and energy quantities. Explicitly,

$$\epsilon_2'N_3'(W)-2\epsilon_3'(W)N_2' \cong\sum{}' \frac{|W_{0m}|^2|W_{0n}|^2(E_n'-E_m')}{(E-E_m')^2(E-E_n')^2}(E_n'-E_m') \times\left[\frac{W_{nn}}{E-E_n'}-\frac{W_{mm}}{E-E_m'}\right], \quad (23)$$

suggesting a high degree of internal cancellation. Internal cancellation is relatively ineffective in reducing the magnitude of the denominator since it is actually a sum of positive-definite terms:

$$\epsilon_2'P_2'-N_2'^2=\tfrac{1}{2}\sum{}' \frac{|W_{0m}|^2|W_{0n}|^2}{(E-E_m')^2(E-E_n')^3} \times(E_n'-E_m')^2. \quad (24)$$

Finally, the formal invariance of the complete perturbation series with respect to uniform displacements of the zeroth order energy spectrum can be demonstrated by a calculation closely resembling that of Eq. (18).

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Degeneracy of the n-Dimensional, Isotropic, Harmonic Oscillator

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We show that what has previously been considered the "accidental" degeneracy in the energy levels of the n-dimensional, isotropic, harmonic oscillator is actually a consequence of its symmetry group. The additional symmetry, beyond the n-dimensional rotation group, arises from the symmetry between the coordinates and momenta.

If the Hamiltonian for a given quantum-mechanical problem is invariant under a group of operations (for example, rotations of the coordinate space), then the eigenfunctions which correspond to each energy eigenvalue form the basis of a representation of this symmetry group. As any representation is a sum of irreducible representations, we see that the degeneracy of an eigenstate is directly related to the dimensionality of the irreducible representations. In many cases we can enlarge the symmetry group so that each energy eigenstate is composed of only one irreducible representation, although all irreducible representations need not appear.[1] In specific cases, however, it is not always easy to find the complete symmetry group. Fock[2] has found the symmetry group for the hydrogen atom and shown that the degeneracy of its energy levels is a necessary consequence of the symmetry properties of its Hamiltonian, but the degeneracy of the n-dimensional, isotropic, harmonic oscillator (hereafter referred to as n-oscillator) has not been understood in terms of the symmetry properties of its Hamiltonian. We shall now show that the degeneracy of its energy levels is a consequence of invariance under the n-dimensional unitary group.

The Hamiltonian operator for the n-oscillator is

$$H=\sum_{k=1}^{n}\left[(p_k^2/2m)+2\pi^2mv^2q_k^2\right]. \quad (1)$$

If we introduce[3] the non-Hermitian operators a_k, which are defined by the relations

$$a_k=\left[1/(2mhv)^{\frac{1}{2}}\right](2\pi mvq_k+ip_k), \quad (2)$$

we obtain

$$H=hv\sum_{k=1}^{n}(a_k^*a_k+\tfrac{1}{2}). \quad (3)$$

The a_k satisfy the commutation relations,

$$\begin{aligned} a_ka_r-a_ra_k=a_k^*a_r^*-a_r^*a_k^*=0,\\ a_ka_r^*-a_r^*a_k=\delta_{kr}. \end{aligned} \quad (4)$$

We shall now show that the quantum mechanical problem of the n-oscillator is invariant under the n-dimensional unitary group. Let us define

$$A_k=\sum_{r=1}^{n}U_{kr}a_r, \quad (5)$$

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[1] H. Margenau and G. P. Murphy, *The Mathematics of Physics and Chemistry* (D. Van Nostrand Company, Inc., New York, 1950), Sec. 15.19.
[2] V. Fock, Z. Physik **98**, 145 (1935).
[3] See, for instance, P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, New York, 1947), Sec. 34.

where U_{kr} is an n-by-n unitary matrix, i.e.,

$$\sum_{r=1}^{n} U_{kr}{}^{*}U_{jr}=\delta_{jk}. \quad (6)$$

Then the new Hamiltonian, $\mathcal{H}$,

$$\begin{aligned}\mathcal{H}&=h\nu\sum_{k=1}^{n}(A_k{}^{*}A_k+\tfrac{1}{2})\\&=h\nu\sum_{k=1}^{n}\left[\sum_{r=1,j=1}^{n}(a_r{}^{*}U_{kr}{}^{*}U_{kj}a_j)+\tfrac{1}{2}\right]\\&=h\nu\sum_{r=1,j=1}^{n}(a_r{}^{*}\delta_{rj}a_j+\tfrac{1}{2}\delta_{rj})\\&=h\nu\sum_{r=1}^{n}(a_r{}^{*}a_r+\tfrac{1}{2})=H, \quad (7)\end{aligned}$$

is equal to the original one, as defined by Eq. (1). We further verify that the A_k satisfy the same commutation relations as the a_k:

$$A_kA_r-A_rA_k=\sum_{j=1,i=1}^{n}[U_{kj}U_{ri}(a_ia_j-a_ja_i)]=0, \quad (8)$$

$$A_k{}^{*}A_r{}^{*}-A_r{}^{*}A_k{}^{*}=0, \quad (9)$$

and

$$\begin{aligned}A_kA_r{}^{*}-A_r{}^{*}A_k&=\sum_{i=1,j=1}^{n}[U_{kj}U_{ri}{}^{*}(a_ja_i{}^{*}-a_i{}^{*}a_j)]\\&=\sum_{j=1}^{n}(U_{kj}U_{rj}{}^{*})=\delta_{kr}. \quad (10)\end{aligned}$$

To construct unitary, irreducible representations of this group, we consider the monomials in n variables:

$$(x_1{}^{f_1}x_2{}^{f_2}\cdots x_n{}^{f_n})/(f_1!f_2!\cdots f_n!)^{\frac{1}{2}}, \quad \sum_{j=1}^{n} f_j=f, \quad (11)$$

which are homogeneous of degree f, where the f_j are non-negative integers. If we think of $(x_1, \cdots, x_n)$ as a unitary vector space, and let the group act on it, then the monomials (11) transform among themselves and hence form the basis for a representation. Weyl[4] shows that this representation is irreducible and unitary. By counting the number of monomials of degree f, we see that the dimensionality of this representation is $[(n+f-1)!/(f)!(n-1)!]$, which is just the degeneracy of the fth level of the n-oscillator.[5]

It is worth noting that these irreducible representations, while sufficient for our purposes, do not exhaust all the inequivalent, irreducible representations of the n-dimensional unitary group. Weyl also shows that not only the symmetric tensors [our monomials, (11)], but also the antisymmetric tensors from the basis of irreducible representations. To see that they need not be equivalent, consider the antisymmetric tensors of the second order for the 3-oscillator. They form a three-dimensional pseudovector representation, which we know is inequivalent to the three-dimensional vector representation (11) already obtained.

That the irreducible representations we have exhibited are the ones actually formed by the eigenfunctions of the n-oscillator may be seen by following the transformation properties of the highest power in the polynomial part of the known solutions (products of Hermite functions) under the real orthogonal subgroup of the unitary group. Under this subgroup it is evident that our irreducible representations have the same transformation properties as do those formed by the known solutions. One can verify by somewhat tedious, but straightforward calculation that the same result is true under the full unitary group. We remark that the remaining terms of the eigenfunctions must, of course, transform properly.

The degeneracy of the energy levels of the n-oscillator is therefore a necessary consequence of the symmetry group of the Hamiltonian.

The author has profited from several conversations with Professor Robert Karplus, who called this problem to his attention.

[4] H. Weyl, *The Theory of Groups and Quantum Mechanics* (Dover Publications, New York, 1932), pp. 139 and 299.
[5] R. H. Fowler, *Statistical Mechanics* (Cambridge University Press, New York, 1955), Sec. 2.21.