The corresponding value of E is obtained from (2):

$$E=E_0+V_{00}+\sum_{i=2}^{2n-1}\epsilon_i+\epsilon_{2n}\bigg/\left\{1-\frac{(\epsilon_{2n-2}-\epsilon_{2n-1})\epsilon_{2n+1}+\epsilon_{2n}^2}{(\epsilon_{2n-2}+\epsilon_{2n-1})\epsilon_{2n}-\epsilon_{2n-1}^2}\right\}$$
$$+\epsilon_{2n+1}\bigg/\left\{1-\frac{(\epsilon_{2n-1}+\epsilon_{2n-2})\epsilon_{2n}+(\epsilon_{2n-1}-\epsilon_{2n-2})\epsilon_{2n+1}-\epsilon_{2n}^2}{\epsilon_{2n-1}^2}\right\}. \quad (6)$$

For the special circumstance, considered by Goldhammer and Feenberg, where

$$\epsilon_{2l+1}=0 \quad \text{for all } l, \quad (7)$$

we can allow such ϵ_i to approach zero in the preceding formulas. Equations (3) and (4) then reduce to the Brillouin-Wigner form for this case:

$$G_n=1, \quad (8)$$

$$E=E_0+V_{00}+\sum_{i=2}^{n}\epsilon_{2i}. \quad (9)$$

Thus, if (7) holds, the Brillouin-Wigner scheme cannot be improved by varying G_n alone.

However, in our second case, if (7) holds, (5) and (6) are replaced by

$$G_{n-1}=G_n=(1-\epsilon_{2n}/\epsilon_{2n-2})^{-1}, \quad (10)$$

$$E=E_0+V_{00}+\sum_{i=1}^{n-1}\epsilon_{2i}+\frac{\epsilon_{2n}}{(1-\epsilon_{2n}/\epsilon_{2n-2})}. \quad (11)$$

In this case, then, by varying G_{n-1} and G_n an improvement on the Brillouin-Wigner procedure is obtained.

Equations (4) and (11) are clearly of the same form. Together, they provide a simple, generally valid prescription for improving the Brillouin-Wigner expansion for the energy: namely, divide the highest order term in the Brillouin-Wigner expansion by 1 minus the ratio of the highest order term to the term of next lower order.[3]

A numerical example illustrating the improvement resulting from this prescription, relative to the usual Brillouin-Wigner procedure, is given in reference 1.

[3] My attention has been called to the following proof that the improved formulas actually reduce the energy: Since the last two terms of Eq. (4) are $(\epsilon_{2n}+\epsilon_{2n+1})(1-\epsilon_{2n+1}^2/\epsilon_{2n}^2)^{-1}$, whereas the corresponding terms of Eq. (2) (with all G's$=1$) are just $\epsilon_{2n}+\epsilon_{2n+1}$, these terms are greater, in absolute value, in Eq. (4) than in Eq. (2). Therefore, if the energy is reduced by the inclusion of these terms in Eq. (2), a greater reduction follows by using Eq. (4).

I am also indebted to P. Goldhammer for the observation that Eq. (4) is exact, in any order n, if the ϵ_i form a geometric progression. A similar remark applies to Eq. (11) and the equation in reference 2.

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Helmholtz Instability of a Plasma*

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A plasma having infinite electrical conductivity and no viscosity is assumed to be in contact with a uniform magnetic field along a plane boundary which is parallel to the field. The behavior of small perturbations of this boundary when the plasma is flowing at velocity v_0 perpendicular to the magnetic field is calculated by linearized theory. Perturbations which only move lines of force parallel to themselves are unstable; for small v_0/c the motion is incompressible and the rate of growth of the perturbation can be obtained from the incompressible hydrodynamic expression by replacing the mass density of each fluid in the hydrodynamic case by the sum of twice the magnetic energy density divided by c^2 and the mass density of each magnetohydrodynamic fluid. The magnetic field is to be considered as a "fluid" having only magnetic mass. It is shown that this analogy holds even in the nonlinear equations for two-dimensional incompressible flow. Perturbations which only bend lines of force are stable, while those which both move lines parallel to themselves and bend them are stable if the bending wavelength is short enough.

INTRODUCTION

HELMHOLTZ instability will be observed in hydrodynamics if two fluids are in relative tangential motion at a sharp plane boundary. Perturbations of the plane boundary are unstable and lead to mixing of the fluids. Another type of instability (Rayleigh instability) occurs if a denser fluid lies in a layer over a less dense one in a gravitational field. An analysis of combined Rayleigh-Helmholtz instability for incompressible fluids is given by Lamb,[1] while

* This work was performed under the auspices of the U. S. Atomic Energy Commission.

[1] H. Lamb, *Hydrodynamics* (Dover Publications, New York, 1945), sixth edition, p. 373.

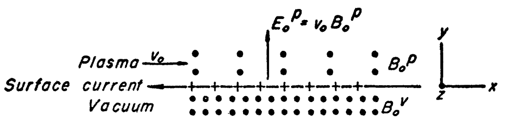

FIG. 1. Unperturbed fields in the "laboratory" frame.

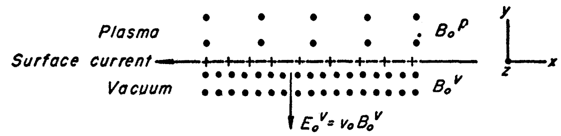

FIG. 2. Unperturbed fields in the plasma frame.

Frieman[2] has studied the compressible Helmholtz case.

In magnetohydrodynamics (MHD), Rayleigh instability has been investigated by Kruskal and Schwarzschild.[3] They took the denser fluid to be a plasma with no viscosity or resistivity, and the lighter fluid to be a magnetic field supporting the plasma in a gravitational field. Such an arrangement is unstable; the denser fluid falls and the lighter (magnetic field) rises to take its place. To extend the analogy between MHD and hydrodynamics it seems interesting to look for instabilities when a plasma flows at right angles to a static magnetic field (Fig. 1). This would be the analog of Helmholtz instability. The unperturbed state of the system is nonstatic in the "laboratory" frame but is static in the frame of the moving plasma, the motion being represented by an electric field in the vacuum normal to the boundary and of magnitude $v_0B_0^v$ (mks units will be used). v_0/c will be assumed small, so the vacuum magnetic field B_0^v is the same in either frame of reference. Since the magnetic fields are assumed different in the plasma and vacuum, there is a surface current in either frame. Also there is a surface charge in either frame, since the unperturbed vacuum electric field is zero in the laboratory frame, while the unperturbed plasma electric field vanishes in the plasma frame (Figs. 1 and 2).

FUNDAMENTAL EQUATIONS

The following formulation is essentially equivalent to that of Kruskal and Schwarzschild.[3] Equations which apply to the plasma are

$$\rho d\mathbf{v}/dt=\mathbf{j}\times\mathbf{B}+f\mathbf{E}-\nabla P, \tag{1}$$

$$\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{v})=0, \tag{2}$$

$$\mathbf{j}=f\mathbf{v}+\sigma(\mathbf{E}+\mathbf{v}\times\mathbf{B}), \tag{3}$$

$$\nabla\times\mathbf{B}-\mu_0\mathbf{j}-\mu_0\epsilon_0\partial\mathbf{E}/\partial t=0, \tag{4}$$

$$\nabla\cdot\mathbf{B}=0, \tag{5}$$

$$\nabla\times\mathbf{E}+\partial\mathbf{B}/\partial t=0, \tag{6}$$

$$\nabla\cdot\mathbf{E}=f/\epsilon_0, \tag{7}$$

$$\left(\frac{1}{P}\right)\frac{dP}{dt}=\left(\frac{\gamma}{\rho}\right)\frac{d\rho}{dt}, \tag{8}$$

where ρ is plasma mass density, $\mathbf{v}$ is velocity, $\mathbf{j}$ is current density, $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic fields, f is charge density, P is pressure, μ_0 and ϵ_0 are the usual mks constants ($\mu_0\epsilon_0=1/c^2$), and γ is c_p/c_v for the plasma gas. Equation (3) is the Ohm's law used. Since σ, the electrical conductivity, is taken as infinite, $\mathbf{E}+\mathbf{v}\times\mathbf{B}=0$ in the plasma. Equation (5) can be omitted, since it is the same as the divergence of (6) if perturbations away from the steady state are assumed to behave as $e^{\omega t}$ and $\omega\neq 0$. Equation (8) is the adiabatic law. Its use requires the assumption that heat flow due to thermal conductivity and heat sources due to $\mathbf{j}\cdot\mathbf{E}$ are negligible. The equation of charge conservation is omitted, since it follows from (4) and (7).

Conditions which apply to the sharp boundary between plasma and vacuum are

$$d\mathbf{n}/dt=\mathbf{n}\times[\mathbf{n}\times(\nabla\mathbf{v})\cdot\mathbf{n}], \tag{9}$$

$$\begin{aligned}\mathbf{n}P=\epsilon_0\{&(\mathbf{E}^p\cdot\mathbf{n})\mathbf{E}^p-(\mathbf{E}^v\cdot\mathbf{n})\mathbf{E}^v\\&-\tfrac{1}{2}\mathbf{n}[(E^p)^2-(E^v)^2]\}+(1/\mu_0)\\&\times\{(\mathbf{B}^p\cdot\mathbf{n})\mathbf{B}^p-(\mathbf{B}^v\cdot\mathbf{n})\mathbf{B}^v\\&-\tfrac{1}{2}\mathbf{n}[(B^p)^2-(B^v)^2]\}-\epsilon_0(\mathbf{n}\cdot\mathbf{v})\\&\times(\mathbf{E}^v\times\mathbf{B}^v-\mathbf{E}^p\times\mathbf{B}^p),\end{aligned} \tag{10}$$

$$\mathbf{n}\cdot(\mathbf{B}^p-\mathbf{B}^v)=0, \tag{11}$$

$$\mathbf{n}\times(\mathbf{E}^p-\mathbf{E}^v)=(\mathbf{n}\cdot\mathbf{v})(\mathbf{B}^p-\mathbf{B}^v). \tag{12}$$

The superscript v denotes vacuum and p the plasma quantity. $\mathbf{n}$ is a unit vector normal to the boundary and directed into the plasma. Equation (10), which is obtained from the electromagnetic stress tensor, says that the total stress (electromagnetic plus hydrostatic) must be continuous across the boundary. The first term is the net electric stress, the second is magnetic, while the third subtracts the amount of stress which is changing the momentum of the electromagnetic field in the boundary. Equations (11) and (12) arise from application of (5) and (6), respectively, to the boundary. In deriving (10) and (12), it is assumed that the directional derivative of a quantity parallel to the boundary is much smaller than the normal derivative—i.e., that the boundary is truly sharp.

Equations (4), (5), (6), and (7) apply to the vacuum. As before, (5) can be omitted. Also, since $\mathbf{j}=0$ and $f=0$, (7) is the same as (4) for $\omega\neq 0$, so that only (4) and (6) are used.

LINEARIZATION AND SOLUTION OF THE EQUATIONS

The basic equations are linearized in the same manner used by Kruskal and Schwarzschild.[3] Each

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[2] E. Frieman, Los Alamos Scientific Laboratory Unclassified Report LA-1608, Sept. 1953 (unpublished).

[3] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) **A223**, 348 (1954).

vector or scalar quantity is assumed to be the sum of a steady-state value and a small perturbation. Products of perturbations are dropped after substitution into the basic equations. The unperturbed solution in the plasma frame is: $\rho=\rho_0$, $\mathbf{v}=0$, $\mathbf{E}^p=0$, $|\mathbf{E}^v|=E_0^v$ (along $-y$), $f=0$, $P=P_0$, $|\mathbf{B}^p|=B_0^p$ and $|\mathbf{B}^v|=B_0^v$, both directed along $+z$, $d\mathbf{n}/dt=0$, and, from (10),

$$\frac{\epsilon_0(E_0^v)^2}{2}+\frac{(B_0^p)^2}{2\mu_0}+P_0=\frac{(B_0^v)^2}{2\mu_0}.$$

Perturbations are assumed to be of the form: amplitude $\times\exp(ilx-my+inz+\omega t)$ in the plasma and amplitude $\times\exp(ilx+qy+inz+\omega t)$ in the vacuum. The signs of the qy and my terms are chosen so that in cases of interest the real parts of m and q are positive. With these substitutions, the plasma equations yield 15 homogeneous scalar equations for the 15 amplitudes, the coefficients being functions of l,m,n,ω, and unperturbed quantities. Similarly the vacuum gives for the six amplitudes six equations containing l, q, n, and ω. From the boundary, ten more equations containing l, m, n, q, and ω are obtained. This system is overdetermined, having 31 equations for 24 amplitudes. When the determinants of the plasma and vacuum systems are set equal to zero, two conditions on l, m, n, q, ω are obtained. A third condition is obtained from the boundary equations; all of the boundary equations were either used or found to reduce to an identity, so that there can be no more than these three conditions on l, m, n, q, ω. This means that ω, for example, can be expressed in terms of l and n, or that one is free to specify the shape of the deformed surface at $t=0$ and then observe what happens in future time.

In terms of the dimensionless symbols $h=n/l$, $\delta=v_0/c$, and $u=i\omega/lc$, the vacuum equation is

$$q^2/l^2=1+h^2-u^2. \tag{13}$$

The plasma equation is

$$\left[h^2-\left(1+\frac{\rho_0}{\epsilon_0(B_0^p)^2}\right)u^2\right]\left\{\left(\frac{m^2}{l^2}-1\right)\times\left[\left(\frac{1}{\gamma}+\frac{\mu_0P_0}{(B_0^p)^2}\right)u^2-\frac{\mu_0\epsilon_0P_0h^2}{\rho_0}\right]-\left(\frac{u^2}{\gamma}-\frac{\mu_0\epsilon_0P_0h^2}{\rho_0}\right)\times\left[h^2-\left(1+\frac{\rho_0}{\epsilon_0(B_0^p)^2}\right)u^2\right]\right\}=0. \tag{14}$$

The boundary equation becomes [after eliminating q^2 but not q by means of (13)]

$$\frac{q}{l}\frac{(B_0^p)^2}{(B_0^v)^2}\left[h^2-\left(1+\frac{\rho_0}{\epsilon_0(B_0^p)^2}\right)u^2\right]=[(\delta+u)^2-h^2]\frac{m}{l}, \tag{15}$$

if δ^2 is neglected compared to unity. These three equations have been obtained with the assumption that ω, m, and q are all different from zero, so that they cannot be trusted if any of these three vanishes. For example, if $\omega=0$, then $\nabla\cdot\mathbf{B}=0$ must be introduced as an additional equation.

The first factor in the plasma equation cannot be zero in cases of interest. If $n=0$ and $l\neq0$, the factor obviously cannot vanish. If $n\neq0$ and $l\neq0$, it is not evident that the factor can be dropped. Therefore, suppose u^2 does equal $h^2[1+\rho_0/\epsilon_0(B_0^p)^2]^{-1}$; then (15) is not satisfied unless $m=0$. But since (13)–(15) are not trustworthy for $m=0$, the original equations for the amplitudes must be re-examined. It turns out that all vacuum and plasma perturbations must vanish if $l\neq0$. If $l=0$, this is actually a permissible solution for ω, in which vacuum perturbations vanish and the boundary is unperturbed from a plane. The only nonvanishing plasma perturbations are v_x, j_y, E_y^p, B_x^p, so that all fluid motion is parallel to the boundary; the motion involves sinusoidal bending of the lines of force in a plane parallel to the boundary and gives wave motion along the lines of force.

The second factor of the plasma equation yields several well-known[4] dispersion expressions for magnetohydrodynamic waves in an infinite plasma. For example, if $n=0$ and m, $l\neq0$, then

$$-\left(\frac{\omega^2}{l^2-m^2}\right)=\frac{\gamma P_0+(B_0^p)^2/\mu_0}{\rho_0+\epsilon_0(B_0^p)^2}. \tag{16}$$

This is a longitudinal wave propagating at right angles to the lines of force without bending them. If $n\neq0$ and m, $l=0$, two solutions are possible:

$$-\frac{\omega^2}{n^2}=\frac{(B_0^p)^2/\mu_0}{\rho_0+\epsilon_0(B_0^p)^2}, \tag{17}$$

which goes to c^2 as $\rho_0\to0$ and is the transverse wave propagating along the lines of force mentioned above. The wave is the analog of a wave in a string having tension $(B_0^p)^2/\mu_0$ and mass per unit length of $[\rho_0+\epsilon_0(B_0^p)^2]$. The other solution is

$$-\omega^2/n^2=\gamma P_0/\rho_0, \tag{18}$$

which is merely a sound wave with fluid motion parallel to the lines of force.

If m/l and q/l are eliminated from (15) by use of (13) and (14), the result is

$$(1+h^2-u^2)^{\frac{1}{2}}(Bh^2-Au^2)=[(\delta+u)^2-h^2]\times[1+(Dh^2-u^2)(Bh^2-Au^2)/(Gh^2-Fu^2)]^{\frac{1}{2}}, \tag{19}$$

where the positive dimensionless quantities A, B, D, F, and G have been introduced:

$$A=[\rho_0+\epsilon_0(B_0^p)^2]/[\epsilon_0(B_0^v)^2],\quad B=(B_0^p)^2/(B_0^v)^2,$$
$$D=\mu_0\epsilon_0P_0\gamma/\rho_0=v_{\text{sound}}^2/c^2,$$
$$G=BD,\quad F=B+D(A-B).$$

[4] H. Alfvén, *Cosmical Electrodynamics* (Oxford University Press, New York, 1950).

The square root to be chosen on each side of (19) is the one with the positive real part, corresponding to perturbations which fall off away from the boundary. These are the only ones of physical interest for semi-infinite plasma and vacuum regions.

Equation (19), which is of eighth degree in u when squared and multiplied by Gh^2-Fu^2, can be solved when $l=0$ ($h=\infty$). In this case it reduces to

$$(1+r)^{\frac{1}{2}}\left\{(Ar+B)+(1+r)^{\frac{1}{2}}\times\left[\frac{(r+D)(Ar+B)}{(Fr+G)}\right]^{\frac{1}{2}}\right\}=0, \quad \text{if} \quad \delta^2\ll 1, \tag{20}$$

where $r=\mu_0\epsilon_0\omega^2/n^2$. The solution $r=-1$ gives $q=0$ by (13) so that the original equations must be re-examined; the result found is that all perturbations must vanish. The solution $r=-B/A$ gives $m=0$ by (14). But this is simply the case where the first factor in (14) vanishes and, as previously stated, it is a permissible solution but does not result in a boundary perturbation. The second factor in (20) gives two more solutions for r. It can be shown that one of these solutions is positive real and the other negative real for nonrelativistic gases (those in which the rest energy of the particles is much greater than their kinetic energy). But $r>0$ is not a solution, since the radicals in (20) are those with positive real parts. Hence the only permissible solution is $r<0$, which means ω is imaginary and the boundary undergoes stable oscillations.

In the more general case where $l\neq 0$, an approximate solution of the eighth degree equation in u can be obtained for small h and δ. Although $h=\delta=u=0$ is a solution, expansion of u in a Taylor series in h and δ about the origin is not permissible, since the partial derivatives $\partial u/\partial\delta$ and $\partial u/\partial h$ are discontinuous at the origin. This can be seen from the fact that $(\partial u/\partial\delta)\delta+(\partial u/\partial h)h$ is different from what is obtained by expanding u in terms of distance along a ray through the origin in the $h-\delta$ plane. The ray expansion is accomplished by letting $\delta=ys$ and $h=xs$ and $u=$ a power series in s, where $s=$ distance from the origin and $y/x=$ slope of the ray. Equating coefficients of the lowest power of s and expressing the result in terms of h and δ gives

$$u=\frac{-\delta\pm i[A\delta^2-(A+1)(B+1)h^2]^{\frac{1}{2}}}{A+1} \tag{21}$$

or

$$\omega=i\frac{v_0 l}{A+1}\pm\frac{[Al^2v_0^2-(A+1)(B+1)n^2c^2]^{\frac{1}{2}}}{A+1}. \tag{22}$$

The discontinuity of the partial derivatives at the origin can be verified from (21). The second term of (22) determines whether instability occurs. If it is real, there are two normal modes, one which grows exponentially and the other which is damped in time. If the second term is imaginary, the system is stable and merely oscillates. If $Al^2v_0^2-(A+1)(B+1)n^2c^2=0$, the system is in a condition of marginal stability. ω is in general complex in the plasma frame, so that unstable modes appear to grow in an oscillatory fashion. In the frame moving at the wave velocity, which can be obtained from the imaginary part of ω, unstable modes grow without oscillating. If v_0 is reversed in sign, it can be seen that the wave moves in the opposite direction with the same speed in the unstable case, while the instability rate is unaffected.

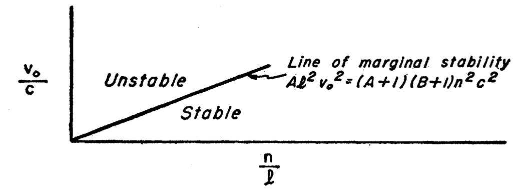

FIG. 3. Stable and unstable regions for v_0/c and n/l small.

If $n=0$ (no bending of lines of force), (22) shows that the system is unstable for all l, ω being larger for shorter wavelengths, as is also true for MHD-Rayleigh instability. If $v_0=0$,

$$\omega^2=-n^2\left(\frac{(B_0^p)^2}{\mu_0}+\frac{(B_0^v)^2}{\mu_0}\right)\bigg/\left(\rho_0+\frac{(B_0^p)^2}{\mu_0c^2}+\frac{(B_0^v)^2}{\mu_0c^2}\right).$$

Except for the two magnetic field terms in the denominator, which are present because light velocity has not been assumed infinite, this agrees with the Kruskal-Schwarzschild[3] result for the case of no gravitational force.

TABLE I. Amplitudes of perturbations when ω, m, and q are all different from zero.

Plasma		
$v_x=-i\mu_0 l\omega\alpha$		
$v_y=\mu_0 m\omega\alpha$		
$v_z=-(in/\rho_0\omega)\{(B_0^p)^2(n^2+\mu_0\epsilon_0\omega^2+l^2-m^2)+\mu_0\rho_0\omega^2\}\alpha$		
$B_x^p=\mu_0 nlB_0^p\alpha$		
$B_y^p=i\mu_0 nmB_0^p\alpha$		
$B_z^p=\mu_0(m^2-l^2)B_0^p\alpha$		
$E_x^p=-\mu_0 m\omega B_0^p\alpha$		
$E_y^p=-i\mu_0 l\omega B_0^p\alpha$		
$E_z^p=0$		
$j_x^p=m(n^2+\mu_0\epsilon_0\omega^2+l^2-m^2)B_0^p\alpha$		
$j_y^p=il(n^2+\mu_0\epsilon_0\omega^2+l^2-m^2)B_0^p\alpha$		
$j_z^p=0$		
$f=0$		
$P=[(B_0^p)^2(n^2+\mu_0\epsilon_0\omega^2+l^2-m^2)+\mu_0\rho_0\omega^2]\alpha$		
$\rho=(\rho_0/\gamma P_0)[(B_0^p)^2(n^2+\mu_0\epsilon_0\omega^2+l^2-m^2)+\mu_0\rho_0\omega^2]\alpha$		
Vacuum		
$B_x^v=-\mu_0(mnl/q)B_0^v\alpha$		
$B_y^v=i\mu_0 mnB_0^v\alpha$		
$B_z^v=-\mu_0(m/q)(n^2+\mu_0\epsilon_0\omega^2)B_0^v\alpha$		
$E_x^v=-\mu_0 m\omega B_0^v\alpha$		
$E_y^v=i\mu_0(ml\omega/q)B_0^v\alpha$		
$E_z^v=0$		
Boundary		
$n_x=-i\mu_0 ml\alpha$	$n_y=0$	$n_z=-i\mu_0 mn\alpha$

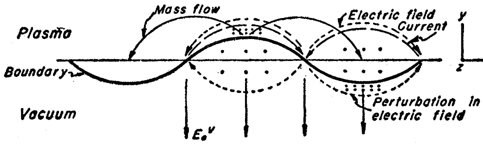

FIG. 4. Fields and flow for $n=0$, $l\neq 0$ (unstable case).

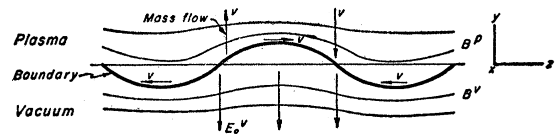

FIG. 5. Fields and flow for $n\neq 0$, $l=0$ (stable case).

The eighth degree equation obtained from (19) has six other roots for u. Two of these were introduced by multiplying through by Gh^2-Fu^2. Two come from (19) with the sign of one side reversed and therefore are also extraneous. The final two, given by $u^2=(A^2-1)/(A^2F-A)$ to lowest order in h and δ, do not satisfy (19) because of the requirement that the radicals have positive real parts. Equation (21) then gives the only two permissible solutions for small h and δ. Figure 3 shows the regions of stability and instability.

ANALOG TO HYDRODYNAMICS

Lamb[1] gives an expression for hydrodynamic Helmholtz instability. If it is applied to a frame of reference in which one fluid is at rest and the other moving with velocity $-v_0$, the result is

$$\omega=\frac{iv_0 l}{1+\rho/\rho'}\pm\frac{(v_0^2 l^2\rho/\rho')^{\frac{1}{2}}}{1+\rho/\rho'}, \tag{23}$$

where ρ is the density of the fluid at rest. It is interesting to note that the approximate MHD solution for ω when $n=0$ can be obtained from (23) by replacing ρ by $\rho_0+\epsilon_0(B_0{}^p)^2$ and ρ' by $\epsilon_0(B_0{}^v)^2$. $\epsilon_0 B^2$ is twice the mass equivalent of a field energy. Now (23) is obtained from linearized theory by assuming incompressible irrotational flow, while from Table I it can be shown that the MHD flow is strictly irrotational if $n=0$, but it is not incompressible, since

$$\nabla\cdot\mathbf{v}=-(P\omega/\gamma P_0)e^{ilx-my+\omega t}.$$

P is the amplitude of the pressure perturbation and equals $[(B_0{}^p)^2(\mu_0\epsilon_0\omega^2+l^2-m^2)+\mu_0\rho_0\omega^2]\alpha$, where α is a parameter which sets the scale of all the perturbations. Also, by (14), l^2-m^2 is proportional to ω^2, so that $\nabla\cdot\mathbf{v}\sim\omega^3$. But by (22), $\omega\sim\delta$, and $\nabla\cdot\mathbf{v}$ is therefore $\sim\delta^3$. Thus to first order in v_0/c, (22) is an incompressible solution.

From the basic equations (1) to (7) one can show that if the lines of force remain straight and parallel and if $\sigma=\infty$,

$$\frac{\partial(\rho+\epsilon_0 B^2)}{\partial t}+\nabla\cdot[(\rho+\epsilon_0 B^2)\mathbf{v}]=-\epsilon_0 B^2\nabla\cdot\mathbf{v}, \tag{24}$$

$$(\rho+\epsilon_0 B^2)\frac{d\mathbf{v}}{dt}+\nabla\left[P+\frac{B^2}{2\mu_0}\left(1-\frac{v^2}{c^2}\right)\right]=\epsilon_0 B^2\mathbf{v}\nabla\cdot\mathbf{v}, \tag{25}$$

where B is scalar and equals the magnetic field. If the terms containing $\nabla\cdot\mathbf{v}$ and v^2/c^2 in (24) and (25) are dropped as being of higher than first order in δ in the present problem, these two equations look like hydrodynamic equations, but with $\rho+\epsilon_0 B^2$ appearing as mass density and $P+B^2/2\mu_0$ as total stress. This is the reason for the similarity of (22) with $n=0$ and (23).

Table I gives the relative amplitudes of the perturbations. The expressions are exact in that they do not involve any approximate ω, and are rigorous solutions of the homogeneous linear equations provided ω, m, and q are all different from zero. α is an arbitrary scale parameter.

Figures 4 and 5 show some features of the fields and flow for $n=0$ and $l=0$, respectively. The figures are only approximate in that the wave motion has been neglected—i.e., the imaginary part of ω has been neglected compared to the real part. This is justifiable, since in practical cases $A\gg 1$.

Extensions of the present work could in principle be made to two fluids with magnetic fields, to $\sigma\neq\infty$, viscosity$\neq 0$, finite dimensions, etc.

ACKNOWLEDGMENTS

I wish to express my thanks to Dr. Clifford Gardner of this Laboratory and to Dr. Martin Kruskal of Princeton University for many helpful ideas in connection with this problem.