2)$$

The last of these relations brings the simplification that if $k_i''>k_F$, then also $k_j''>k_F$ so that the allowed (unoccupied) states in the integral (9.10) all lie outside the sphere $k_i''=k_F$, and

$$E(\mathbf{k}_i+\mathbf{k}_j-\mathbf{k}_i'')=E(k_i'')>E_F, \quad (10.3)$$
$$E(k_j)=E(k_i)<E_F. \quad (10.4)$$

The solution is best illustrated by an example. We take $v_{ij}(r)$ to be a Yukawa interaction, which we write in terms of the effective range theory[37] in the form[38]

$$Mv_{ij}=-\frac{3.56s}{br}e^{-2.12r/b}, \quad (10.5)$$

where b is the intrinsic range which we choose to be 2.5×10^{-13} cm, and s is the strength paramater. We choose $s=1$, which means that the scattering length for the two-body system at zero energy is infinite, corresponding to exact resonance at zero energy.[39] It will be convenient to introduce

$$a=b/2.12=1.19\times10^{-13}\text{ cm}; \quad (10.6)$$

then

$$Mv_{ij}=-\frac{1.68}{ar}e^{-r/a}. \quad (10.7)$$

The Fourier transform (9.5) of (10.7) is

$$Mw_{ij}(q)=4\pi\times1.68\frac{a}{1+(qa)^2}. \quad (10.8)$$

For small q, this is proportional to the range a of the Yukawa potential. The potential matrix for a Serber

[36] The more general case, $\mathbf{k}_i+\mathbf{k}_j\neq0$, is at present being considered by Mr. Thouless of the Cavendish Laboratory, Cambridge.

[37] It is a great advantage to use this theory so that one may be sure that all parameters are chosen consistently, in agreement with each other and with the evidence from two-body experiments.

[38] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949), especially Eq. (4.4)

[39] Actually, a slightly greater value of s would be better since we have to take the average of singlet and triplet interaction.

force becomes, when we use (9.6), (9.8), and (10.2):

$$(\mathbf{k}_i'\mathbf{k}_j'|M\tilde{v}_{ij}|\mathbf{k}_i\mathbf{k}_j)_N$$
$$=2\pi\times 1.68a\left[\frac{1}{1+a^2(\mathbf{k}_i-\mathbf{k}_i')^2}+\frac{1}{1+a^2(\mathbf{k}_i-\mathbf{k}_j')^2}\right]. \quad (10.9)$$

For the energies we use the same approximation as Brueckner,

$$E(k)=k^2/2M^*, \quad (10.10)$$

where the "effective mass" M^* is to be determined by the condition of self-consistency, and we introduce the new notation

$$v'=-M(\tilde{v}_{ij})_N, \quad G'=-M\tilde{G}_{ij}, \quad (10.11)$$

so that these quantities are now positive. We further drop the reference to the state of particle j which is given by (10.2) and the subscript i in $\mathbf{k}_i$, etc. Then (9.10) becomes

$$(\mathbf{k}'|G'|\mathbf{k})=(\mathbf{k}'|v'|\mathbf{k})+(2\pi)^{-3}(M^*/M)$$
$$\times\int_{k_F} d^3k''(\mathbf{k}'|v'|\mathbf{k}'')\frac{1}{k''^2-k^2}(\mathbf{k}''|G'|\mathbf{k}). \quad (10.12)$$

This is very similar to Brueckner,[12] Eq. (7), except for the lower limit k_F on k''. To start with, we shall now make the further simplifying assumption that $k=0$; then the two terms in (10.9) will become equal.

We shall now show that the integral in (10.12) is numerically rather small compared with the first term on the right-hand side of (10.12). To show this, we shall use an iteration procedure: We *assume* that $G'=v'$ in first approximation, we insert this into the integral and evaluate it, and then show that indeed the result is small. We calculate in particular the diagonal term

$$(\mathbf{k}|G'|\mathbf{k}=0)=1.68\times 4\pi a\left[1+(2\pi)^{-3}(1.68)(4\pi a)\right.$$
$$\left.\times\frac{M^*}{M}\times 4\pi\int_{k_F}^{\infty}\frac{k''^2dk''}{k''^2}\frac{1}{(1+k''^2a^2)^2}\right]. \quad (10.13)$$

The integral can easily be evaluated as a function of

$$\alpha=k_Fa=1.29\times 1.19=1.53, \quad (10.14)$$

(see 9.12, 10.6), and the result is

$$\frac{(k|G'|k=0)}{(k|v'|k=0)}=1+\frac{2}{\pi}(1.68)\frac{M^*}{M}\frac{1}{2}\left(\arctan\frac{1}{\alpha}-\frac{\alpha}{1+\alpha^2}\right)$$
$$=1+0.064M^*/M. \quad (10.15)$$

Thus even if $M^*=M$, the integral is only about 6% of the first-order term, $(\mathbf{k}|v'|\mathbf{k})$.

This means that the *Born approximation is well justified*, a most remarkable result. The reason for this result is of course the exclusion principle, which greatly reduced the value of the integral. Indeed, this effect of the exclusion principle arises from the same circumstances as the rapid convergence of Brueckner's "linked cluster" expansion,[15] namely the fact that the matrix elements of the two-body interaction for momentum changes of the order of k_F are small. The small result for the second term in (10.15) justifies, of course, the use of the iteration procedure for solving (10.12). It further justifies the use of the rough approximation (10.10) for the energy in the denominator.

It is easy to see that the contribution of the second term in (10.15) would be even smaller if the density had been chosen higher. This would give a large value of k_F and of α, and in this limit the term in parentheses in (10.15) becomes $2/(3\alpha^3)$, and

$$\frac{(k|G'|k=0)}{(k|v'|k=0)}=1+0.36\alpha^{-3}\frac{M^*}{M}. \quad (10.16)$$

The contribution thus decreases inversely as the density of the nucleus, and the Born approximation becomes increasingly good at high density because the Pauli principle takes more and more effect.[40]

On the other hand, if the exclusion principle had not been taken into account, k_F would have to be replaced by zero. Then α in (10.14) is zero, and (10.15) is replaced by

$$1+0.84M^*/M. \quad (10.17)$$

In this case, the deviation from Born approximation is large, and the iteration procedure is no longer applicable. This is the case treated by Brueckner,[12] and in this case it is necessary to solve the integral equation (10.12) explicitly, instead of simply carrying out a quadrature. Brueckner was able, in his case, to reduce the integral equation to a differential equation in coordinate space, a reduction which becomes more complicated if the Pauli principle is taken into account.[23] However, a differential equation is still more complicated than a quadrature, so that the exclusion principle still remains a simplification.

It might be argued[41] that a good approximation could still be obtained by solving Eq. (10.12) without taking the Pauli principle into account. Indeed, we have shown that a good approximation to (10.12) is obtained if G' in the integral is replaced by v' and then the Pauli principle is taken into account. The total potential energy of the nucleus is then proportional to

$$\int d^3k_id^3k_j(\mathbf{k}_i\mathbf{k}_j|\tilde{G}|\mathbf{k}_i\mathbf{k}_j)$$
$$=\int d^3k_id^3k_j(\mathbf{k}_i\mathbf{k}_j|\tilde{v}|\mathbf{k}_i\mathbf{k}_j)$$
$$+(2\pi)^{-3}\int d^3k_id^3k_jd^3k_i''d^3k_j''|(\mathbf{k}_i\mathbf{k}_j|\tilde{v}|\mathbf{k}_i''\mathbf{k}_j'')|^2$$
$$\times\frac{\delta(\mathbf{k}_i''+\mathbf{k}_j''-\mathbf{k}_i-\mathbf{k}_j)}{E_i''+E_j''-E_i-E_j}, \quad (10.18)$$

[40] The same conclusion has been reached by W. J. Swiatecki. Starting from perturbation theory and taking into account the exclusion principle, he finds that the Born approximation is quite good at the observed nuclear density while it would be very poor at low density. I am indebted to Dr. Swiatecki for sending me his manuscript before publication.

[41] See, e.g., Brueckner, reference 12, footnote 5.

where the integral over k_i'' and k_j'' goes only over unoccupied states. Now, in this form, nothing would be changed if the integral over k_i'' and k_j'', were extended also over occupied states, because interchange of the integration variables $k_i''k_j''$ with k_ik_j would give the same contribution with opposite sign. However, three things are wrong with this argument. First, the energies in the denominator in (10.18) will be different according to whether the Pauli principle is taken into account or not, so that the contribution of the unoccupied states $k_i''k_j''$ will be different in the two cases (it will be less when the Pauli principle is not taken into account because the effective mass will then be lower). Secondly, while the first iteration provides a good solution of (10.12) when the Pauli principle is taken into account, this is not so when it is disregarded [see Eq. (10.17)]. In this latter case, then, there is no reason for (10.18) to be a good approximation to the energy of the nucleus. Third, the interchange of integration variables is only possible if *both* k_i'' and k_j'' are occupied. A large contribution comes from k_i'' occupied, k_j'' empty or *vice versa*, and this is not compensated by interchange of initial and intermediate states.

The small contribution of the integral in (10.12) has been demonstrated only for the case $k=0$. For $k\neq 0$, the denominator k''^2-k^2 in (10.12) becomes smaller. However, crude estimates show that the increase of importance of the second term in (10.12) with k is very slow, and only when k is *very* close to k_F does it become appreciable. Mr. Thouless of Cambridge University has done quantitative calculations which show that the correction to the potential energy of a particle remains small even near k_F, although it is then of course larger than for $k=0$.

In correspondence with K. A. Brueckner, the following interesting comparison has been developed between the solution with and without exclusion principle (EP). According to (10.18), the *average* value of the second term in the iteration expansion of G is the same in both solutions [disregard the third argument given below Eq. (10.18)]. From (10.15) and (10.17), we know that for $k=0$ this second term is larger without EP, leading to a lowering of the energy level $k=0$. For $k=k_F$, the reverse must then be the case, i.e., the energy level lies higher without the EP than with it. [Indeed, in some cases, the highest occupied level calculated without EP turned out to lie above zero energy, which is incompatible with Eq. (9.25).] Generally, it follows that the nucleon levels calculated without EP will be too much spread out in energy; in other words, that the effective mass will turn out too low. This in turn will reduce the value of the second iterated term in G and thus the error made by neglecting the EP. The error in the average binding energy due to neglect of EP should be less than in the effective mass.

Our method of solution of (10.12) holds of course only for "well-behaved" interactions $v_{ij}(r)$, such as the "classical" shapes Yukawa, exponential, Gaussian, and square well. It obviously cannot hold if the interaction contains a repulsive core, because in this case the Fourier transform (6.5) of the potential is infinite (and repulsive). The integral equations (9.10) and (10.12) are still alright if v is interpreted as the limit of a finite repulsive core. But the solution by iteration becomes meaningless; therefore Brueckner *et al.*, who were mostly interested in repulsive-core potentials, could not make use of our iteration procedure. Instead, it is necessary to transform the equations (9.10) and (9.12) back to coordinate space. In this form, the equations are more complicated with the exclusion principle than without, and it is therefore natural that Brueckner in his first solution[12] neglected the EP.

A solution of the problem of a pure repulsive core with Pauli principle has been obtained by Bethe and Goldstone,[23] using a suitable coordinate space equation. Brueckner (private communication) has obtained a numerical solution for a more realistic case, *viz.*, a deep and narrow attractive well outside a repulsive core, and has found that in this case the effect of the exclusion principle is not large.

Finally, we want to consider briefly the complications discussed in Sec. III. These can easily be taken into account if the integral equations can be solved by iteration, as seems to be the case for our simple potentials. We may as a first approximation set $G_{ij}=v_{ij}$, obtain the potential energy $V(k)$ from (9.14), and set $E(k)=k^2/2M+V(k)$, for all k. Then, in second approximation, we set $G_{ij}=v_{ij}$ under the integral in (9.10)—as we did in solving (10.12)—and insert the just-obtained values of $E(k)$ in the denominator of (9.10). To this approximation, the complications of Sec. III have no influence on the reaction matrix and on the energy of a nucleon in the chosen configuration. Now we use the same (second) approximation to calculate the reaction matrix for an excited state; according to (3.7), we get, with the notation of (10.12):

$$(\mathbf{k}_i'\mathbf{k}_l'|G_{il}'|\mathbf{k}_i\mathbf{k}_l^0;\mathbf{k}_i^0\mathbf{k}_j^0,\mathbf{k}_j)=(\mathbf{k}_i'\mathbf{k}_l'|v_{il}'|\mathbf{k}_i\mathbf{k}_l^0)_N+(2\pi)^{-3}\int d^3k_i''(\mathbf{k}_i'\mathbf{k}_l'|v_{il}'|\mathbf{k}_i''\mathbf{k}_l'')_N$$

$$\times\frac{M}{E(k_i'')+E(\mathbf{k}_i+\mathbf{k}_l^0-\mathbf{k}_i'')+E(\mathbf{k}_j)-E(k_i^0)-E(k_l^0)-E(k_j^0)}(\mathbf{k}_i''\mathbf{k}_l''|v_{il}'|\mathbf{k}_i\mathbf{k}_l^0)_N. \quad (10.19)$$

The diagonal elements of this matrix,

$$(\mathbf{k}_i\mathbf{k}_l^0|G_{il}'|\mathbf{k}_i\mathbf{k}_l^0;\mathbf{k}_i^0\mathbf{k}_j^0,\mathbf{k}_j),$$

are then inserted into (3.5) to give the (second approximation to) the energy of the state in which nucleons i and j are excited. Since the second term of (10.19) will again be a few (maybe 3)[42] percent of the first, its *dependence* on the empty states k_i^0, k_j^0 and on the extra occupied state k_j will be only a fraction of this amount; let us say, about one percent for the common choices of these three states. Thus in practice the energy of the state k_i will depend on the supernumerary quantum numbers k_i^0, k_j^0, k_j only to about 1%, and to this accuracy it *is* possible to define $V(k_i)$ and $E(k_i)$. In third approximation, then, the energies of the excited states k_i, k_j as calculated from (10.19) can be inserted in the denominator of (9.10), and at the same time $\bar{G}$ in (9.10) replaced by the second approximation calculated in (10.15). Since the second term in (10.15) is only about 5% of the first, and the approximation of $\mathcal{E}(k_i,k_j;k_i^0,k_j^0)$ of (3.5) by the denominator in (9.10) is good to about 1%, the error in the energy of the *ground* state caused by this approximation will be about 0.05% which is completely negligible.

Thus the complications discussed in Sec. III are of no practical importance for well-behaved nucleon interactions such as the Yukawa interaction discussed in this section. However, for potentials with a repulsive core these complications may be much more important.

XI. INTERACTION IN S STATES ONLY

For an adequate treatment of the saturation problem, it is necessary to use nuclear forces with a repulsive core. A method for solving the integral equation (9.10) with such forces and with the Pauli principle is given by Bethe and Goldstone.[23] However, much work will be required before reliable results can be obtained. It is therefore desirable, for a preliminary orientation, to investigate a potential which gives saturation without having a repulsive core.

Such a potential is suggested by Brueckner's paper[12] on the evaluation of his theory. He uses an interaction which exists only in S states, which he justifies on physical grounds.[43] Now such an interaction, even if it contains no repulsive core,[44] will always lead to saturation. This can be seen most easily by a statistical argument. A given nucleon in a large (infinite) nucleus will interact essentially with all those nucleons which are in S states with respect to the given nucleon, and are within a sphere whose radius is equal to the range of the nuclear forces, b. Thus the number of interactions per nucleon is proportional to the number of S states within a sphere of radius b, which in turn is proportional to k_Fb. Therefore the attractive potential energy is proportional to k_F and hence to the cube root of the density. [This will be shown quantitatively in (11.17) and (11.19).] The kinetic energy is proportional to k_F^2. The combination of two such terms clearly leads to a minimum of the total energy at some definite value of k_F and hence to saturation. The difference from the case of an ordinary or a Serber force is, of course, that in these latter two cases the potential energy per nucleon is proportional to the density, and thus to k_F^3, so that no minimum of the total energy exists.

Interaction in S states only does, of course, not correspond to an ordinary potential, but we may speak of the potential acting between two nucleons in an S state and denote it by $v_{ij}(r)$. We must now derive the matrix elements $(\mathbf{k}_i'\mathbf{k}_j'|v_{ij}|\mathbf{k}_i\mathbf{k}_j)_P$. It is convenient to use center-of-mass coordinates so that $\mathbf{k}_j=-\mathbf{k}_i$ and then to drop the subscript i. The desired matrix element is then the spherically symmetric part of the Fourier transform of $v_{ij}(r)$, or in other words, the average of the Fourier transform over all directions of the final momentum $\mathbf{k}'$, keeping the direction of the incident wave fixed. Thus we have, using a notation similar to (9.8) and (9.5):

$$(\mathbf{k}'|\tilde{v}_{ij}|\mathbf{k})_P=\int\frac{d\omega'}{4\pi}w(\mathbf{k}-\mathbf{k}'). \tag{11.1}$$

Now w depends only on

$$q=|\mathbf{k}-\mathbf{k}'|, \tag{11.2}$$

and if θ is the angle between $\mathbf{k}$ and $\mathbf{k}'$,

$$d\omega'=2\pi\sin\theta d\theta=2\pi qdq/kk', \tag{11.3}$$

so that

$$(\mathbf{k}'|\tilde{v}_{ij}|\mathbf{k})_P=(2kk')^{-1}\int_{|k-k'|}^{k+k'}qdqw(q). \tag{11.4}$$

Since the S state has even parity, exactly the same result will be obtained if the forces are Majorana exchange or Serber forces.

The integral in (11.4) gives a slightly simpler result if we use an exponential potential rather than the Yukawa interaction between two nucleons. Again using the effective range formula of Blatt and Jackson,[38] Eq. (4.4), we have

$$Mv_{ij}(r)=-\frac{1.446s}{a^2}e^{-r/a}, \tag{11.5}$$

where $a=b/3.54$, with b the intrinsic range of Blatt and Jackson. Taking again $b=2.5\times10^{-13}$ cm, we have $a=0.706\times10^{-13}$ cm. The strength parameter s will again be set equal to one (exact resonance). Then

$$-Mw(q)=-4\pi\int Mv_{ij}(r)r^2dr\frac{\sin qr}{qr}$$
$$=8\pi\times1.446a(1+a^2q^2)^{-2}, \tag{11.6}$$

[42] Since the denominator in (10.19) contains three excitation energies rather than the two in (9.10), the value of the integral term in (10.19) is estimated to be only $\frac{2}{3}$ of that in (10.12).

[43] If one assumes a Serber force, there is no interaction in P states. D-state interaction is small at the normal nuclear density provided there is a strong tensor force [Brueckner (private communication)].

[44] Brueckner's S-state interaction does contain a repulsive core.

and (11.4) becomes, with the notation (10.11):

$$(k'|v'|k)=\frac{1.446\times 8\pi a}{2kk'}\int\frac{qdq}{(1+a^2q^2)^2}$$
$$=\frac{1.446\times 8\pi a}{[1+a^2(k-k')^2][1+a^2(k+k')^2]}. \quad (11.7)$$

Now it was shown in Sec. X that the reaction matrix G' is nearly equal to the potential matrix v'. We are then interested in the diagonal elements $k'=k$ of v', and we get

$$(k|v'|k)=\frac{1.446\times 8\pi a}{1+4a^2k^2}. \quad (11.8)$$

Going back to the laboratory system, we have $2\mathbf{k}=\mathbf{k}_i-\mathbf{k}_j$, which we shall denote by $\mathbf{K}$, and

$$M(\mathbf{k}_i'\mathbf{k}_j'|v|\mathbf{k}_i\mathbf{k}_j)$$
$$=-\frac{1.446\times 8\pi a}{1+a^2K^2}(2\pi)^3\delta(\mathbf{k}_i'+\mathbf{k}_j'-\mathbf{k}_i-\mathbf{k}_j). \quad (11.9)$$

The potential acting on one particle k_i is, from (9.14),

$$-MV(k_i)=-3(2\pi)^{-3}\int d^3k_j(\mathbf{k}_i\mathbf{k}_j|G_{ij}'|\mathbf{k}_i\mathbf{k}_j)$$
$$=\frac{1.446\times 24\pi a}{(2\pi)^3}\frac{2\pi}{k_i}\int_0^{k_F}k_jdk_j\int_{|k_i-k_j|}^{k_i+k_j}\frac{KdK}{1+a^2K^2}. \quad (11.10)$$

This can easily be evaluated by integrating first over k_j and then over K. If we use

$$\kappa=k_ia, \quad \kappa_F=k_Fa, \quad (11.11)$$

the result is

$$-MV(k_i)=\frac{1.446\times 6}{\pi a^2}\left[\frac{\kappa_F^2-\kappa^2+1}{4\kappa}\ln\frac{1+(\kappa_F+\kappa)^2}{1+(\kappa_F-\kappa)^2}\right.$$
$$\left.+\kappa_F-\arctan(\kappa_F+\kappa)-\arctan(\kappa_F-\kappa)\right]. \quad (11.12)$$

In the limit $\kappa=0$ we get for the expression in square brackets in (11.12)

$$[\ ]=2(\kappa_F-\arctan\kappa_F), \quad (11.13)$$

while in the limit $\kappa=\kappa_F$ the result is

$$[\ ]=\frac{1}{4\kappa_F}\ln(1+4\kappa_F^2)+\kappa_F-\arctan 2\kappa_F. \quad (11.14)$$

If, in addition, κ_F is very small, both of these expressions reduce to $\frac{2}{3}\kappa_F^3$ which can easily be checked directly from (11.10). In this limit (i.e., for very low density), the potential V does not depend on velocity.

For the adopted nuclear constants, $k_F=1.29\times 10^{13}$ and $a=0.706\times 10^{-13}$, we have

$$\kappa_F=0.91, \quad (11.15)$$
$$[\ ]=0.346 \quad \text{for } \kappa=0,$$
$$[\ ]=0.244 \quad \text{for } \kappa=\kappa_F. \quad (11.6)$$

Thus the potential decreases by a factor of 1.42 from the slowest to the fastest nucleon.

For very high density, we have

$$[\ ]=2(\kappa_F-\pi/2) \quad \text{for } \kappa=0,$$
$$[\ ]=\kappa_F-\pi/2 \quad \text{for } \kappa=\kappa_F, \quad (11.17)$$

so that there is then a factor of 2 between slowest and fastest nucleon. As was shown in Sec. X, the assumption $G=v$ is best in this limit.

The most interesting quantity is the *average* of the potential V over the Fermi sphere [one-half of this quantity is the average potential energy per nucleon, see Eq. (9.15)]. This is obtained by integrating first over k_i and k_j and last over K. The result is

$$-MV_{Av}=-M\int_0^{k_F}V(k_i)k_i^2dk_i\Big/\int k_i^2dk_i$$
$$=\frac{1.446\times 6}{\pi a^2}\left[-\frac{1}{4\kappa_F}+\frac{3}{2\kappa_F}+\frac{1}{\kappa_F}\left(\frac{3}{4}+\frac{1}{16\kappa_F^2}\right)\right.$$
$$\left.\times\ln(1+4\kappa_F^2)-2\arctan 2\kappa_F\right]. \quad (11.18)$$

For small κ_F, the expression in square brackets is of course again $\frac{2}{3}\kappa_F^3$. For large κ_F, it is

$$[11.18]=\tfrac{3}{2}(\kappa_F-\tfrac{2}{3}\pi) \quad (11.19)$$

which approaches the mean of the two values given in (11.17). For $\kappa_F=0.91$, it is

$$[11.18]=0.280 \quad (11.20)$$

closer to the lower of the two values in (11.16). [11.18] means the expression in square brackets in Eq. (11.18).

The kinetic energy of any nucleon is

$$MT=\tfrac{1}{2}k^2=\frac{1}{2a^2}\kappa^2, \quad (11.21)$$

and the average kinetic energy is

$$MT_{Av}=\tfrac{1}{2}(\tfrac{3}{5}k_F^2)=\frac{3}{10a^2}\kappa_F^2. \quad (11.22)$$

The binding energy per nucleon is conveniently written in the form

$$W=\frac{3}{10Ma^2}\{-\kappa_F^2+(4/3)\alpha[11.18]\}. \quad (11.23)$$

where

$$\alpha=15\times 1.446/2\pi=3.452, \quad (11.24)$$

TABLE I. Nuclear energies at two densities in Mev (exponential interaction, no repulsive core).

	Observed	Optimal	Optimal, large
1. Density:			
2. κ_F (in 10^{13} cm^{-1})	0.91	2.92	$\alpha \gg 1$
3. r_0 (in 10^{-13} cm)	1.18	0.367	$1.07/\alpha$
4. Mean kinetic energy	20.7	213	α^2
5. Mean potential energy (negative)	32.3	278	$2\alpha^2$
6. Mean binding energy W	11.6	65	α^2
7. Slowest nucleon, binding energy	79.7	775	$(16/3)\alpha^2$
8. Fastest nucleon, negative potential energy	56.3	420	$(8/3)\alpha^2$
9. Fastest nucleon, kinetic energy	34.5	355	$(5/3)\alpha^2$
10. Fastest nucleon, negative eigenvalue	21.8	65	α^2
11. Variation of potential energy	23.4	355	$(8/3)\alpha^2$
12. Ratio to variation of kinetic energy, r	0.68	1.00	1.6
13. M^*/M	0.595	0.500	0.384

when we recall that the average interaction energy per nucleon is one-half of the V_{Av} given in (11.18).

If we insert for [11.18] the asymptotic value (11.19), the maximum of W occurs at

$$\kappa_F = \alpha \tag{11.25}$$

[this is the reason for the choice of the coefficient in (11.23)]. For the actual value of α, 3.452, the asymptotic formula (11.19) is not accurate enough, and the minimum of (11.18) is found to occur at $\kappa_F = 2.92$. This value of κ_F is more than three times the "observed" value $\kappa_F = 0.91$, Eq. (11.15). In other words, inserting the actual strength of the nuclear forces and using interaction in S states only, we obtain a nuclear radius (in equilibrium) more than three times too small, a nuclear density about thirty times too large. [Using (11.25) and (11.24) would be even worse.]

In Table I, we give the numerical values of various physical quantities for $\alpha = 3.452$ and two different values of the density, one the observed density of nuclear matter ($\kappa_F = 0.91$) and the other the density which makes W in (11.18) a maximum (called "optimal density" in Table I, $\kappa_F = 2.92$). All energies for these two cases are given in Mev, and the range of the exponential interaction is assumed to have the value $a = 0.706 \times 10^{-13}$ cm.

In the last column of Table I, we have considered yet another case of still less physical meaning, but of some mathematical interest. We have assumed that the nuclear forces are increased in strength so that α is larger than (11.24). Then the asymptotic formulas (11.19) and (11.20) *do* become valid, and all physical quantities can be expressed in a simple manner in terms of α. In this column, all energies are in units of

$$B = 3/10 M a^2, \tag{11.26}$$

which has the value 25.0 Mev for $a = 0.706 \times 10^{-13}$ cm.

The first three rows of the table describe the case considered. The next three give the energy of an average nucleon, *viz.*, its kinetic, potential, and total binding energy; in particular, line 5 is the average potential energy per nucleon, or one-half of the average of the one-particle potential V. The seventh row gives the negative eigenvalue of a nucleon of zero momentum, and the following three refer to the fastest nucleon. Of these rows, line 8 gives the negative of the one-particle potential for $\kappa = \kappa_F$.

Line 11 gives the change of the one-particle potential from the slowest to the fastest nucleon, and line 12 the ratio of this change to the kinetic energy of the fastest nucleon (line 9). Assuming the variation of potential energy to be quadratic in κ, the effective mass is then $M^*/M = 1/(1+r)$, and is given in the last line of the table.

We first note that, according to Sec. IX, the negative eigenvalue of the fastest nucleon (line 10) should be equal to the average binding energy W (line 6), and this is indeed true for the two last columns in which the density (i.e., κ_F) has been adjusted to make W a maximum. For the "observed" nuclear density (and the adopted values of a and α), W is not a minimum and lines 10 and 6 therefore do not agree. This case, therefore, is not a solution of the self-consistent problem (Sec. IX) and the numbers in the table for this case are actually not meaningful.

It has already been pointed out that the maximum of W occurs at a very high density. Indeed, $\kappa_F = 2.92$ is 3.2 times larger than the "observed" κ_F of 0.91, and therefore the density is about 33 times too high. This shows that the assumption of interaction in S states only, while it formally gives saturation, does not give saturation at the correct density at all. It is to be hoped that the introduction of a repulsive core will improve this situation—indeed, this surely must lead to an equilibrium density far lower than corresponds to $r_0 = 0.367$ because for this value of r_0 the repulsive core from one nucleon would almost reach the next nucleon, if the conventional value $r_c = 0.6 \times 10^{-13}$ is assumed for the radius of the core.

At the high density $\kappa_F = 2.92$, all energies are naturally very high. The binding energy per nucleon, 65 Mev, may be compared with the volume energy of 15.5 Mev per nucleon deduced by Green and Engler[45] from empirical data on nuclear binding energies. At that, the theoretical binding energy is a relatively small difference between the potential and kinetic energy, each of which is over 200 Mev for an average nucleon. The eigenvalue of the slowest nucleon is even larger, *viz.*, almost 800 Mev. It is clear that none of these numbers have any relation to actual nuclear energies.

We now consider the results at the observed nuclear density, $\kappa_F = 0.91$. This is not legitimate since W is not a maximum. We could imagine, however, that an added core might change the position of the maximum of W so as to give the correct density, and might at the same

[45] A. E. S. Green and N. A. Engler, Phys. Rev. **91**, 40 (1953).

time not change W itself very much. Assuming hopefully that this can be achieved, the binding energy W of 11.6 Mev is now in reasonable agreement with the observed value of 15.5 Mev, closer in fact than our crude assumptions about the nuclear interaction warrant. Thus there is some hope that if the theory can be made to yield the correct density, it will then also yield a good value for the binding energy.

Our model, just like Brueckner's,[12] gives a large increase of the one-particle potential V with momentum. This increase is quite comparable with that of the kinetic energy. Indeed, the ratio of the two (line 12) varies from 0.7 to 1.6 for the three cases considered, increasing with increasing density. For the case $\kappa_F = 2.92$, the ratio is accidentally exactly 1. The corresponding effective masses range from about 0.6 to 0.4. If we consider the case of the observed density as significant, even though W is not maximized, we should expect an effective mass of about 0.6 which is somewhat larger than that reported by Brueckner.[12]

XII. CLUSTER TERMS

In Sec. IV it has been shown that the Brueckner method will work satisfactorily provided that certain terms, which we have called cluster terms, are small. The most important of these is the correction to the energy,

$$\Delta E = (\Phi_0 | \sum_{ij} I_{ij} L_{ij} | \Phi_0) = \sum_{ij} (I_{ij}\Phi_0, L_{ij}\Phi_0), \qquad (12.1)$$

where Φ_0 is the model wave function of the ground state, the sum goes over all pairs of nucleons, and L_{ij} includes all terms of the operator F_{ij}, Eq. (4.3), which are "linked" to the pair ij. Another interesting quantity is ΔE_i which is the contribution to (12.1) from clusters which involve one particular particle i. If the chosen configuration is degenerate, we shall be interested in the matrix elements of $\sum I_{ij}L_{ij}$ between two degenerate configurations; these will be considered in Sec. XVI. In the present section, we shall evaluate the cluster term of Eq. (12.1).

It was shown in (7.2) and (7.8) that the only nonvanishing matrix elements of I_{ij} starting from the chosen configuration C are those in which both nucleons i and j become excited. Hence $I_{ij}\Phi_0$ contains only such model states in which i and j are excited while all other nucleons are in their chosen states. L_{ij} of Sec. IV may be expanded thus:

$$L_{ij} = 1 + \sum_{l\neq i} \frac{Q}{e_{il}} I_{il} + \sum_{l\neq i} \frac{Q}{e_{jl}} I_{jl} + \sum_{l\neq i} \frac{Q}{e_{il}} I_{il} \sum_{m\neq i} \frac{Q}{e_{lm}} I_{lm}$$
$$+3 \text{ other terms involving 2 factors } I$$
$$+\text{terms involving 3 or more factors } I. \qquad (12.2)$$

In the energy denominators e, the nucleons excited at each stage have been put in evidence. The first term inserted in $L_{ij}\Phi_0$ of Eq. (12.1) gives Φ_0 which is clearly orthogonal to $I_{ij}\Phi_0$. In the second (third) term, l must be different from j (i), so that $I_{il}\Phi_0 (I_{jl}\Phi_0)$ is again orthogonal to $I_{ij}\Phi_0$. Only the next term in (12.2) can give a nonvanishing result,

$$\Delta E_3 = \sum_{ijk} \left(\Phi_0 \left| I_{ij} \frac{Q}{e_{ji}} I_{jk} \frac{Q}{e_{ki}} I_{ki} \right| \Phi_0 \right). \qquad (12.3)$$

In this term, the three nucleons i, j, and k are successively excited. There are of course also terms in which more nucleons are successively excited and deexcited.

The term (12.3) was first derived by BL, Eq. (20). It was investigated by Brueckner[15] who called it a cluster term. According to Brueckner, ΔE_3 is the contribution to the energy of a cluster of three particles which interact simultaneously. When we constructed the G matrix, in Eq. (3.1) or (9.10), the influence of particles other than the two interacting particles i,j was already taken into account, but only insofar as it could be expressed as an over-all potential which influenced the excitation energies $E(n_i'') - E(n_i^0)$ of the interacting particles i and j. Thus (12.3) (and higher cluster terms) are required only to take into account the *fluctuations* of the influence of other particles (in this case k) on the interaction of two particles i and j. From this description it becomes plausible that ΔE_3 should be small: It is necessary that while i and j interact strongly, a third particle k be so close that it also has a much higher interaction with them than the average. This is very unlikely, no matter what the nuclear density is: If it is high, then many nucleons are always close to particles i and j, and their action can be approximated by an average. If the density is low, and if no actual clusters are formed (see, however, the discussion below), then it is geometrically very improbable that three nucleons come close enough together so that all of them interact strongly—in this case, the influence of the other particles on i and j can be omitted altogether. This argument is clearly an essential point for the success of the Brueckner method. Whether it is valid can be decided only by actual calculation of ΔE_3 and possibly higher cluster integrals.

Such an actual calculation has been carried out by Bruckner[15] and will be repeated in this section. Like Brueckner, we shall assume for the present that the interaction is of the Serber type, (8.20), and has the spatial dependence of a Yukawa potential, (10.5). Other possibilities will be discussed at the end of this section.

Let us assume that the nucleons i,j,k have respectively the orbitals (spatial wave functions) a,b,c in the chosen configuration. Further we assume that the interactions take place in the order ij, jk, ki, as shown in (12.3); thus the order of the three orbitals a,b,c is meaningful. Since two or even all three orbitals may be alike, N^3 different choices of the orbitals a,b,c are possible if $N = A/4$ is the total number of orbitals occupied.

We must first discuss the influence of spin and charge. Considering a, b, and c different (the other case being unimportant), there are 64 different spin-charge combinations which may be associated with the three orbitals in the chosen configuration. We may call the three spin-charge wave functions α,β,γ; they are initially associated with a,b,c, respectively, so that the order of α,β,γ is meaningful. Several cases may arise:

1. The spin-charge states α,β,γ may all be different; there are 24 possible choices of α,β,γ corresponding to this case. Then it can be shown (see Appendix) that in the two intermediate states the spins must be the same as in the chosen state: In the first intermediate state, the wave functions of the three nucleons are $a'\alpha$, $b'\beta$, and $c\gamma$, in the second $a'\alpha$, $b\beta$, and $c'\gamma$. That is to say, we can choose the orbitals a',b',c' of the intermediate states (keeping in mind momentum conservation, however), but we have no further choice of the spins. If we disregard the difference between G' and v', as is reasonable for the Yukawa interaction according to Sec. X, then each matrix element I_{ij}, I_{jk} and I_{ki} in (12.3) becomes the matrix element of the interaction v.

2. Two of the spin-charge states may be the same; there are 36 choices of this type. It is shown in the Appendix that for a Serber force the result is zero if either $\alpha=\beta$ or $\alpha=\gamma$, and that each of the 12 states $\beta=\gamma$ gives a contribution which is the negative of that from the states of type 1. This is due to the fact that there must be three exchanges of spin between orbitals.

3. If all three spin-charge states are the same (4 cases), the result is zero.

Thus, for a Serber force, the result is as if there were only 12 rather than 64 spin-charge states for any triple of orbitals a,b,c. This factor, reducing the result, was not taken into account of Brueckner. We also propose to omit it for the moment because (a) this facilitates comparison with Brueckner, (b) the factor depends on the special assumption of a Serber force, and (c) the negative contribution of the spin distributions of type 2 seems to depend on the fact that an odd number of interactions is involved, hence the small result for 3-particle clusters may not be repeated for the 4-particle type.

We now consider again an infinite nucleus so that the initial nucleon orbitals are defined by their momenta $\mathbf{k}_i$, $\mathbf{k}_j$, and $\mathbf{k}_k$. Because of momentum conservation, only one further momentum can be chosen; we take it to be the momentum change of nucleon j in the first transition and call it $\mathbf{q}$. Then after the first transition we have

$$\mathbf{k}_i'=\mathbf{k}_i-\mathbf{q};\quad \mathbf{k}_j'=\mathbf{k}_j+\mathbf{q};\quad \mathbf{k}_k'=\mathbf{k}_k, \tag{12.4}$$

and after the second transition

$$\mathbf{k}_i'=\mathbf{k}_i-\mathbf{q};\quad \mathbf{k}_j''=\mathbf{k}_j;\quad \mathbf{k}_k''=\mathbf{k}_k+\mathbf{q}. \tag{12.5}$$

The initial states $\mathbf{k}_i$, $\mathbf{k}_j$, and $\mathbf{k}_k$, which lie inside the Fermi sphere, and the momentum change $\mathbf{q}$, must all be chosen so that the intermediate states $\mathbf{k}_i'$, $\mathbf{k}_j'$ and $\mathbf{k}_k'$ lie outside the Fermi sphere.[46]

The energy will again be assumed to depend quadratically on k, Eq. (10.10). The Fourier transform of the potential is given by (10.8), and we shall use the abbreviation

$$f(\mathbf{q})\equiv f(q)=1/[1+(qa)^2]. \tag{12.6}$$

Then with our assumptions and notations, the three-particle cluster term gives a result similar to Brueckner's[15] Eqs. (59) and (62). The contribution per particle is

$$\begin{aligned}\Delta E_3/A=&-(2\pi\times 1.68a)^3(2M^*)^2M^{-3}64(2\pi)^{-12}\\ &\times\int^{k_F}d^3k_i\int^{k_F}d^3k_j\int^{k_F}d^3k_k\int d^3q\\ &\times[f(\mathbf{q})+f(\mathbf{k}_j-\mathbf{k}_i+\mathbf{q})][f(\mathbf{q})+f(\mathbf{k}_k-\mathbf{k}_j)]\\ &\times[f(\mathbf{q})+f(\mathbf{k}_k-\mathbf{k}_i+\mathbf{q})][(\mathbf{k}_i-\mathbf{q})^2+(\mathbf{k}_j+\mathbf{q})^2\\ &-k_i^2-k_j^2]^{-1}[(\mathbf{k}_i-\mathbf{q})^2+(\mathbf{k}_k+\mathbf{q})^2-k_i^2-k_k^2]^{-1}\\ &\times\left[4(2\pi)^{-3}\int^{k_F}d^3k\right]^{-1},\end{aligned} \tag{12.7}$$

where it is understood that $|\mathbf{k}_i-\mathbf{q}|$, $|\mathbf{k}_j+\mathbf{q}|^2$ and $|\mathbf{k}_k+\mathbf{q}|$ must all be greater than k_F. The use of 2π in the first factor, rather than 4π, takes into account the factor $\frac{1}{2}$ in the Serber force.[47] The factor $(2\pi)^{-12}$ goes with the four integrals over momentum space, and 64 represents the possible spin-charge states of the three initial nucleons k_i, k_j, and k_k. The denominator $4(2\pi)^{-3}\int d^3k$ reduces the result to an energy per nucleon. The negative sign arises from the fact that there are three attractive interactions.

Now the matrix elements (12.6) decrease rapidly with increasing q. Therefore, if the z axis is chosen to be in the direction $\mathbf{q}$, the most important contributions will come from initial momenta $\mathbf{k}_j$ and $\mathbf{k}_k$ which have a positive z component, and momenta $\mathbf{k}_i$ with a negative z component, and particularly from states which lie

[46] P. C. Martin and J. Goldstone have independently pointed out to me that there is still another possible type of 3-particle cluster term: After the first intermediate state (12.4) is reached, the nucleon k fills the void left by nucleon i, while nucleon j takes up the momentum balance. This gives the alternative second intermediate state

$$\mathbf{k}_i''=\mathbf{k}_i-\mathbf{q};\quad \mathbf{k}_j''=\mathbf{k}_j+\mathbf{k}_k-\mathbf{k}_i+\mathbf{q};\quad \mathbf{k}_k''=\mathbf{k}_i.$$

The contribution from this alternative is likely to be of the same order as that from (12.5). [The possibility of letting nucleon k fill the state $\mathbf{k}_j$ does not give anything new because in this case $\mathbf{k}_j''=\mathbf{k}_k+\mathbf{q}$, $\mathbf{k}_k''=\mathbf{k}_j$, which is the same as state (12.5).]

[47] Brueckner has an additional factor $\frac{1}{8}$ which presumably is meant to take into account that each pair ij, jk, and ki in (12.3) is to be counted only once—as is well known, the sum over all pairs ij is equivalent to one-half the sum over i and j independently However, in our case, the three states i,j, and k are distinguishable by the order in which the nucleons get excited and de-excited. Thus i is the nucleon which gets excited in the first step but de-excited only in the last. Therefore we should let each of the three nucleons i, j and k *independently* assume each of the possible A states.

close to the surface of the Fermi sphere—the latter assumption will also reduce the resonance denominators in (12.7). Thus the main contribution will come from rather restricted regions of the Fermi spheres for $\mathbf{k}_i$, $\mathbf{k}_j$, and $\mathbf{k}_k$, and this is the main reason why the contribution ΔE_3 will turn out to be small.

If we assume that only these restricted regions of the Fermi sphere contribute appreciably, two simplifications are possible. First, the exchange terms $f(\mathbf{k}_j-\mathbf{k}_i+\mathbf{q})$ and $f(\mathbf{k}_k-\mathbf{k}_i+\mathbf{q})$ can be neglected because the momentum change in each case is about $2k_F$, and $f(2k_F)$ is only about 0.1 [see Eq. (10.14)]. However, the last exchange term, $f(\mathbf{k}_k-\mathbf{k}_j)$, may be large. Secondly, the denominator may be approximated by

$$(\mathbf{k}_i-\mathbf{q})^2+(\mathbf{k}_j+\mathbf{q})^2-k_i^2-k_j^2\approx 2qk_F(\mu_i+\mu_j), \quad (12.8)$$

where $\mu_j=\cos\theta_j$ [θ_j being the angle between $\mathbf{k}_j$ and the positive z axis ($\mathbf{q}$ direction)], and $\mu_i=\cos\theta_i$ (θ_i being the angle between $\mathbf{k}_i$ and the negative z axis). According to the argument of the last paragraph, both μ_i and μ_j are between 0 and 1. We have assumed in (12.8) that k_i, k_j, and k_k are all close to k_F. In (12.8), a term quadratic in q is neglected; if it were included, the convergence of the integral for large q would become more rapid [see Eq. (12.20)]. With the same approximations, $d^3k_i=2\pi k_F^2 dx_i d\mu_i$, where $x_i=k_F-k_i$ may go from 0 to $q\mu_i$, again neglecting terms of order q^2, a neglect which again overestimates the integral. Putting also $d^3q=4\pi q^2 dq$, we get then

$$\frac{\Delta E_3}{A}=-\frac{16}{(2\pi)^3}\frac{M^{*2}}{M^3}(1.68a)^3\frac{k_F^4}{\frac{1}{3}k_F^3}\int_0^\infty\frac{q^2dq}{q^2}f^2(q)$$
$$\times\int_0^1\int\int d\mu_i d\mu_j d\mu_k\int_0^{q\mu}\int\int dx_i dx_j dx_k$$
$$\times\frac{1}{\mu_i+\mu_j}\frac{1}{\mu_i+\mu_k}[f(q)+f(k_F(\mu_j-\mu_k))]. \quad (12.9)$$

The integrations over x_i, x_j, and x_k may be carried out, and f inserted from (12.6):

$$\frac{\Delta E_3}{A}=-\frac{6}{\pi^3}\left(\frac{M^*}{M}\right)^2(1.68a)^3\frac{k_F}{M}\int_0^\infty\frac{q^3dq}{(1+q^2a^2)^2}$$
$$\times\int_0^1\int\int\frac{\mu_i d\mu_i\mu_j d\mu_j\mu_k d\mu_k}{(\mu_i+\mu_j)(\mu_i+\mu_k)}$$
$$\times\left[\frac{1}{1+q^2a^2}+\frac{1}{1+\alpha^2(\mu_j-\mu_k)^2}\right], \quad (12.10)$$

with α given by (10.14).

Brueckner has considered only the first term in the square brackets. The second term is somewhat more difficult to integrate; in fact, if (12.10) is taken literally, the integral over q diverges for this term. But it is clear from the discussion above that the integrand has been greatly overestimated for $q>k_F$; therefore it is reasonable to cut off the integral at about k_F, and in this case the contributions of the two terms in the square bracket will be about the same. Since only an estimate is wanted at present, we have considered only the first term in the bracket, integrating up to ∞, and have then doubled the final result.

The integrals over μ_j and μ_k are then elementary and alike, and the whole μ integral becomes

$$\int_0^1\mu_i d\mu_i\left[1-\mu_i\ln\frac{1+\mu_i}{\mu_i}\right]^2=0.089. \quad (12.11)$$

The μ_i integral has here been evaluated numerically, which is probably the quickest way, and quite reliable since the integrand is almost constant at about 0.1 over the range from $\mu_i=0.2$ to 1. (Brueckner's result is 0.078.) The q integral gives $1/4a^4$, and

$$\frac{\Delta E_3}{A}=-2\left(\frac{3}{2\pi^3}\right)\left(\frac{M^*}{M}\right)^2(1.68)^3 0.089\frac{k_F}{Ma}, \quad (12.12)$$

where the factor 2 has been inserted to take into account the second term in the bracket in (12.10).

This result is proportional to k_F, as is Brueckner's Eq. (73). Thus the three-particle clusters will not be very important for large k_F where the main part of the potential energy will be proportional to k_F^3 if Serber forces are assumed; this is in agreement with our previous argument that these terms represent fluctuations in the effect of a third particle on the interaction of a pair and that the fluctuations become (relatively) smaller at higher density. For low density, on the other hand, although the *absolute* value of (12.12) decreases (in accord with the discussion in the beginning of this section), its *relative* importance becomes greater. This means that the independent-nucleon picture is then relatively more perturbed by the "cluster" term. As the density is lowered, a point will be reached where the cluster terms are so important that the approximation scheme of this paper no longer applies. It will then be a better starting point to assume that the nucleons associate in *actual* clusters, i.e., that we have a number of α particles widely separated from each other. That this is so will also be indicated by the fact that the energy of a dilute nucleus, calculated according to the prescriptions of this paper, will turn out to be higher than that of an assembly of noninteracting α-particles. Thus we see that the tendency to cluster formation at low density will be automatically indicated by the formalism. Conversely, the calculation at normal nuclear density indicates (see below) that the cluster terms are not important. This we consider as the best possible proof of the "shell-model" approximation used in this paper. We do not think it is necessary to make an additional *physical* assumption that clusters are

absent[48] at normal nuclear density, but we consider this a purely mathematical result of the theory.

With our adopted constants, $k_F/Ma=45$ Mev, and (12.12) becomes

$$\Delta E_3/A=-1.8(M^*/M)^2 \text{ Mev}. \quad (12.13)$$

Tentatively, we may adopt for M^*/M the value obtained in Sec. XI, Table I, for the observed nuclear density, *viz.*, 0.6. Then

$$\Delta E_3/A=-0.66 \text{ Mev}. \quad (12.14)$$

This is approximately 100 times Brueckner's result, Eq. (74),

$$(\Delta E_3/A)_{\text{Brueckner}}=-0.007 \text{ Mev}. \quad (12.15)$$

Of this discrepancy, a factor of 8 is explained by the counting of states.[47] A factor of 2 disappears in Brueckner's calculation between his Eqs. (71) and (73). Another factor of 2 was introduced by us in (12.12) to take into account the second term in the bracket in (12.10). A factor of $0.089/0.078=1.14$ comes from the evaluation of the angular integral (12.11). There is some difference in the choice of constants, which accounts for a factor 1.3 in the same direction. Finally, our evaluation of Brueckner's Eq. (73) with his constants gives 0.013 Mev.

Our result (12.14) is small but not spectacularly so. It will be greatly reduced if we take into account the previously deduced fact that actually only 12, rather than 64, spin-charge states contribute; then

$$\Delta E_3/A=-0.12 \text{ Mev}, \quad (12.16)$$

which gives about 1% correction to the binding energy. Since this is no longer entirely negligible, it would be desirable to repeat the calculation more accurately, especially the integration of (12.7). The actual result will be probably somewhat smaller than 0.12 Mev.

The result may be compared with the first-order average potential energy which is about 30 Mev at the actual nuclear density (Sec. XI, Table I), and with the second-order contribution which comes from the difference between G' and v' in (10.12) and which is 4% of the first order (with the assumed value $M^*/M=0.6$) and hence about 1.2 Mev. Thus the third-order term, (12.16), falls rather well in line with the first two orders. This result is far more plausible than that implied in Brueckner's papers: in his calculation, the second order was about as large as the first because the Pauli principle had not been taken into account, while the third-order correction (12.15) was about 1 part in 4000 of the potential energy.

Our separation into first, second, third, · · · order is of course only valid for "well-behaved" potentials for which the G matrix can be expanded into a rapidly convergent series in powers of v, as shown in Sec. X. For potentials with repulsive cores this expansion is impossible because the matrix elements of v do not exist while those of G do. The cluster expansion, on the other hand, remains meaningful. Brueckner *et al.* considered primarily potentials with repulsive cores; therefore the idea of comparing the cluster expansion with the expansion of G could not occur to them. However, it remains gratifying that for potentials for which G *can* be expanded, the various orders of this expansion and of the cluster expansion form a sequence which converges quite regularly.

We may then expect the next order, i.e., the four-particle cluster term, to be again about 10% of the third order. If we assume that there is no benefit from spin factors, i.e., nothing comparable to the factor 12/64 of the three-particle clusters, (an assumption which is quite unreasonable), the fourth order may be about 10% of (12.14) which would make it about half as large as the third order. The higher order terms would then be definitely smaller. The sign of each order is negative: In the nth order, (12.1) contains n interaction terms (negative), and $n-1$ energy denominators (also negative). This result as well as the estimates of orders of magnitude should of course be checked by actual calculation.

Even though our result (12.16) for the three-particle cluster terms is not spectacularly small, it is still small enough to make the calculation without this correction very accurate. Moreover, this correction is merely a correction in the total binding energy and will probably not affect very much the relative positions of nuclear energy levels.

That ΔE_3 is so small is of course largely due to the Pauli principle, as already pointed out by Brueckner. The action of this principle greatly reduces the part of momentum space which can contribute appreciably. Only the halves of the Fermi spheres for which k_{jz} and k_{kz} are positive, and k_{iz} negative, will contribute, and the main contributions come from the neighborhood of the "poles" of the Fermi spheres, i.e., k_{jz}, k_{kz}, and $-k_{iz}$ nearly equal to k_F. These geometrical restrictions make the integral (12.11) as small as 0.09. The restrictions, of course, exist only because the Fourier transform of the potential, (12.6), is quite small when the momentum of the nucleon changes from one side of the Fermi sphere to its opposite pole, i.e., when $q=2k_F$. The great effect of the Pauli principle therefore again depends on the relatively high density of the nucleus, compared to the relatively long range of nuclear forces.

We shall now consider some other forms of the interaction between particles. The first is the Coulomb force. In this case, the factors $(1+q^2a^2)^{-1}$ in (12.10) will be replaced by q^{-2}. Then the integral will diverge for small q, a result which is clearly due to the long range of the Coulomb force. It might possibly be remedied by using the scattering matrix G' for the Coulomb field, rather than the potential matrix v'.

The second type of interaction is one which is more

[48] Contrary to R. J. Eden, reference 20, Sec. III.

important for nuclei, *viz.*, a repulsive core. In this case, the matrix I in (12.3) must not be replaced by the interaction matrix v but must be used directly. BG[23] show that the scattering matrix for two nucleons whose center of mass is at rest may be written

$$(\mathbf{k}', -\mathbf{k}'|G'|\mathbf{k}, -\mathbf{k}) = \int w_k(y) e^{-i\mathbf{k}'\cdot\mathbf{y}} d^3y, \quad (12.17)$$

where w does not depend strongly on k. If the radius of the repulsive core r_c is small so that $k_F r_c < 1$ (which is well fulfilled for the actual dimensions), then

$$w_k(y) = \delta(y - r_c)/r_c, \quad (12.18)$$

where δ is the ordinary, one-dimensional δ function. Then G' is independent of k, and presumably also independent of the assumption that the center of mass is at rest, and we have with abbreviated notation

$$(\mathbf{k}'|G'|\mathbf{k}) = 4\pi \sin(k' r_c)/k'. \quad (12.19)$$

This takes the place of (10.8), and k' is nearly equal to q.

The main concern one might have in connection with a repulsive core is that the contribution from large q might diverge, or at least be very large. We therefore investigate (12.7), with the changed expression (12.19) for the matrix element, for large q. In this case, the previous approximation (12.8) for the energy denominator is not satisfactory but the energy denominators may instead be replaced by $2q^2$ each. The integrals over k_i, k_j, and k_k each give a factor $(4\pi/3)k_F^3$. The effective mass M^* for high q should be put equal to M. The term $-2\pi \times 1.68a[f(q) + f(\mathbf{k}_j - \mathbf{k}_i + \mathbf{q})]$ and the corresponding term with $\mathbf{k}_k$, will each be replaced by (12.19). Further, $-2\pi \times 1.68a f(\mathbf{k}_k - \mathbf{k}_j)$ will be replaced by the limit of (12.19) for small k', *viz.*, $4\pi r_c$, and $f(q)$ may be neglected by comparison. Finally, we change the 64 in (12.7) to the more correct value 12, see above. Then (12.7) is replaced by

$$\frac{\Delta E_3}{A} = \frac{12}{M}(2\pi)^{-9}\left(\frac{4\pi}{3}k_F^3\right)^2 (4\pi)^3 r_c 4\pi \times \int \frac{q^2 dq}{4q^4} \frac{\sin^2(qr_c)}{q^2}. \quad (12.20)$$

This integral clearly converges at large q and the main contribution comes from near the lower limit. This limit we put arbitrarily equal to $2k_F$, which is about the point where our approximations become seriously wrong. Then, assuming $k_F r_c \ll 1$, we obtain

$$\frac{\Delta E_3}{A} = \frac{4}{3\pi^3}\frac{k_F^2}{M}(k_F r_c)^3. \quad (12.21)$$

Using $r_c = 0.6 \times 10^{-13}$ cm, and our old value $k_F = 1.29 \times 10^{13}$ cm^{-1}, this gives 1.37 Mev. However, for these values of the constants, $k_F r_c$ is actually not very small, and more accurate evaluation gives a correction factor of about 1/3, or $\Delta E_3/A = 0.45$ Mev. This is larger than the result (12.16) for the attractive Yukawa potential but still a rather small correction, leaving the Brueckner theory a very good approximation.

It will be noted that (12.21), arising from a repulsive interaction, is positive while (12.16), coming from an attractive interaction, is negative. There will thus be partial compensation of these contributions, as was pointed out by Brueckner.[49] Further calculations will be necessary to establish the actual correction terms from 3-particle and larger clusters; for the time being, the best estimate is a few hundred kev, and the sign is uncertain.

XIII. THE DEPENDENCE ON THE MASS NUMBER A

Brueckner[15] has shown that the leading term in the energy of a large nucleus is proportional to its mass number A. This result—which is correct under his assumptions—has given rise to considerable discussion because it is well known that the energy of certain systems is not proportional to the number of particles; e.g. the energy of an atom with Z electrons is proportional to $Z^{7/3}$. Only if the forces in the system saturate should the actual binding energy be proportional to A.

The point is that Brueckner tacitly assumes that the density of particles is held constant as the size of the system is increased. With this assumption, the result $E \sim A$ is most plausible, in fact nearly obvious.[50] Now as we have discussed in Sec. IX, calculating the energy at given density gives merely a formal solution of the problem. To get the actual self-consistent solution, we must find the minimum of the energy as a function of density. Now if the forces saturate, then the density tends to a definite limit for large number of particles, and then the actual binding energy will also be proportional to A. This corresponds to the observed behavior of nuclei. If, however, the forces do not saturate, the density itself will increase with A, and the binding energy per particle will do likewise.

A convenient example of a nonsaturating force is the gravitational interaction, i.e., an attractive potential $-g^2/r$ between any two particles. The self-consistent problem for this case is easy to formulate, and not very difficult to solve. If the number of particles is very large, the Fermi statistical method may be used, and the problem then becomes similar to the Fermi-Thomas statistical distribution of the electrons in an atom. There are two differences, however: (a) the interaction between the particles is attractive rather than repulsive, and (b) there is no central body attracting the particles.

[49] Private communication through R. J. Eden.

[50] Nevertheless, this result represented great progress. The Brillouin-Wigner perturbation theory expansion, in which the energy denominators are $E - H_0$, with E the exact eigenvalue and H_0 the unperturbed Hamiltonian, gives in fourth-order terms proportional to A^2, in the sixth-order proportional to A^3, etc. [K. A. Brueckner (private communication) and BC]. In BC it was shown for the first time that these divergences with A, which must be spurious on physical grounds, do in fact cancel if E is replaced by the unperturbed energy.

Exactly the same problem as ours occurs in the theory of the density distribution in a white dwarf star[51] in which we have a degenerate electron gas held together by gravitational forces.[52]

If relativistic effects are neglected, we can easily calculate the radius R of the sphere containing A particles interacting by gravitation. The kinetic energy per particle is proportional to k_F^2, where k_F is the Fermi momentum. The potential energy per particle is the gravitational potential which is proportional to A/R, where R is the radius of the sphere. The density is $\rho\sim A/R^3$ and k_F^3 is therefore proportional to A/R^3; hence the potential energy is proportional to $A^{2/3}k_F$ and the total energy proportional to

$$\mathcal{E}=\tfrac{1}{2}k_F^2-cA^{2/3}k_F, \tag{13.1}$$

where c is a constant. The minimum of this expression is obtained for

$$k_F=cA^{2/3}. \tag{13.2}$$

The other quantities then depend on A as follows:

$$R\sim A^{1/3}/k_F\sim A^{-1/3},\quad \rho\sim AR^{-3}\sim A^2,\quad \mathcal{E}\sim A^{4/3}. \tag{13.3}$$

The volume of such a sphere is thus inversely, rather than directly, proportional to the number of particles. The radius behaves just as in the Fermi-Thomas, atom where the mean radius is proportional to $Z^{-1/3}$ and the total energy goes as $A^{7/3}$, which is also the same as in the Thomas-Fermi atom.

The behavior just described will also be obtained from the Brueckner method because it, just like the Hartree method, has the Fermi-Thomas model as its limit for high particle density. Thus the actual binding energy in our example is by no means proportional to A, but to a much higher power; but this does not necessarily affect Brueckner's proof that at given density, the energy is indeed proportional to A.

This last statement has to be qualified, however, by the condition that the energy per particle must be finite in an infinite system if the density is kept constant. This condition remains satisfied if the interaction does not saturate but has finite range. But for Coulomb forces, with their effectively infinite range, it is not satisfied: The gravitational potential at the center of a sphere of radius r at constant density is proportional to r^2 and therefore does not tend to a finite limit as r increases.

XIV. TERMS OF ORDER $1/A$

In Sec. III we stated that certain correction terms were of relative order $1/A$, and hence negligible in comparison with the main term for large nuclei. We shall now discuss these terms. In this section, we shall consider only those matrix elements which conserve momentum, as is required for an infinite nucleus; those which do not conserve momentum will be discussed in the next section. We shall first discuss the diagonal elements of the correction terms for the chosen configuration, and then, very briefly, the nondiagonal terms which mix the model states.

We shall here consider the terms w_2 and w_3, Eqs. (4.15) and (4.16). These terms were already considered by BL, Sec. III. We shall follow their argument, but with the important difference that we consider the matrix element

$$(\Phi_C,w_m\Phi_C), \tag{14.1}$$

while they used

$$(\Psi_C,w_m\Phi_C). \tag{14.2}$$

It was shown in footnote 25 that the use of (14.2) is incorrect, and we shall show now that (14.1) is much simpler and gives a much smaller result than (14.2).

The diagonal element of w_2 is

$$\begin{aligned}(\Phi_C,w_2\Phi_C)&=-\sum_{ij}\Big(\Phi_C,\bar{G}_{ij}\underset{e}{Q}I_{ij}L_{ij}\Phi_C\Big)\\&=-\sum_{ij}\Big(\underset{e}{Q}\bar{G}_{ij}\Phi_C,I_{ij}L_{ij}\Phi_C\Big).\end{aligned} \tag{14.3}$$

But $\bar{G}_{ij}$, according to its definition, (7.4), (7.10), and (7.11), can excite at most one nucleon out of the chosen configuration. On the other hand, I_{ij} operating on Φ_C must excite *both* nucleons. Therefore, at least if L_{ij} is replaced by 1, Eq. (14.3) gives exactly zero. The first nonvanishing contribution is of the three-particle cluster type, *viz.*:

$$\sum_{ijk}\Big(\Phi_C,\bar{G}_{ij}\underset{e_{ij}}{Q}I_{ij}\underset{e_{jk}}{Q}I_{jk}\underset{e_{ki}}{Q}I_{ki}\Phi_C\Big), \tag{14.4}$$

where $\bar{G}_{ij}$ excites either nucleon i or j only. Of the I operators, at least one must violate momentum conservation; $\bar{G}$ does the same, and it is shown in Sec. XV that each such operator introduces at least a factor $A^{-\frac{1}{3}}$. Thus (14.4) will be at least a factor A smaller than the analogous term (14.7) below, and is therefore entirely negligible. In BL, on the other hand, this term gives the main contribution, Eq. (26), which they evaluate to be about 10 Mev for the whole nucleus (their Appendix B).

The diagonal element of w_3 is

$$\begin{aligned}(\Phi_C,w_3\Phi_C)&=-\sum_{ij}\Big(\Phi_C,v_{ij}\underset{e_{ij}}{Q}\bar{G}_{ij}L_{ij}\Phi_C\Big)\\&=-\sum\Big(\bar{G}_{ij}\underset{e_{ij}}{Q}v_{ij}\Phi_C,L_{ij}\Phi_C\Big).\end{aligned} \tag{14.5}$$

The operator v_{ij}, in contrast to $\bar{G}_{ij}$ in (14.3), can excite both nucleons i and j, and the operator $\bar{G}_{ij}$ will then leave the two nucleons excited. The structure of (14.5)

[51] S. Chandrasekhar, Monthly Notices Roy. Astron. Soc. **95**, 207 (1935).

[52] It does not matter that the gravitation acts primarily on the nuclei in the white dwarf because nuclei and electrons are closely tied together by the (much stronger) electric forces.

is therefore analogous to the cluster term (12.1) except that I_{ij} has been replaced by a more complicated expression,

$$-\bar{G}_{ij}\frac{Q}{e_{ij}}v_{ij}. \tag{14.6}$$

The lowest order nonvanishing contribution to (14.5) is therefore, in analogy to (12.3), the three-particle cluster term

$$w_{33}=-\sum\left(\Phi_C,v_{ij}\frac{Q}{e_{ij}}\bar{G}_{ij}\frac{Q}{e_{ij}}I_{jk}\frac{Q}{e_{ki}}I_{ki}\Phi_C\right). \tag{14.7}$$

Now (14.7), or the more general expression (14.5), can easily be combined with (12.3) or (12.1), respectively, if we replace v_{ij} by I_{ij}. As we have seen in Sec. X, the v matrix is not very different from the G matrix if the potential is well-behaved, and the elements of v_{ij} which lead to the excitation of two nucleons are essentially equal to the corresponding elements of the I matrix, Eq. (7.5). Writing then

$$L_{ij}-1=\frac{Q}{e_{ij}}L_{ij}' \tag{14.8}$$

[the term 1 in F_{ij} gives no contribution to either (12.1) or (14.5)], we get for the *sum* of (12.1) and (16.5):

$$\left(\Phi_C,\sum_{ij}I_{ij}\left(\frac{Q}{e_{ij}}-\frac{Q}{e_{ij}}\bar{G}_{ij}\frac{Q}{e_{ij}}\right)L_{ij}'\Phi_C\right)$$
$$\approx\left(\Phi_C,\sum_{ij}I_{ij}\frac{Q}{e_{ij}+\bar{G}_{ij}}L_{ij}'\Phi_C\right), \tag{14.9}$$

where in the last line it has been assumed that $\bar{G}_{ij}\ll e_{ij}$, as will be shown to be indeed the case. Thus the only effect of w_3 is to modify the first denominator in the cluster terms by adding $\bar{G}_{ij}$.

Now $\bar{G}_{ij}$ is essentially the same as the diagonal element of v_{ij}. This was calculated in (6.4) and found to be proportional to Ω^{-1} or A^{-1}. This shows that the $\bar{G}_{ij}$ in (14.9) gives a correction of the order of $1/A$, since e_{ij} is independent of A. The term w_3 is therefore of order $1/A$ compared with the cluster terms, and using the estimates of Sec. XII for the latter, we see that w_3 is of the order of a few hundred kev *for the whole nucleus*, which is clearly negligible.

The reason for the smallness of w_3 is the occurrence of the operator $\bar{G}_{ij}$ in (14.5). This operator is of order Ω^{-1}, which has to be compared with the denominator e_{ij} which occurs associated with it and which is independent of Ω. It may be argued that the I_{ij} occurring in (12.1) are also of order Ω^{-1}: But I_{ij}, being a nondiagonal element, leaves open the choice of the momentum change $\mathbf{q}$, (see Sec. VI), and the number of choices of $\mathbf{q}$ is proportional to Ω. On the other hand, $\bar{G}$ is a diagonal element and thus permits no choice of final state.

In spite of the smallness of w_3, it is interesting to investigate the meaning of the change of the denominator in (14.9). According to (7.3),

$$(n_in_j|\bar{G}_{ij}|n_in_j)=(n_in_j|G_{ij}|n_in_j), \tag{14.10}$$

and this is the first term in ΔG_{ij}, Eq. (3.6), which is a small correction to the excitation energy $-e_{ij}$, Eq. (3.4). Thus, according to (14.9), the interaction between the two excited nucleons ij should be omitted in the first (but not the other), resonance denominator e_{ij} when calculating the cluster term ΔE_3. This takes into account *all* the small $(1/A)$ corrections of Sec. IV provided we put $v=G$ in these corrections.

The nondiagonal terms,

$$(\Phi_B,w_m\Phi_C), \tag{14.11}$$

are important because they indicate to what extent the "improved" model wave function Φ', Eq. (4.6), differs from the simple model wave function Φ_C defined in Sec. II. We have shown in Sec. IV and in EB that w_1 has essentially no nondiagonal elements if the chosen configuration is nondegenerate. Therefore we have to consider only w_2 and w_3.

It can be shown fairly easily that w_2 and w_3 cause an admixture of "foreign" configurations B in the improved model wave function Φ' which is only of order $1/A$. The corresponding second-order perturbation of the energy is then also of order $1/A$; a rough estimate yields

$$\Delta E_{2,3}{}^{(2)}=\frac{\sum|\Phi_B,(w_2+w_3)\Phi_C|^2}{E_C-E_B}\approx-\frac{2\text{ Mev}}{A}, \tag{14.12}$$

which is completely negligible.

XV. TERMS NOT CONSERVING MOMENTUM

In all our quantitative calculations, e.g., in Secs. VI and IX–XIV, we have assumed that the matrix elements of v_{ij} and G_{ij} are different from zero only if momentum is conserved, as in (2.8). This is of course not strictly true for a finite nucleus. In this section we shall estimate the magnitude of the matrix elements which do not conserve momentum, particularly those in which only one nucleon changes its state.

For this investigation, it is convenient to replace G_{ij} by v_{ij} (Sec. X). Then the required matrix elements are

$$(n_in_j'|v_{ij}|n_in_j)_P$$
$$=\int d\tau_i d\tau_j\psi^*(n_i,\mathbf{r}_i)\psi^*(n_j',\mathbf{r}_j)v_{ij}(\mathbf{r}_i-\mathbf{r}_j)$$
$$\times\psi(n_i,\mathbf{r}_i)\psi(n_j,\mathbf{r}_j). \tag{15.1}$$

It is convenient to define a quasi-potential acting on particle j, thus:

$$u_{ij}(\mathbf{r}_j)=\int d\tau_i|\psi(n_i,\mathbf{r}_i)|^2v_{ij}(\mathbf{r}_i-\mathbf{r}_j). \tag{15.2}$$

In a nucleus of volume Ω, $|\psi|^2$ is of order Ω^{-1} and $u(\mathbf{r}_j)$ for any given point $\mathbf{r}_j$ will be of the same order, due to the assumed finite range of v_{ij}. The *diagonal* element of v_{ij},

$$(n_i n_j|v_{ij}|n_i n_j)=\int d\tau_j u(\mathbf{r}_j)|\psi(n_j,\mathbf{r}_j)|^2, \quad (15.3)$$

will then be of the same order Ω^{-1} since $|\psi|^2$ is of order Ω^{-1} and the whole nuclear volume Ω gives a positive contribution.

To estimate the nondiagonal terms, we use closure, thus:

$$\sum_{n_j'}|(n_i n_j'|v_{ij}|n_i n_j)|^2$$
$$=\sum_{n_j'}\left|\int d\tau_j \psi^*(n_j',\mathbf{r}_j)u(r_{ij})\psi(n_j,\mathbf{r}_j)\right|^2$$
$$=\int d\tau_j |u(r_j)|^2|\psi(n_j,\mathbf{r}_j)|^2. \quad (15.4)$$

Since u and $|\psi|^2$ are each of order Ω^{-1}, the integrand is Ω^{-3} and the integral over the nuclear volume is Ω^{-2}. Thus the sum in (15.4) is of the same order of magnitude as the single term $n_j'=n_j$ which is the square of (15.3). In other words, all the nondiagonal terms together contribute about as much to (15.4) as the single diagonal term.

The number of nondiagonal terms n_j', however, is proportional to Ω since $\sum_{n_j'}$ may be replaced by $\Omega(2\pi)^{-3}\int d^3k_j'$. Hence each individual term n_j' gives a contribution of order Ω^{-3} to (15.4), or the matrix element (15.1) is of order $\Omega^{-3/2}$. This compares with Ω^{-1} for the matrix elements which conserve momentum (Sec. VI), and thus shows that indeed the elements in which only a single nucleon is excited, become small for large nuclei. The same is true for matrix elements in which both nucleons are excited but momentum is not conserved.

If we calculate, for example, the contribution to the cluster term ΔE_3 which comes from matrix elements which do not conserve momentum, then at least two such elements must be involved, and the resulting contribution is at least by a factor A smaller than that calculated in Sec. XII.

An even closer estimate than (15.4) may be made for the nondiagonal matrix elements of the one-nucleon potential V. With our assumption $G_{ij}=v_{ij}$, and with the definition (15.2), we have

$$V(\mathbf{r}_j)=\sum_i u_{ij}(\mathbf{r}_j)=\int d\tau v(\mathbf{r}-\mathbf{r}_j)\sum_i|\psi(n_i,\mathbf{r})|^2$$
$$=\int d\tau\rho(\mathbf{r})v(\mathbf{r}-\mathbf{r}_j), \quad (15.5)$$

where ρ is the total density of nucleons. Now for a large nucleus, ρ is very nearly constant ($=\rho_0$) over the interior of the nucleus. Only in a surface layer of relative volume of order $A^{-1/3}$, does ρ become appreciably smaller than ρ_0. To estimate the nondiagonal elements (7.9) of V, we therefore form, similarly to (15.4):

$$\sum_{n_j'\neq n_j}|(n_j'|V|n_j)|^2$$
$$=\int d\tau[V(r_j)]^2|\psi(n_j,\mathbf{r}_j)|^2-\langle V\rangle^2$$
$$=\int d\tau[V(r_j)-\langle V\rangle]^2|\psi(n_j,\mathbf{r}_j)|^2, \quad (15.6)$$

where

$$\langle V\rangle=\int V(r_j)|\psi(n_j,\mathbf{r}_j)|^2 d\tau_j \quad (15.7)$$

is the expectation value of the potential. Now in the interior, $V(r_j)-\langle V\rangle$ is of order $A^{-1/3}$ so that the contribution of the interior to (15.6) is of order $A^{-2/3}$. The main contribution comes from the surface layer and is proportional to the relative volume of that layer, $A^{-1/3}$.

Thus the sum is (15.6) is of order $A^{-1/3}$, and is smaller by this factor than the diagonal term $(n_j|V|n_j)^2$. Since the number of terms is again proportional to A, an individual term $(n_j'|V|n_j)$ will only be of order $A^{-2/3}$ compared with the diagonal term.

XVI. PROBLEMS FOR A FINITE NUCLEUS

In Secs. II, III, and VII we have developed the general method for a finite just as for an infinite nucleus. In Secs. XIV and XV we have shown that the correction terms which may arise for a finite nucleus, are small. Nevertheless, many further problems will have to be solved before reliable quantitative results can be obtained.

The first step in treating a finite nucleus is the choice of a potential to start the self-consistent calculation of Sec. II. This is far more difficult than in the Hartree method for two reasons: First, the whole potential concept is somewhat invalidated by the complications discussed in Sec. III, but we have shown at the end of Sec. X that these complications are not very important quantitatively, at least for "well-behaved" interactions. Second, we have to know a potential *matrix* $(\mathbf{r}'|V|\mathbf{r})$, rather than a simple potential $V(r)$.

To determine the potential matrix in a finite nucleus combines the problems of finding the Hartree potential for an atom and the Brueckner potential for an infinite nucleus. The Hartree potential is diagonal in position, $V(r)$, and when transformed to momentum space, depends only on the difference $|\mathbf{k}'-\mathbf{k}|$. The Brueckner potential for an infinite nucleus (Secs. IX to XI) is a function of momentum only, $V(k)$, and if transformed to coordinate space, will depend only on the distance $|\mathbf{r}'-\mathbf{r}|$; these two are therefore complementary. The potential matrix in a finite nucleus will be neither

diagonal in position nor in momentum, and will thus depend on $\mathbf{r}'-\mathbf{r}$ as well as $\mathbf{r}'+\mathbf{r}$, and on $\mathbf{k}'+\mathbf{k}$ as well as $\mathbf{k}'-\mathbf{k}$.

There remains, however, a simplifying feature which will probably be very important for the actual solution. The dependence on $\mathbf{r}'+\mathbf{r}$ is designed to simulate the conventional potential well; the potential V will therefore drop when $\frac{1}{2}|\mathbf{r}'+\mathbf{r}|$ exceeds the nuclear radius $R\approx r_0A^{1/3}$. On the other hand, the dependence on $\mathbf{r}'-\mathbf{r}$ is to represent the momentum-dependence of the Brueckner potential $V(k)$ for an infinite nucleus. Now $V(k)$ changes appreciably when k goes from 0 to k_F; therefore the Fourier transform $V(\mathbf{r}'-\mathbf{r})$ will change appreciably when $|\mathbf{r}'-\mathbf{r}|$ goes from 0 to $1/k_F$, which is about r_0. For a large nucleus, therefore, much larger distances are involved in $|\mathbf{r}'+\mathbf{r}|$ than in $|\mathbf{r}'-\mathbf{r}|$, and the two dependences can be separated (except in the region of the nuclear surface). For a small nucleus, the determination of $(\mathbf{r}'|V|\mathbf{r})$ is likely to be much more difficult.

The wave functions of the nucleons will be determined mainly by the large-scale behavior of V, i.e. by its dependence on $\mathbf{r}'+\mathbf{r}$. For a first approximation, therefore, one can probably take simply

$$(\mathbf{r}'|V|\mathbf{r})=V(r)\delta(\mathbf{r}'-\mathbf{r}),$$

in other words an old-fashioned Hartree type potential, and determine the wave functions in this potential by solving (2.3). In other words, one may go back to the standard procedure in shell-model theory. Of course, once the wave functions of the nucleons have been determined, one must then find the reaction matrix using (3.1) and one will thus obtain a velocity-dependent potential, just as for an infinite nucleus. This potential will *not* agree with the Hartree potential used at the start, but when wave functions are calculated in this new potential, it is hoped that they will not differ too much from those calculated in the first approximation.

The accurate determination of the potential matrix $(\mathbf{r}'|V|\mathbf{r})$ in the region of the nuclear boundary is obviously a very difficult problem. To solve it, it will probably be best to consider first the problem of an infinite plane nuclear boundary and to solve the self-consistent problem for this. Then if the normal to the boundary is in the x direction, we may write

$$(\mathbf{r}_2|V|\mathbf{r}_1)=V(\tfrac{1}{2}(x_1+x_2), x_1-x_2, r_{12}). \qquad (16.1)$$

It has been suggested by Skyrme[53] that one might consider the dependence on r_{12} to be similar to that for an infinite nucleus whose density is the same as that found at $\frac{1}{2}(x_1+x_2)$. Solution of this problem of the nuclear boundary layer will yield the surface energy of the nucleus.[53]

Once the problem of the plane nuclear boundary is solved, the potential matrix for a finite nucleus may be assumed to depend on the position coordinates $\mathbf{r}$, $\mathbf{r}'$ in the boundary layer in the same manner, i.e., it should be a good approximation to neglect the curvature of the nuclear surface. From the potential, one may then obtain the wave functions solving the Schrödinger equation (2.3).

Let us now assume that the wave functions of the individual nucleons have been determined. For the moment, we shall assume that the potential $(\mathbf{r}'|V|\mathbf{r})$ is spherically symmetric, i.e., that it depends only on r, r', and the angle between them (more about this later). Then the wave equation (2.3) separates in polar coordinates, and the nuclear wave functions $\psi(n_i,\mathbf{r}_i)$ should be the well-known shell model wave functions. In order to obtain the observed dependence on j, it must be assumed that V contains a term coupling spin and orbit, and it will be one of the tasks of the theory to derive such a term from the observed interaction between two nucleons.

If the chosen configuration C contains only closed shells, and corresponds to the ground state, the further calculation is essentially the same as for an infinite nucleus. It should be mentioned again, however, that configuration interaction is already contained in the method (reaction matrix); therefore the state of a closed-shell nucleus must *not* be considered as a mixture of various configurations, in an endeavor to lower its eigenvalue. The model wave function corresponds to one configuration C only.

Essentially the same statements hold for a nucleus containing one nucleon outside closed shells, or one less than a closed shell. This is analogous to the problem of an alkali atom.

If there are two or more nucleons in an incomplete shell, the situation is more complicated, just as in the corresponding case in the theory of atomic spectra. There are then many configurations of the nucleus which have the same energy, at least as long as we only consider the sum of the eigenvalues $E(n_i^0)$ of the individual nucleons and disregard the interaction of the nucleons outside closed shells with each other. We therefore have to consider degenerate configurations; in fact the degeneracy is much higher than in the atomic case.

The most important consequence of degeneracy is that w_1, Eq. (4.14), now has many more nonvanishing matrix elements than in the absence of degeneracy, and that these appear in a much lower order of approximation. The operator $1-Q$ in (4.14) has the value unity for *all* configurations which are degenerate with the chosen configuration. We may thus consider the matrix element of (4.14) between the chosen configuration C and any other configuration B which is degenerate with it, *viz.*,

$$(\Phi_B,w_1\Phi_C)=\sum_{ij}(\Phi_B,I_{ij}L_{ij}\Phi_C). \qquad (16.2)$$

Now in contrast to the diagonal element (12.1) of w_1, the nondiagonal element (16.2) will in general not

[53] T. H. R. Skyrme (to be published).

vanish even if we set $L_{ij}=1$, its first approximation according to its definition (12.2). Then, in first approximation,

$$(\Phi_B, w_1\Phi_C)=\sum_{ij}(\Phi_B, I_{ij}\Phi_C). \quad (16.3)$$

This is nonzero if B and C differ in the quantum states of two nucleons, which is indeed the most common case for two degenerate configurations. Assume, for example, that there are exactly two nucleons outside a closed shell of 50 nucleons; then these two nucleons may be in the $3s$, $2d$, or $1g_{7/2}$ shell. A state of total angular momentum 0 (S state) may be obtained from the configurations $3s^2$, $2d^2$, or $1g^2$. To go from the first to the second of these configurations, the two nucleons (i and j) have to be moved from an s to a d state which corresponds exactly to the matrix element (16.3). If there are more than two nucleons outside closed shells, we do not need to invoke the degeneracy of nucleon states of different l, such as the $3s$, $2d$, and $1g$ above, but even if all nucleons are within the same shell nl, there are many different configurations leading to the same total orbital momentum and spin, L and S, of the whole nucleus, which "interact" through matrix elements of the type (16.3).

Expression (16.3) is entirely analogous to the expression for the interaction of degenerate configurations in conventional shell-model theory. The only difference is that in the conventional theory $I_{ij}=G_{ij}$ is replaced by v_{ij}, and according to Sec. X this does not make much difference for well-behaved potentials. For potentials with a repulsive core, it does make a difference, and in fact only the "modern" expression (16.3) is meaningful while the old-fashioned one with v_{ij} would not be. But the details of I_{ij} or v_{ij} do not actually matter; we are interested only in the matrix element (16.3) which depends on some over-all behavior of I_{ij}. This I_{ij} may well be replaced by an equivalent well-behaved potential which has the same matrix elements.

This is the more so since the most important matrix elements of (16.3),

$$(n_i'n_j'|G_{ij}|n_in_j), \quad (16.4)$$

are those in which the states of the nucleons i and j (initial as well as final) are not very different. Then the product of the four wave functions involved in (16.4) will not change very rapidly with the distance r_{ij}, and the exact dependence of G_{ij} on r_{ij} will not matter much.

Thus our theory fully justifies[54] the conventional procedure of shell-model theory: the interaction between degenerate configurations is given by a very similar expression, and considerable freedom is allowed in the choice of the quasipotential v_{ij} which is to replace I_{ij}. Of course, it will be desirable in future shell-model calculations to choose v_{ij} so as to be compatible with the G_{ij} which has to be used in the Brueckner theory for an infinite nucleus, and which is derived from the interaction between two free nucleons in the manner described in Sec. X.

While the interaction between degenerate configurations thus reduces essentially to the conventional shell-model expression, the interaction between nondegenerate configurations is treated completely differently. In the conventional shell model, the matrix elements of the interaction corresponding to transitions from C to *any* other configuration would have to be calculated, and the resulting Hamiltonian matrix reduced to diagonal form, an essentially impossible task.[55] In our theory, all this part of the work is done in the first part of the procedure, i.e., when the scattering matrix G_{ij} for the two-nucleon system is determined by solving (3.1). After this is done, the configurations B which are not degenerate with C need no longer—in fact *must* no longer—be considered when the problem of the whole nucleus is set up. Thus the Hamiltonian matrix which has to be diagonalized for the nucleus is finite rather than infinite, and contains only the degenerate states. (Obviously, only states of the same total angular momentum J, and the same $J_z=M$, need to be considered.) Thus Brueckner's method separates the essentially insoluble problem into two simpler ones, that of determining the G matrix and that of reducing the Hamiltonian matrix for degenerate states only.

When solving (3.1) for a finite nucleus, the wave functions $\psi(n_i,\mathbf{r}_j)$ are of course shell-model wave functions, i.e., radial functions times spherical harmonics, not plane waves. However, because of the Pauli principle, the sum over n_i'' and n_j'' in (3.1) contains only unoccupied states which may be expected to be states of high momentum and to be not very different for a finite and for an infinite nucleus. Stated somewhat differently, the sum in (3.1) makes G_{ij} different from v_{ij} at small distances r_{ij}, of order r_0 or less, and at these distances the over-all behavior of the wave functions, i.e., whether they are plane waves or spherical shell-model wave functions, should not matter very much. Thus it is to be hoped that the results from infinite nuclei can be taken over with little change.

We have made an essential distinction between degenerate and nondegenerate configurations. The question arises how to distinguish these in practice, especially because nucleon states like the $3s$, $2d$, and $1g_{7/2}$ mentioned above are almost but not exactly degenerate. Fortunately, there is great latitude in the definition of the operator Q in Sec. III, and this makes it a matter of our choice which of the configurations we wish to consider degenerate, and which not. For those in the former category, we set $Q=0$, which means that they are not taken into account as possible intermediate

[54] When calculating operators other than the energy, it is of course necessary to take into account that the actual nuclear wave function differs from the model wave function, as has been described in Sec. VI.

[55] In practice, only a few other configurations are considered. This practice, adopted only for the sake of simplicity, is justified by our method.

states in calculating the scattering matrix G from (3.1); they will then give nonvanishing matrix elements (16.3) and thus have to be taken into account in the final Hamiltonian matrix of the complete nucleus. For the configurations which we wish to consider as nondegenerate with the former class, we set $Q=1$; then they will contribute as intermediate states to the scattering matrix G in (3.1) but will not contribute elements (16.3) to the final Hamiltonian matrix.

It will presumably be convenient to include all configurations arising from particles in the same shell (in the sense of the shell model) as degenerate, and all others as nondegenerate. Thus if we have n particles outside the 50-shell, we distribute these in all possible ways over the 20 available places in the shells $3s$, $2d$, and $1g_{7/2}$ (and possibly also $h_{11/2}$?), and consider all configurations arising in this way as degenerate. If one particle is put into the next higher shell, or a particle removed from the 50-shell and put into the *sdg* shell, the resulting configuration is considered as not degenerate with the former class, and assigned the value $Q=1$. Then the smallest denominators occurring in (3.1) will be equal to the energy difference between two successive shells. From the estimates in Sec. X it is likely that the contribution to G from configurations which have the smallest possible denominator will not be very large, and this fact justifies their inclusion among the "nondegenerate" configurations.

In contrast to the case of an infinite nucleus, the one-nucleon levels in a finite nucleus are discrete and the number of distinct shells is rather small. If we consider the nucleon shells $n=1, 2, 3, \cdots$ as closed at 2, 8, 20, 28, 50, 82, 126, $\cdots$ nucleons, then from $n=4$ up the total number of nucleons up to shell n is given by

$$N(n)=\tfrac{1}{3}n(n^2+5), \tag{16.5}$$

and the number of shells for a given mass number is roughly

$$n=(3A/2)^{1/3}\sim A^{1/3}. \tag{16.6}$$

The energy spacing between successive shells therefore decreases with A only as $A^{-1/3}$, and it is this slow decrease which makes the shell model so useful. The number of nucleons per shell goes up as $A^{2/3}$.

Furthermore, the matrix elements of v_{ij} or G_{ij} vary greatly from one nucleon pair to another. Two nucleons which have the same spatial wave function (and differ in spin or charge) will have a very large interaction, decreasing with atomic weight more slowly than $1/A$. Two nucleons in different shells, on the other hand, will have a considerably smaller interaction. Our estimates in Secs. VI and XV of the size of nuclear matrix elements must therefore not be taken seriously for individual matrix elements but hold only on the average.

The interaction between a pair of nucleons in the same shell depends greatly on the symmetry of the wave function with respect to the two nucleons. This establishes a connection with the theory of multiplets developed by Wigner.[56] The only difference is that in our theory the multiplet structure, caused by the symmetry of the wave function, is subordinate to the shell structure for the individual nucleons, just as it is in atomic spectroscopy. This is of course quite familiar from the conventional treatments of the shell model. It is quite possible that the shell model with symmetry effects considered in second approximation, becomes a better approximation as the size of the nucleus increases, and that for very small nuclei (up to carbon, say) the original Wigner procedure, of considering symmetry first, is preferable.

The interaction between nucleons can presumably separate levels of the nucleus with different numbers of spatially symmetric pairs[56] very greatly, probably by more than the energy difference between successive nucleon shells. It may therefore be desirable to redefine the concept of "degenerate configurations" to take this symmetry effect at least partially into account. This will be important only when there are many nucleons in an incomplete shell.

If there are many nucleons in an incomplete shell, the "collective model" of Bohr and Mottelson[57] may become applicable. In our theory this means that a self-consistent solution will not be obtained for a spherically symmetric potential $(\mathbf{r}'|V|\mathbf{r})$ but for one of ellipsoidal shape. Since the self-consistent method is equivalent to a variational principle,[20] we may also say that the ellipsoidal shape gives a lower energy to the nucleus. Clearly this will happen only for certain distributions of the nucleons over the possible m values (magnetic quantum number) in the incomplete shell.

We have frequently stated that our method is applicable not only to the ground state but also to other states of the nucleus. For instance, all the configurations which are considered "degenerate" in first approximation, will be calculated simultaneously by diagonalization of the Hamiltonian matrix of the nucleus, as is customary in conventional shell-model theory. But also states in which one or more nucleons are excited to higher shells can be calculated. In this case, some of the denominators in the fundamental equation (3.1) will be negative because the intermediate state can have lower energy than the now chosen configuration. But this makes no difference: the main point is that states degenerate with the chosen one are always excluded by the operator Q so that no vanishing denominators can occur. Of course, the higher the excitation, the higher the degeneracy, so that it soon becomes impossible in pratice to diagonalize the Hamiltonian. But at least a reliable method is provided for calculating statistical properties, such as the spread in energy of all the levels which arise for instance from a given

[56] E. P. Wigner, Phys. Rev. **51**, 106 (1937). For a simple exposition, see J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), Chap. VI.

[57] A. Bohr and B. R. Mottelson, Physica **18**, 1066–1078 (1952).

excitation of one nucleon, and perhaps also the distribution of these levels in energy. This will give a better basis for calculations of level density, and it may also be relevant to the giant resonance in the photoeffect.[58]

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APPENDIX. SPIN CONSIDERATIONS FOR THREE-PARTICLE CLUSTER TERMS

In this appendix, we shall consider the possible values of the spins of the three nucleons which interact according to Sec. XII. We shall consider wave functions which describe both spin and charge of a nucleon, and denote them by α, β, γ for the three nucleons in the cluster. The orbitals of the three nucleons will be denoted by a, b, c. Then $(a\alpha,b\beta,c\gamma)$ denotes a possible assignment of spin-charge to the three orbitals; $(a\beta,b\gamma,c\alpha)$ would be another. The part of the wave function referring to the three nucleons (denoted 1, 2, 3) is the Slater subdeterminant

$$\psi_0=(a\alpha,b\beta,c\gamma)=\begin{vmatrix} a(1)\alpha(1) & b\beta(1) & c\gamma(1) \\ a(2)\alpha(2) & b\beta(2) & c\gamma(2) \\ a(3)\alpha(3) & b\beta(3) & c\gamma(3) \end{vmatrix}. \quad (A.1)$$

First step.—We operate on the wave function (A.1) with the Serber operator

$$v=\tfrac{1}{2}v_{ab}(1+P_{Mab}), \quad (A.2)$$

which gives

$$v\psi_0=\tfrac{1}{2}v_{ab}[(a\alpha,b\beta,c\gamma)+(b\alpha,a\beta,c\gamma)], \quad (A.3)$$

and take the matrix element leading to the first intermediate state

$$\psi_1=(a'\alpha',b'\beta',c\gamma). \quad (A.4)$$

Clearly, particle $c\gamma$ must remain in the same state. It is sufficient to take the symmetry into account in the initial state, so we write

$$\psi_1=a'\alpha'(1)b'\beta'(2)c\gamma(3). \quad (A.5)$$

Further, it is sufficient to take the part of (A.3) which assigns the state $c\gamma$ to particle 3, then we have

$$\begin{aligned} v\psi_0&=\tfrac{1}{2}v_{ab}[a\alpha(1)b\beta(2)+b\alpha(1)a\beta(2) \\ &\quad -b\beta(1)a\alpha(2)-a\beta(1)b\alpha(2)]c\gamma(3) \\ &=\tfrac{1}{2}v_{ab}[a(1)b(2)+b(1)a(2)] \\ &\quad \times[\alpha(1)\beta(2)-\beta(1)\alpha(2)]c\gamma(3). \end{aligned} \quad (A.6)$$

Clearly, the interaction is zero if $\alpha=\beta$ as also shown in Sec. VIII. The integral over the coordinates of the third particle gives unity, and there is at this point no restriction on the spin γ.

The matrix element $(\psi_1,v\psi_0)$ for $\alpha\neq\beta$ will give a non-vanishing result in two cases:

I. If $\alpha'=\alpha$, $\beta'=\beta$, the matrix element comes from the term $\alpha(1)\beta(2)$ in (A.6) and is

$$(\psi_1,v\psi_0)=\tfrac{1}{2}(a'b'|v|ab)+\tfrac{1}{2}(b'a'|v|ab). \quad (A.7)$$

II. If $\alpha'=\beta$, $\beta'=\alpha$, the matrix element comes from $-\beta(1)\alpha(2)$ in (A.6) and is equal to (A.7) *with opposite sign.*

Second step.—Next we let particles b and c interact, in accord with the form of the three-cluster operator (12.3). This operation must bring a particle back to state $b\beta$. Thus the second intermediate state is

$$\psi_2=(a'\alpha',b\beta,c'\gamma'). \quad (A.8)$$

Since b and c interact, their spins in state ψ_1 must be different, according to (A.6). Hence we must have

$$\gamma\neq\beta'. \quad (A.9)$$

However, it is permissible that $\gamma=\alpha'$ since particle a does not interact at this stage. We discuss separately the two cases defined above.

I. If $\beta'=\beta$, then (A.9) shows that $\gamma\neq\beta$. Then we must also have $\gamma'=\gamma$, Eq. (A.8), and the matrix element is

$$\tfrac{1}{2}(bc'|v|b'c)+\tfrac{1}{2}(c'b|v|b'c). \quad (A.10)$$

II. The two interacting particles have the spins $\beta'=\alpha$ and γ in the first intermediate state, so we must have $\gamma\neq\alpha$. In the second intermediate state, their spins are β and γ'. Since the interaction does not contain spin, the spins γ, α must match the spins β, γ'. Since we know that $\alpha\neq\beta$, the only possibility is that $\gamma=\beta$; hence we must have $\gamma'=\alpha$ and the states of the three particles are

$$\psi_0=(a\alpha,b\beta,c\beta), \quad (A.11)$$

$$\psi_1=(a'\beta,b'\alpha,c\beta), \quad (A.12)$$

$$\psi_2=(a'\beta,b\beta,c'\alpha). \quad (A.13)$$

Then in going from ψ_1 to ψ_2, the spins of the last two particles get interchanged, as they did in the first step of Case II. Thus, according to (A.6), the matrix element for the second step is the negative of (A.10).

Third step.—We let particles c and a interact, thus getting back to the initial state ψ_0.

[58] D. H. Wilkinson, *Proceedings of the Glasgow Conference on Nuclear Physics, 1954* (Pergamon Press, London, 1955), p. 161.

Case I. Since the configuration in the second intermediate states is $(a'\alpha,b\beta,c'\gamma)$, application of (A.6) shows that $\gamma\neq\alpha$. Thus in case I, all three spins must be different. The matrix element is

$$\tfrac{1}{2}(ac|v|a'c')+\tfrac{1}{2}(ac|v|c'a'). \qquad (A.14)$$

Case II. The second intermediate state is given by (A.13). The requirement for (A.6) to be nonvanishing is now $\alpha\neq\beta$ which is fulfilled anyway. The spins α and β are interchanged again, so there is again a negative sign.

General argument.—We consider a triplet of orbitals a, b, c whose order is determined by the order of excitation. We have then the following possible spin assignments:

1. All 3 spins may be different. There are $4\times3\times2=24$ possible assignments of spins to orbitals in this class. We then obtain Case I, and the complete matrix element in the cluster is the product of (A.7), (A.10), and (A.14). This has been used in Sec. XII.

2. Two spins are equal, the third different.

(a) Particle b and c have equal spins; there are $4\times3=12$ different assignments in this class. Then we have the situation of case II, Eqs. (A.11)–(A.13). The matrix element is the same as in case I, except for three negative signs.

(b) Particle b has the different sign. In this case, the matrix element is zero as can be seen by letting ac interact first. (12 possible assignments.)

(c) Particle c has the different spin. Again matrix element zero as shown by (A.6). 12 possible assignments.

3. All three spins equal. 4 possible values of this spin. Matrix element zero.

The total result is therefore (24–12) times the product of (A.7), (A.10), and (A.14).