PHYSICAL REVIEW VOLUME 103, NUMBER 4 AUGUST 15, 1956

# Electric-Monopole Transitions in Atomic Nuclei*

E. L. CHURCH,† *Frankford Arsenal, Philadelphia, Pennsylvania*

AND

J. WENESER,‡ *Brookhaven National Laboratory, Upton, New York*

(Received April 23, 1956)

Low-energy electric-monopole or $E0$ transitions ($\Delta\mathbf{I}=0$, no) proceed solely by internal conversion, with zero units of angular momentum transferred to the ejected electron. Single gamma-ray emission of this multipole order is strictly forbidden. Electric-monopole pair production is possible for transition energies greater than $2mc^2$.

It is pointed out that (1) the electric-monopole mode of de-excitation is available between any two equal-parity states of the same spin, zero or otherwise, (2) in such cases, $E0$ internal conversion may compete favorably with the paralleling magnetic-dipole and electric-quadrupole transitions in heavy nuclei, and (3) monopole matrix elements may be particularly useful in the study of nuclear structure.

The relative and absolute conversion properties of electric-monopole transitions have been calculated relativistically, including the effects of the finite nuclear size and bound-state atomic screening. These results have been used to analyze the available experimental data on the $2+\rightarrow 2+$ transitions in $Hg^{198}$, $Pt^{196}$, and $Pt^{192}$, and to determine upper limits for the monopole matrix elements. These upper limits appear appreciably smaller than the values for the matrix elements of well-known monopole transitions of the $0+\rightarrow 0+$ type.

The possible significance of these results is considered with reference to current nuclear models.

## INTRODUCTION[1]

TRANSITIONS between nuclear levels occur by the competing processes of gamma-ray emission and internal conversion. For transition energies greater than $2mc^2$, pair production is also possible. These processes are usually analyzed in terms of multipoles (parity and angular-momentum representation). A given nuclear transition may consist of a mixture of several multipoles, consistent with the conservation of parity and vector angular momentum.[2] For other than monopole transitions, the rate of internal conversion is proportional to the rate of gamma-ray emission, since they involve the same nuclear matrix elements to lowest order. The proportionality constant is the internal-conversion coefficient. The emission of a single electric-monopole gamma ray, however, is strictly forbidden by the transverse nature of the electromagnetic field.

Low-energy electric-monopole or $E0$ transitions[2] ($\Delta\mathbf{I}=0$, no) proceed solely by internal conversion, with zero units of angular momentum transferred to the ejected electron. For energies greater than $2mc^2$, monopole pair production is also possible. The existence of monopole transitions between two states of zero spin and the same parity has long been known (zero-zero transitions).[3] However, it is now pointed out that (1) the electric-monopole mode of de-excitation is available between any two equal-spin states of the same parity, zero or otherwise, (2) in such cases, $E0$ internal conversion may compete favorably with the paralleling $M1$ and $E2$ transitions in heavy nuclei, and (3) monopole matrix elements are sensitive to the finer details of the nuclear wave functions, and may be especially useful in the study of nuclear structure.

In this paper we confine our attention to the internal-conversion mode of electric-monopole de-excitation. The relevant properties of $E0$ transitions are described below, and are used to analyze the meager experimental data now available. Monopole matrix elements are then discussed with reference to current nuclear models. Discussion of the formulation of the problem and details of the calculations are given in the appendix.

## PROPERTIES

(A) The most conspicuous property of low-energy electric-monopole transitions is the fact that they proceed solely by internal conversion. The absolute transition probability for $E0$ conversion, $\mathcal{W}$, may be written as the product of an electronic factor, $\Omega$, and the square of the nuclear "strength parameter," $\rho$, which contains the nuclear matrix elements. The electronic factor, $\Omega$, is independent of the spin of the nuclear states involved. The relative conversion in various atomic shells and subshells is essentially independent of nuclear properties.

The separation of the $E0$-conversion probability into electronic and nuclear factors is not as well defined as for the conversion of higher multipoles, nor is the electronic factor, $\Omega$, completely independent of nuclear properties. Physically, the monopole transition interaction vanishes except while the electron is within the nuclear charge distribution, and hence it is the electron wave functions within the nucleus which enter into the calculation of $\Omega$. These in turn, depend on the average static nuclear charge distribution, in a manner discussed in the appendix. Assuming such an average charge

* Work supported in part by the U. S. Atomic Energy Commission.

† Guest Scientist at Brookhaven National Laboratory.

‡ Now on leave at the University of Illinois, Urbana, Illinois.

[1] Preliminary reports of this work have already appeared: E. L. Church and J. Weneser, Phys. Rev. 100, 943, 1241(A) (1955).

[2] $2^L$-pole electric ($EL$) and magnetic ($ML$) transitions carry off $L$ units of angular momentum ($|\Delta\mathbf{I}|=L$), and obey the parity rules $\pi_i\pi_f=(-1)^L$ and $-(-1)^L$, respectively.

[3] R. H. Fowler, Proc. Roy. Soc. (London) A129, 1 (1930).

distribution, we find

$$\mathcal{W}=\Omega\rho^2,$$

$$\rho=\sum_p \int \phi_f^* \left[\left(\frac{r_p}{R}\right)^2-\sigma\left(\frac{r_p}{R}\right)^4+\cdots\right]\phi_i d\tau, \quad (1)$$

where $\phi_i$ and $\phi_f$ are the initial and final nuclear wave functions, $r_p$ is the position vector of the $p$th proton, and $R$ is the appropriately defined nuclear radius. As shown in the appendix, it is sufficient for the present purposes to take the nuclear charge distribution as uniform over a sphere. The numerical coefficient $\sigma$ appearing in (1) depends on the derivatives of the electron wave functions at the origin, and hence slightly on the nuclear transition energy and the electron shell involved. Graphs of this coefficient are given in the appendix, where it is shown that in any reasonable case, $\sigma$ is less than 0.1. In the usual discussions of electric-monopole conversion, only the leading term in (1) is considered because of the smallness of $\sigma$ and higher coefficients. It should be emphasized, however, that the different nuclear matrix elements appearing in (1) are related only within the framework of a specific nuclear model. If the experimentally determined values of $\rho$ are small, then at least the leading term in (1) is small, but the higher terms may not be negligible.

(B) The reduced monopole-conversion probability, $\Omega=\mathcal{W}/\rho^2$, has been computed for various atomic shells, using for the electron wave functions the relativistic Dirac solutions for a uniformly charged sphere of radius $R=1.20\times10^{-13}A^{\frac{1}{3}}$ cm.[4] As in beta decay and $K$ capture, these wave functions are relatively insensitive to the details of the nuclear charge distribution and the magnitudes of the static nuclear moments. The bound-electron functions have also been corrected for the effects of atomic screening. The details of these calculations and a list of formulas are given in the appendix.

Figure 1 presents the reduced transition probability for monopole conversion in the $K$ shell, $(\mathcal{W}/\rho^2)_K$, as a function of the nuclear transition energy, $k$, for various atomic numbers. Since $E0$ transitions between nonzero spin states may be in competition with $M1$ and $E2$ transitions, it is convenient to compare the $E0$ $K$-shell conversion probability with the corresponding $E2$ and $M1$ transition probabilities. This is done in Fig. 2 for a 511-kev transition for various atomic numbers. The $E2$ and $M1$ transition probabilities shown are computed in the "Weisskopf" approximation,[5]

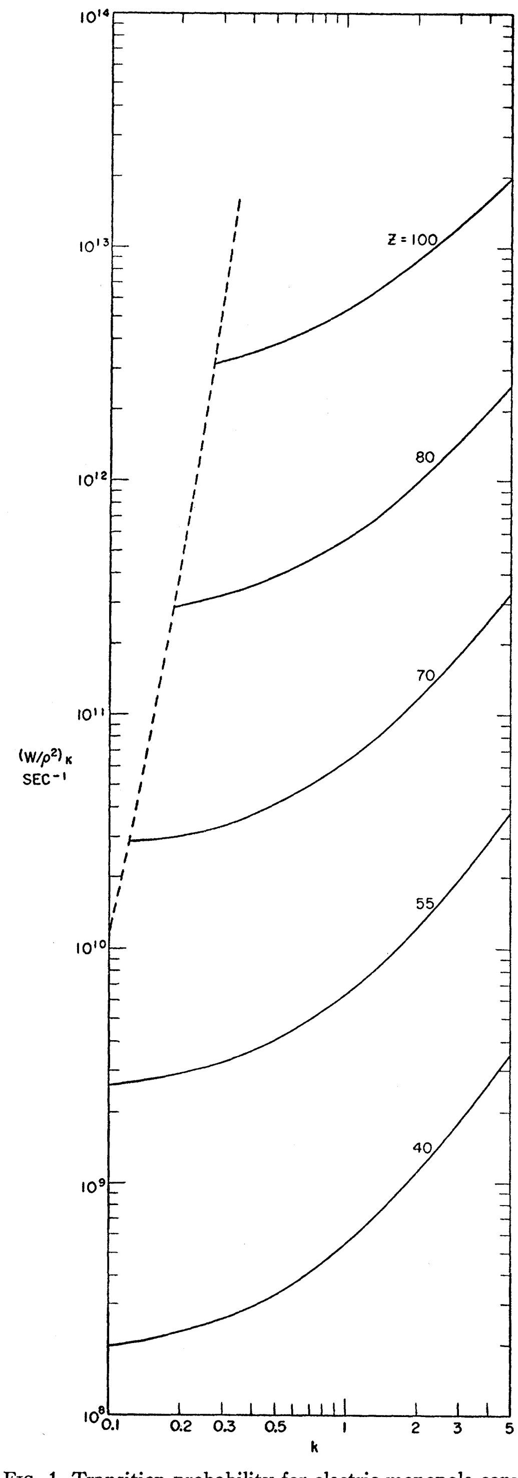


FIG. 1. Transition probability for electric-monopole conversion in the $K$ shell divided by $\rho^2$. Results are given for various atomic numbers as functions of the nuclear transition energy, $k$, in units of $mc^2$. The nuclear strength parameter, $\rho$, is defined by Eq. (1). These results include the effects of the finite nuclear size and bound-state atomic screening on the electron wave functions. The dashed line indicates the threshold for $K$ conversion, in the immediate vicinity of which the indicated results may be uncertain because of the neglect of continuum screening.

[4] As shown in the appendix, the reduced conversion probability, $\Omega$, varies approximately as $R^{4\gamma}$, where $\gamma$ lies between unity ($Z=0$) and $\sim$0.7 ($Z=100$).

[5] V. F. Weisskopf, Phys. Rev. **83**, 1073 (1951). The $M1$ gamma-ray transition probability indicated in Fig. 2 differs from the result quoted in this reference by the factor $(4/3)^2[(\mu_p-\frac{1}{2})/(\mu_p+1)]^2=0.65$. See S. A. Moszkowski, reference 6, Chap. 13; also M. Goldhaber and J. Weneser in *Annual Reviews of Nuclear Science* (Annual Reviews, Inc., Stanford, 1955), Vol. 5, p. 1.

using the $K$-shell conversion coefficients of Rose *et al.*[6] The $E0$ conversion probability is given for $\rho=1$. Values of $\rho$ of this order would correspond to single-proton transitions with complete overlap of the initial and final nuclear wave functions—analogous to the "Weisskopf" estimates for the higher multipoles. It follows from Fig. 2 that $E0$ conversion may effectively compete with $E2$ and $M1$ transitions in heavy elements. There are several additional factors which may make it of even greater practical importance. First, in any experiment involving the measurement of conversion-electron intensities, $E0$ conversion will appear in competition with $E2$ and $M1$ conversion, which are generally only a few percent of their corresponding gamma-ray intensities. Second, the relative importance of monopole conversion is appreciably enhanced at lower energies, since the "Weisskopf" $M1$ and $E2$ gamma-ray transition probabilities vary as $k^3$ and $k^5$, respectively, while the $E0$ dependence, as illustrated in Fig. 1, is much weaker. Finally, there is a large group of nuclei in the heavy-element region exhibiting $2+\rightarrow 2+$ transitions, in which the $M1$ components are severely attenuated.[7] In such cases the possibly dominant magnetic-dipole component may be absent.

It appears entirely feasible to determine experimental values of monopole matrix elements of $I\pm\rightarrow I\pm$ transitions corresponding to values of $\rho$ as much as an order of magnitude smaller than the "Weisskopf" value of $\sim 1$.

(C) The striking increase of the reduced $E0$ transition probability with atomic number indicated in Figs. 1 and 2 has a simple physical origin. Electric-monopole transitions occur via the Coulomb coupling of the atomic electrons and the nuclear protons. Since the monopole moment of the nucleus is constant outside the nuclear volume, the interaction leading to $E0$ conversion vanishes except within the nucleus where the electrons directly sense the variations in proton distribution. Conversion then occurs most strongly in those atomic shells and subshells which are prominent at the origin, namely, those involving zero units of orbital angular momentum. The reduced transition probability might then be expected to increase roughly as $(R/a_Z)^3$ or $Z^4$. As is well known, however, the high kinetic energy of the atomic electrons near the nucleus gives rise to a large relativistic enhancement of the $l=0$ parts of the Dirac wave functions at small radii, especially in heavy elements. These effects lead to the indicated rapid increase of $\Omega$ with atomic number. The dependence on the nuclear transition energy enters only through the magnitude of the continuum wave functions near the origin, and accounts for the relatively slight energy dependence shown in Fig. 1.

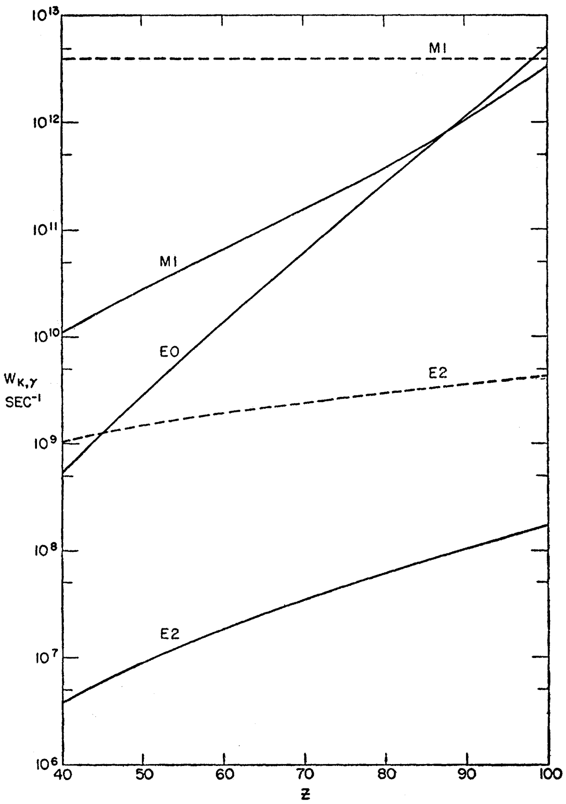


FIG. 2. Transition probability for electric-monopole ($E0$) conversion in the $K$ shell as a function of atomic number for a transition energy of one $mc^2$. These results have been derived from Fig. 1 by assuming $\rho=1$. The analogous "Weisskopf" estimates for the $M1$ and $E2$ gamma ray (dashed lines) and $K$-conversion probabilities (solid lines), are included for comparison. The latter are based on the unscreened calculations of the corresponding $K$-shell conversion coefficients of Rose *et al.* (reference 6).

Conversion occurs predominantly in $s_{\frac{1}{2}}$ subshells ($K$, $L_{\mathrm{I}}$, $M_{\mathrm{I}}$, $\cdots$), since the wave functions of the other subshells are always much weaker in the vicinity of the nucleus. Since the density of the $s_{\frac{1}{2}}$ wave functions at the origin decreases with increasing shell number, high $K/L$ conversion ratios are expected. In the nonrelativistic limit, for example, $K:L_{\mathrm{I}}:M_{\mathrm{I}}\sim 1:1/8:1/27$, with negligible conversion in other subshells. This behavior is similar to that expected for $M1$ transitions. In computing conversion ratios for a given monopole transition, the strength parameters for the various atomic shells and subshells are assumed to be the same—a reasonable assumption in view of the expected relative unimportance of higher terms in the expansion of $\rho$, and the fact that the coefficient $\sigma$ in (1) is constant to first order. Theoretical $K/L$ conversion ratios for $E0$ transitions are given in Fig. 3. Except for the possible effects of screening on the continuum wave functions, these values are insensitive to the nuclear transition

[6] M. E. Rose in *Beta and Gamma-Ray Spectroscopy*, edited by K. Siegbahn (Interscience Publishers, Inc., New York, 1955), Chap. 14. Interpolations were made with the aid of the figures in Rose, Goertzel, and Perry, Oak Ridge National Laboratory Report ORNL-1023, 1951 (unpublished). Errors indicated for the theoretical values of $\alpha_{M1}$ and $\alpha_{E2}$ quoted in the text are those due only to assumed uncertainties of 1–2% in their graphical interpolation.

[7] G. Scharff-Goldhaber and J. Weneser, Phys. Rev. 98, 212 (1955).

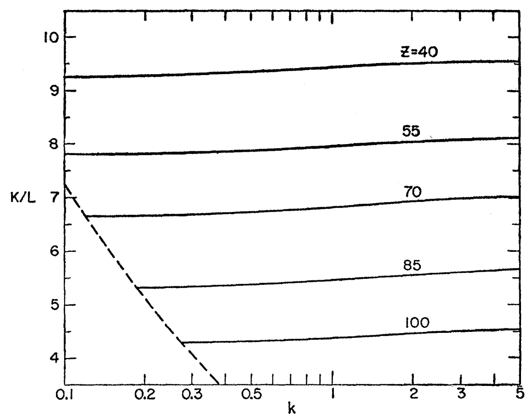


FIG. 3. Relative electric-monopole conversion in the $K$ and (total) $L$ shell for various atomic numbers as a function of the nuclear transition energy, $k$, in units of $mc^2$. The dashed line indicates the threshold for $K$ conversion, in the immediate vicinity of which the indicated results may be uncertain because of the neglect of the effects of atomic screening on the continuum electron wave functions.

energy down to $K$ threshold, and are apparently almost identical with those expected for $M1$ conversion.[8] In contrast, $E2$ transitions exhibit $K/L$ conversion ratios which increase rapidly with increasing energy.

Since relative conversion in various subshells is frequently useful in assigning the multipolarities of low-energy transitions in heavy nuclei, the $L_{\mathrm{I}}/L_{\mathrm{II}}$ conversion ratios for $E0$ transitions have been computed, and are given in Fig. 4. Values for $M1$ transitions (dashed lines) are included for comparison.[8] $E0$ transitions are seen to exhibit very slight $L_{\mathrm{II}}$ conversion except for high-energy transitions in heavy elements, where it is enhanced by the relativistic admixture of an $l=0$ component into the electron wave functions. $L_{\mathrm{III}}$ conversion, on the other hand, is always completely negligible—the ratio $L_{\mathrm{III}}/L_{\mathrm{II}}$ being less than $10^{-6}$ over the region considered. The smallness of the $L_{\mathrm{III}}$ conversion arises from the fact that the wave functions involve at least one unit of orbital angular momentum, which lowers the $L_{\mathrm{III}}/L_{\mathrm{II}}$ conversion ratio by a factor of $\sim(R/\lambda\!\!\!^{-}_{ce})^4$ where $\lambda\!\!\!^{-}_{ce}\equiv\hbar/m_e c$. Again in sharp contrast with $E0$ and $M1$ conversion, $E2$ transitions exhibit comparable $L_{\mathrm{II}}$ and $L_{\mathrm{III}}$ conversion, with $L_{\mathrm{I}}/L_{\mathrm{II}}$ increasing with increasing energy.[8]

In the $M$ shell, both $E0$ and $M1$ conversion occur almost entirely in the $M_{\mathrm{I}}$ subshell, with $M_{\mathrm{II}}$ weakly represented. For $E0$ transitions, one expects $L_{\mathrm{I}}/M_{\mathrm{I}}\sim 3$ in cases of practical interest. For low-energy transitions in heavy nuclei, where such subshell conversion is experimentally resolvable, $E2$ conversion occurs principally in the $M_{\mathrm{II}}$ and $M_{\mathrm{III}}$ subshells. Generalizations of the above may be given for higher shells.[9]

(D) In a nuclear transition between two equal-parity states of the same spin, only $E0$, $M1$, and $E2$ transitions need be considered, since the effects of higher-order transitions are negligible. A direct consequence of the presence of an $E0$ component is the presence of an excess of conversion electrons above that expected from the $M1$ and $E2$ transitions alone. A convenient measure of the intensity of the monopole component is $\epsilon^2$, the ratio of the rate of $E0$ *conversion* to the rate of $E2$ *gamma-ray* emission. The analogous ratio of the rates of $M1$ gamma emission to $E2$ gamma emission is defined here[10] as $\delta^2$. Since the intensity contributions of the multipole components of a mixed transition are additive, the relationship between these quantities is simply

$$\epsilon^2=(\alpha_{\mathrm{exp}}-\alpha_{E2})-\delta^2(\alpha_{M1}-\alpha_{\mathrm{exp}}), \qquad (2)$$

where $\alpha_{\mathrm{exp}}$ is the experimentally observed conversion coefficient, and $\alpha_{E2}$ and $\alpha_{M1}$ are the conversion coefficients of the pure $E2$ and $M1$ transitions, respectively. In practice, $\alpha_{\mathrm{exp}}$ is obtained by a measurement of the net conversion coefficient, $\delta^2$ is determined from gamma-gamma directional-correlation experiments involving the mixed transition, and $\alpha_{E2}$ and $\alpha_{M1}$ are taken from theoretical calculations. It is perhaps worth noting

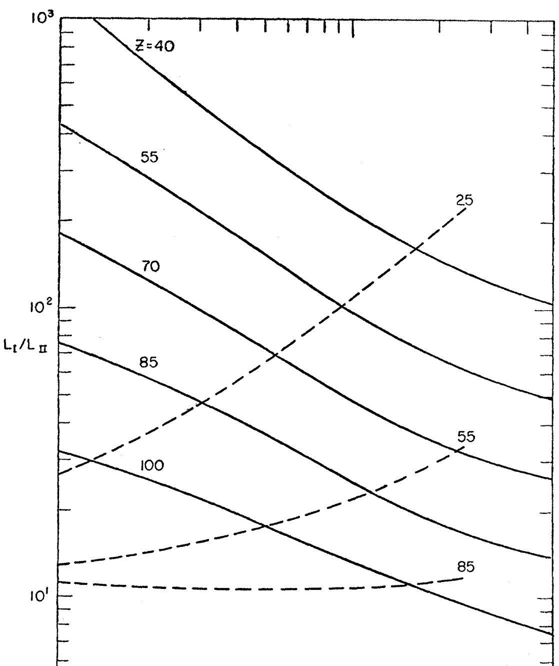


FIG. 4. Relative electric-monopole conversion in the $L_{\mathrm{I}}$ and $L_{\mathrm{II}}$ subshells for various atomic numbers as a function of the nuclear transition energy, $k$, in units of $mc^2$. The dashed lines represent the analogous results for $M1$ transitions according to the screened calculations of Rose *et al.* (reference 8). Conversion of $E0$ transitions in the $L_{\mathrm{III}}$ subshell is completely negligible relative to the $L_{\mathrm{II}}$ subshell.

[8] Rose, Goertzel, and Swift, (privately circulated tables, 1955–6). See also reference 6.

[9] E. L. Church and J. E. Monahan, Phys. Rev. 98, 718 (1955).

[10] The present definition, $\delta^2=M1/E2$, is the reciprocal of that employed, for example, by L. C. Biedenharn and M. E. Rose, Revs. Modern Phys. 25, 729 (1953).

that in heavy elements, where the parentheses in (2) are positive, a simple measurement of the $K$-conversion coefficient is sufficient to place an upper limit on the strength of the monopole contribution, independent of the amount of $M1$ admixture. Expressions analogous to (2) are easily written for conversion ratios.

Gamma-gamma directional correlations would be unaffected by an $E0$ component in one of the transitions. In a directional correlation experiment involving conversion electrons (e.g., $e_K-\gamma$ correlation), however, the $E0$ component would be directly observable. In the case of an $E0$-$M1$-$E2$ mixture, there is interference between the $K$-conversion electrons of the $E0$ and $E2$ components, but none between the $E0$ and $M1$ electrons. The $E0$-$E2$ interference term appearing in the coefficient of $P_2$ (cos$\theta$) is then proportional to $\epsilon$ and not its square. Such correlation measurements might, therefore, provide a more sensitive means for determining the amount of $E0$ mixing than the measurement of conversion coefficients. The theoretical correlation functions for such cases are in the process of computation, and will appear in a subsequent publication in collaboration with M. E. Rose.

## EMPIRICAL EVIDENCE

(A) Any study of electric-monopole transitions involves their identification by means of the properties described above, the determination of the magnitude of their nuclear strength parameters, and the interpretation of these values with reference to specific nuclear models. Until now, data on $E0$ transitions were obtained only from a study of isolated zero-zero transitions, proceeding either by simple internal conversion, or, if the transition energy is greater than $2mc^2$, in combination with pair production.[11,12] Monopole matrix elements are also available from electron-excitation studies.[13] Data of this type are relatively well known, and are included in Table I. A number of other pairs of 0+ levels are known, but no results appear to be available on the transition rates between them.

Since the electric-monopole mode of de-excitation is available between any two states of the same spin and parity, one may also obtain data on monopole transitions from a much wider class of transitions. For example, a recent study of the systematics of even-even nuclei indicates that there exists a large and regular class of heavy nuclei having a 0+ ground state, and 2+ first and second excited states.[7] In most cases the cross-over transition is unimportant, and the main decay of these nuclei is (apparently) a simple cascade of a mixed $M1+E2$ transition between the 2+ levels, followed by a pure $E2$ transition to the ground state. It is observed that the $M1$ component of the former is generally attenuated, and that the transition probability of the $E2$ ground-state transition is greater than the "Weisskopf" estimate by a large and regular factor.[14] Because of the attenuation of the masking $M1$ component, heavy nuclei of this type provide a favorable region for the search for new data on $E0$ transitions. Although data on the transitions between the 2+ states are meager, sufficient results are available for the preliminary analysis of three cases, as discussed below. As yet there appear to be no systematics on equal-spin states in odd-mass nuclei.

(B) $Hg^{198}$ is an even-even nucleus having 2+ first and second excited states, located 412 and 1089 kev above the 0+ ground state. Directional-correlation studies of the cascading 677- and 412-kev gamma rays indicate that the former is an $M1+E2$ mixture with[15] $\delta^2=0.67\pm0.15$, and that the latter is pure $E2$. The measured net half-life of the ground-state transition is $2.1\times10^{-11}$ second.[14] The $K$-shell internal-conversion coefficient of the mixed transition has been measured as[16] $(2.24\pm0.19)\times10^{-2}$. The theoretical calculations of Rose *et al.*[6] indicate that the $K$-shell conversion coefficients of the pure $E2$ and $M1$ components are $(1.12\pm0.02)\times10^{-2}$ and $(4.64\pm0.10)\times10^{-2}$, respectively. Substitution of these values into (2) shows that $\epsilon_K^2$ is zero to within the limits of error, with the quoted extremes corresponding to $\epsilon_K^2\leqslant2.3\times10^{-3}$.

To obtain a quantitative estimate of the transition probability of the $E0$ component of the 677-kev transition, the gamma-ray transition probability of the $E2$ component must be known. Unfortunately, no direct measurement of the lifetime of this transition is available. However, the discussion of the spectra of 0–2–2 nuclei in terms of a "free-vibration" model by Scharff-Goldhaber and Weneser[7] indicates that aside from the usual fifth-power energy dependence, the 2–2 quadrupole transition should be twice as fast as the 2–0 ground-state transition. Willets and Jean[17] have recently proposed a strong-coupling "shape-unstable" collective model of these same nuclei, which may also be shown to lead to a similar factor of between 1.5 and 2. For our purposes we adopt a simple "factor-of-two" rule. The transition probability of the 677-kev transition is then $\sim$24 times that of the 412-kev ground-state transition, corresponding to an $E2$ gamma half-life of $\sim9.1\times10^{-13}$ second for the former. This result, in combination with the previous limit on $\epsilon_K^2$ and the data in Fig. 1, yields the limit $\rho\leqslant1/14$. The upper limit indicated corresponds to the extremes of the quoted limits of error on the values of $\alpha$ and $\delta^2$.

(C) $Pt^{196}$ is also an even-even nucleus of the 0–2–2 type, with 2+ levels lying 354 and 685 kev above the ground state. The decay of the 685-kev level has been

[11] S. D. Drell, Oak Ridge National Laboratory Report ORNL-792, 1950 (unpublished).
[12] S. D. Drell and M. E. Rose, Progr. Theoret. Phys. Japan **7**, 125 (1952).
[13] L. I. Schiff, Phys. Rev. **98**, 1281 (1955).
[14] See, for example, A. W. Sunyar, Phys. Rev. **98**, 653 (1955).
[15] D. Schiff and F. R. Metzger, Phys. Rev. **90**, 849 (1953); C. D. Schrader, Phys. Rev. **92**, 928 (1953).
[16] Elliott, Preston, and Wolfson, Can. J. Phys. **32**, 153 (1954).
[17] L. Wilets and M. Jean, Phys. Rev. **102**, 788 (1956).

shown to proceed entirely via the 331 to 354-kev cascade. Gamma-gamma directional correlation experiments indicate[18] $\delta^2=(4.71\pm0.88)\times10^{-2}$ for the 331-kev transition, and its $K$-conversion coefficient has been measured as[19] $(5.9\pm0.4)\times10^{-2}$. This value, in combination with Rose's theoretical $K$-shell conversion coefficients[6] of $(5.00\pm0.05)\times10^{-2}$ and $(2.51\pm0.03)\times10^{-1}$ for the pure $E2$ and $M1$ components, respectively, again indicates a vanishing monopole contribution. The quoted limits of error correspond to the limit $\epsilon_K^2\leqslant6.4\times10^{-3}$. In order to translate this in terms of the $E0$ matrix element, the lifetime of the competing $E2$ component must be estimated. Based on the known lifetime of the analogous 412-kev transition in $Hg^{198}$, the 354-kev ground-state transition in $Pt^{196}$ is expected to have an $E2$ gamma half-life of $\sim5.0\times10^{-11}$ second.[20] The $E2$ gamma half-life of the 331-kev transition estimated with the aid of the "factor-of-two" rule discussed in the case of $Hg^{198}$, is then $\sim3.5\times10^{-11}$ second. Combination of this result with the previous limit on $\epsilon_K^2$ and the results of Fig. 1, yields the upper limit $\rho\leqslant1/34$ for the 2–2 transition in $Pt^{196}$.

(D) $Pt^{192}$ is very similar to $Pt^{196}$, with 2+ levels lying 316 and 612 kev above the 0+ ground state. Values of $\delta^2=(2.6\pm1.0)\times10^{-2}$ and $\alpha_K=(6.5\pm1.0)\times10^{-2}$ have recently[21,22] been reported for the 296-kev 2–2 transition. These values, in conjunction with Rose's estimates[6] of $(6.50\pm0.07)\times10^{-2}$ and $(3.40\pm0.03)\times10^{-1}$ for the $E2$ and $M1$ $K$-conversion coefficients, respectively, lead to the upper limit $\epsilon_K^2\leqslant6.5\times10^{-3}$. The $E2$ gamma half-life of the 316-kev transition may be estimated as in the case of $Pt^{196}$, and is calculated to be $\sim8.9\times10^{-11}$ second.[20] The $E2$ gamma half-life of the 296-kev 2–2 transition is then estimated to be $\sim6.3\times10^{-11}$ second. These values lead to the upper limit $\rho\leqslant1/45$. Although the quoted upper limit on $\epsilon_K^2$ is higher in this case than for $Hg^{198}$, the corresponding limit for $\rho$ is appreciably less because of the lower transition energy and the $k^5$ energy dependence assumed for the $E2$ transition probability.

(E) The results of the above analysis of $E0$ transitions are presented in Table I. It is seen that monopole transitions of the 2–2 type apparently have strength parameters, $\rho$, significantly less than unity, and which, in fact, are zero to within the limits of experimental error. These estimates, however, depend critically on a knowledge of the lifetimes of the second excited states and the internal-conversion coefficients of the competing $M1$ and $E2$ transitions. In the cases considered, the lifetimes have not yet been directly measured. In the absence of such data it has been assumed that the reduced transition probability[20] of the $E2$ component of the 2–2 transition is twice that of the 2–0 ground-state transition in the same nucleus.[23] The omission of this factor would *lower* the quoted upper limits of the strength parameters by ~30%.

The conversion coefficients of the pure $M1$ and $E2$ components have been taken from the relativistic calculations of Rose *et al.*,[6] which neglect the effects of the finite nuclear size and the small corrections due to atomic screening.[8] There is evidence[24] that Rose's values of $\alpha_{M1}$ for the $K$ shell may be too large by ~35% for $Z\sim80$. Sliv *et al.*[25] have shown that effects of this magnitude may be attributed to the attenuation of the electron wave functions at small radii due to the finite dimensions of the nuclear charge distribution, and predict a simultaneous lowering of $\alpha_{E2}$ for the $K$ shell by ~3% in the same region. If the previously quoted values of the $M1$ and $E2$ $K$-shell coefficients are lowered by these amounts, the upper limits of the strength parameters for $Hg^{198}$, $Pt^{196}$, and $Pt^{192}$ become 1/6, 1/26, and 1/37, respectively. Thus, at least in the case of the

TABLE I. Empirical data on $E0$ transitions.

| Element | Transition | Energy (Mev) | Method of measurement | Strength parameter See Eq. (1) |
|---|---|---|---|---|
| $C^{12}$ | $0\rightarrow0$ | 7.68 | Electron-scattering cross section[a] | $\sim1/2$ |
| $O^{16}$ | $0\rightarrow0$ | 6.06 | Pair-production lifetime[a] | $\sim1/2$ |
| $Ge^{72}$ | $0\rightarrow0$ | 0.69 | Conversion lifetime[b] | $\sim1/9$ |
| $Pt^{192}$ | $2\rightarrow2$ | 0.30 | Conversion-coefficient and directional-correlation measurements[c] | $\leqslant1/45$ |
| $Pt^{196}$ | $2\rightarrow2$ | 0.33 | | $\leqslant1/34$ |
| $Hg^{198}$ | $2\rightarrow2$ | 0.68 | | $\leqslant1/14$ |
| $Po^{214}$ | $0\rightarrow0$[d] | 1.42 | Conversion lifetime[e] | (1/20)? |

[a] L. I. Schiff, Phys. Rev. **98**, 1281 (1955).
[b] The measured half-life is $\sim3\times10^{-7}$ second. See M. Goldhaber and R. D. Hill, Revs. Modern Phys. **24**, 179 (1952).
[c] The indicated values are based on the theoretical calculations of the $M1$ and $E2$ $K$-shell conversion coefficients of Rose *et al.*,[6] and the assumption that the reduced transition probability of the competing $E2$ gamma ray is twice that of the $2\rightarrow0$ ground-state transition in the same nucleus. The omission of the latter factor would lower the quoted upper limits on $\rho$. The indicated upper limits for these strength parameters must be considered as tentative. (See text.)
[d] The observed $K:L:M$ conversion ratios are in good agreement with the $E0$ assignment: Latyshev, Sliv, Barchuk, and Bashilov, Izvest. Akad. Nauk. S.S.S.R., Ser. Fiz. **13**, No. 3, 340 (1949) [see Physics Abstracts **52**, 7253 (1949)]. In addition, D. Alburger and A. Hedgran, Arkiv Fysik **7**, 423 (1954), have placed limits on the $L$-subshell conversion consistent with this assignment.
[e] R. H. Fowler, Proc. Roy. Soc. (London) **A129**, 1 (1930). Adopting his estimate of the branching between the $E0$ conversion and the competing alpha decay, the $K$ ejection half-life of this transition is estimated to be $\sim2\times10^{-10}$ second. The alpha lifetime has been inferred from the known half-life of $1.64\times10^{-4}$ second of the ground state of $Po^{214}$, which presumably also decays to the ground state of $Pb^{210}$. Although both of these alpha decays are of $0\rightarrow0$ type, the 1.42-Mev level may have a very different structure from the ground state, so that the use of the simple Gamow factor may lead to a considerable error in estimating the alpha lifetime of the excited state. Because of the indirectness of this estimate of the conversion lifetime, the quoted value of $\rho$ may only be significant to within an order of magnitude.

[18] R. M. Steffen (private communication) quoted in reference 19. See also R. M. Steffen, Phys. Rev. **89**, 665 (1953).

[19] M. T. Thieme and E. Bleuler, Phys. Rev. **101**, 1031 (1956).

[20] It is assumed that the $E2$ gamma-ray transition probability varies as $k^5Z^2A^{4/3}$ (see reference 14). Coulomb-excitation data of P. H. Stelson and F. K. McGowan [Phys. Rev. **99**, 112 (1955)] and McClelland, Mark, and Goodman [Phys. Rev. **97**, 1191 (1955)], indicate gamma half-lives of $\sim3.5\times10^{-11}$ and $\sim1.6\times10^{-10}$ second, respectively, for the 331-kev transition in $Pt^{196}$.

[21] H. W. Taylor and R. W. Pringle, Phys. Rev. **99**, 1345 (1955); (private communication, 1955).

[22] Baggerly, Marmier, Boehm and DuMond, Phys. Rev. **100**, 1364 (1955); (private communication, October, 1955).

[23] There is as yet no experimental evidence for or against this factor in these or other nuclei.

[24] A. Wapstra and G. Nijgh, Nuclear Phys. **1**, 245 (1956).

[25] L. A. Sliv, Zhur. Eksptl. i Teort. Fiz. **21**, 770 (1951); L. Sliv and M. Listengarten, Zhur. Eksptl. i Teort. Fiz. **22**, 29 (1952); L. A. Sliv (private communication, 1955).

platinum isotopes, the experimental data still indicate that the monopole matrix elements are strongly attenuated relative to the "Weisskopf" value of $\rho\sim 1$.

In view of the paucity of experimental data and the uncertainties in their interpretation, the quantitative estimates of the upper limits of the strength parameters of the 2–2 transitions quoted in Table I must be taken as indicative rather than conclusive. The need for further experimental data is apparent.

## MATRIX ELEMENTS

The study of electric-monopole matrix elements provides data on nuclear transitions not obtainable by other means. Although the presently available experimental data are insufficient to allow generalizations to be made, the upper limits of the monopole matrix elements of the $E0$ components of the $2+\rightarrow 2+$ transitions in $Pt^{192}$ and $Pt^{196}$ given in Table I are distinctly less than the moderate values observed for the $0+\rightarrow 0+$ transitions in $C^{12}$ and $O^{16}$. The qualitative behavior of monopole matrix elements are indicated below for various nuclear models. It is found that for the models considered, all terms in the expression for the monopole strength parameter (1) are subject to the same selection rules, since these arise from their common angular dependences.

In the strict shell model, $E0$ transitions are allowed only if they involve the transition of no more than a single particle, and this only betweeen two states of the same $l$ and $j$. Since this requires a jump through at least two major shell closings, such transitions would not normally be encountered among the low-lying states, although strength parameters of the order of unity would be expected for proton transitions of this kind. It has recently been proposed that the 0–0 transitions in $C^{12}$ and $O^{16}$ are of this type.[26,27] Nuclear states of equal spin and parity can be constructed within a shell by combining particles of given configurations in groups having different internal symmetries (seniority or coupling of the subshells), although monopole transitions between pure states of this type are forbidden, since the monopole operator is diagonal in the angular variables. These rules, however, may be relaxed by configuration mixing, but only if this mixing involves differing radial dependences. Still, it would appear very difficult to obtain strength parameters appreciably greater than 0.1 on the basis of configuration mixing alone. This conclusion is borne out by the explicit calculations of Schiff[13] for $C^{12}$. However, such a mechanism may be proposed in explanation of the moderate value of $\rho$ observed for $Ge^{72}$.

The hydrodynamical model of the nucleus proposed by Bohr and Mottelson,[28] which attributes dynamical properties to the nuclear core, has been successfully applied to the study of heavy nuclei. If attention is confined to the $Y_2$ modes of deformation, the core contributions to the $E0$ transition operator may be expanded as a power series in the collective deformation parameter[28] $\alpha_{2\mu}$. The leading term of this expansion is a constant and cannot give rise to transitions. Since a linear term is absent, only quadratic and higher terms remain. It should be noted that with

$$R=R_0[1+\alpha_{2\mu}Y_{2\mu}],$$

the nuclear volume is not constant to order $\alpha_{2\mu}^2$. The introduction of the higher terms in $R$ necessary to keep the volume constant introduces corrections in the quadratic and cubic terms in the $E0$ matrix element. Although the significance of terms of such high order is doubtful in existing theories, it is still interesting to estimate their orders of magnitude for comparison with empirical results.

These higher terms in the monopole matrix element have been evaluated[29] both for the "free-vibration" model[7] and the strong-coupling "shape-unstable" model.[17] These models are designed to describe the 0–2–2 even-even nuclei of which $Pt^{192}$, $Pt^{196}$, and $Hg^{198}$ may be examples. It is convenient to discuss these models in terms of the collective deformation parameters $\beta$ and $\gamma$ introduced by Bohr and Mottelson.[28] In this case the quadratic term in the expansion of the monopole operator is proportional to the scalar $\beta^2=\sum|\alpha_{2\mu}|^2$. It follows that this term does not contribute to the $E0$ matrix element between the first and second $2+$ states, since the wave functions describing these states are orthogonal in the $\gamma$ coordinate, and the $E0$ operator is diagonal in the $\gamma$ coordinate. In these models, therefore, contributions only come from the cubic and higher terms. The cubic term in the operator expansion is proportional to $\beta^3\cos 3\gamma$ and does lead to a nonvanishing result. On forming the matrix element of this operator, and evaluating the relevant parameters from the lifetime of the $E2$ transition between the first $2+$ and the ground $0+$, we obtain $\rho\sim 1/150$ for the "free-vibration" model, and $\rho\sim 1/300$ for the "shape-unstable" model of the nuclei considered. We see, then, that the higher terms in the operator expansion do lead to small contributions, thereby at least qualitatively explaining the smallness of the empirical $\rho$ values for the 2–2 transitions in these nuclei. It would be of considerable interest to obtain explicit experimental values of $\rho$ for such transitions rather than upper limits.

Because of the smallness of the monopole strength parameters derived on the basis of the present theories, moderate mixing of states produced by deviations from the simple models considered cannot change the qualitative agreement with experiment.

The "free-vibration" model allows a $0+$ excited state in the vicinity of the second $2+$ state. In the "shape-

[26] P. J. Redmond, Phys. Rev. **101**, 751 (1956).
[27] R. Ferrell and W. Visscher, Phys. Rev. **102**, 450 (1956).
[28] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).
[29] J. Weneser and E. Church, Bull. Am. Phys. Soc. Ser. II, **1**, 181 (1956).

unstable" model there are two possible excited 0+ states, one of which is the analog of the 0+ state described by the "free-vibration" model. In both models the $E0$ transition between this excited 0+ state and the 0+ ground state proceed via the quadratic term in the operator expansion, since the wave functions have no $\gamma$ dependence. This then leads to $\rho\sim\frac{1}{5}$ for either model. The other 0+ of the "shape-unstable" model is $\gamma$ orthogonal to the ground state, and so can proceed only via the cubic and higher terms. Its strength parameter is, then, two orders of magnitude smaller than for the 0–0 transition proceeding via the quadratic term. Empirical values for these $E0$ 0+→0+ matrix elements would also be of considerable interest.[30]

### ACKNOWLEDGMENTS

We thank Dr. J. E. Monahan for his assistance in the earlier phases of this work, Dr. M. Goldhaber for helpful discussions, and Dr. R. D. Hill for constructive criticism. Betty Oppenheim's assistance with the numerical calculation is also acknowledged.

### APPENDIX

(A) The interaction between the atomic electrons and the nuclear charges can be written in terms of a multipole expansion. Since a gauge transformation can be carried out separately for each multipole order, the gauge question can also be separately decided. In the calculation of the $E0$ transition probability it is convenient to use the Coulomb gauge, which describes the interaction via the retarded transverse field plus the instantaneous Coulomb force. The $E0$ multipole, zero angular momentum, and even parity, can then come only from the instantaneous Coulomb terms, since the transverse terms contain only multipoles of higher angular momenta.

The Coulomb interaction between the atomic electrons and the nuclear charge distribution is responsible both for the existence of the eigenstates of the atomic system and for transitions between them. To illustrate the separation of these effects, we assume a simplified Hamiltonian in which the electron and nuclear protons interact only via the Coulomb interaction, i.e.,[31]

$$H=H(\text{nuclear})+H(\text{electron})-\sum_{p,e}\frac{\alpha}{|r_p-r_e|},\qquad(3)$$

where $H$(nuclear) depends only on the nuclear coordinates and describes the internal nuclear states, and $H$(electron) is the usual Dirac Hamiltonian for the electrons. The Coulomb term describes the interaction between the nuclear and electron systems. This description is adequate for the monopole conversion problem.

[30] D. Alburger and B. Toppel, Phys. Rev. **100**, 1357 (1955), report measurements on the 1.14-Mev 0→0 cross-over transition in $Pd^{106}$ which lead to the estimate $\rho\leqslant 1/7$ for this transition.

[31] Relativistic units, $\hbar=m=c=1$, are used throughout the following discussion. Hence $e^2=\alpha\approx 1/137$.

The Hamiltonian (3) can be rewritten to regain the usual description of atomic physics as

$$H=[H(\text{nuclear})]+\left[H(\text{electron})-\sum_e\alpha\int\frac{q(r)}{|r-r_e|}d\tau\right]+\left[\sum_e\alpha\int\frac{q(r)}{|r-r_e|}d\tau-\sum_{p,e}\frac{\alpha}{|r_p-r_e|}\right],\qquad(4)$$

where the quantity $q(r)$ represents a stationary average of the nuclear charge distribution. The first term on the right of (4) is the nuclear Hamiltonian, and the second is the usual atomic Hamiltonian describing the interaction of the atomic electrons with the average nuclear charge distribution. The third term is a correction term, the effects of which can be treated by perturbation theory. The effects of this term on the steady states of the coupled system are patently negligible, although in the present description it is entirely responsible for transitions between them. Since only

$$H'=-\sum_{p,e}\frac{\alpha}{|r_p-r_e|}\qquad(5)$$

in the perturbation term depends on nuclear coordinates, it alone need be considered in calculating the transition probability. The quantity $q(r)$ is chosen to best represent the average nuclear charge distribution, and so to form the optimum basis for a perturbation calculation. It might be defined as the diagonal charge operator, $\sum_p\int|\phi_j|^2\delta(r_p'-r)d\tau'$, which, of course, differs from one nuclear state, $j$, to another. However, these differences will be very slight for low-lying states, and for our purposes it is sufficient to take an empirically determined nuclear charge distribution.

(B) The explicit monopole-transition matrix element is the matrix element of the $L=0$ part of $H'$ given in (5). One finds

$$\langle i|H'(L=0)|f\rangle=-\sum_{p,e}\alpha\left[\int d\tau_{\text{nuc}}\int_0^{r_p}d\tau_{\text{el}}\phi_f^*\psi_f^*\frac{1}{r_p}\phi_i\psi_i+\int d\tau_{\text{nuc}}\int_{r_p}^{\infty}d\tau_{\text{el}}\phi_f^*\psi_f^*\frac{1}{r_e}\phi_i\psi_i\right]$$
$$=-\sum_{p,e}\alpha\int d\tau_{\text{nuc}}\int_0^{r_p}d\tau_{\text{el}}\phi_f^*\psi_f^*\left(\frac{1}{r_p}-\frac{1}{r_e}\right)\phi_i\psi_i,\qquad(6)$$

where the $\phi$'s are the nuclear wave functions, the $\psi$'s are the electron functions, and $r_p$ and $r_e$ are the proton and electron position vectors, respectively. Since the region of electron integration is confined to small dimensions (within the nuclear volume), it is convenient to expand the radial parts of the Dirac electron wave functions about $r_e=0$. For all "reasonable" charge distributions [those for which $r^2q(r)\to 0$ at the origin], these are of the form

$$\text{"large" component } (g_{s\frac{1}{2}}, f_{p\frac{1}{2}}) = C(1+ar_e^2+\text{higher order terms}), \quad (7)$$

$$\text{"small" component } (f_{s\frac{1}{2}}, g_{p\frac{1}{2}}) = C(0+br_e+\text{higher order terms}).$$

Substitution of (7) into (6) yields

$$\langle i|H'(L=0)|f\rangle = \tfrac{1}{6}\alpha C_i C_f^* R^2 \rho, \quad (8)$$

where as previously [Eq. (1)],

$$\rho = \sum_p \int \phi_f^* \left[\left(\frac{r_p}{R}\right)^2 - \sigma\left(\frac{r_p}{R}\right)^4 + \cdots\right]\phi_i d\tau, \quad (9)$$

and

$$\sigma = -(3/10)(a_i + b_i b_f^* + a_f^*)R^2. \quad (10)$$

For any "reasonable" charge distribution

$$\sigma_{s\frac{1}{2}, p\frac{1}{2}} = (R^2/15)[(W-V)^2 + (k\pm 1)(W-V) + \tfrac{1}{4}(3k\mp 4)(k\pm 2)], \quad (11)$$

where $W$ is the total energy of the bound electron, $k$ is the nuclear transition energy, and $V$ is the electrostatic potential at the center of the nucleus (of the order of $-\alpha Z/R$). Numerical values of the coefficient $\sigma$ for the $K$ conversion of a 511-kev transition are plotted in Fig. 5 for various nuclear charge distributions. Since $|V|\gg W$ in general, $\sigma\sim(VR)^2/15$, and is very nearly independent of the converting atomic shell and the nuclear transition energy. Because of the smallness of the coefficient $\sigma$, the term in the strength parameter (9) in which it appears is customarily neglected.

(C) A useful approximation for the electron wave functions is the "point-nucleus" approximation, in which the $a$'s and $b$'s in (7) are set equal to zero, and the $C$'s are taken as the values of the corresponding Coulomb wave functions for a point nucleus, but evaluated at a distance equal to the nuclear radius $R$. For the usual case $\alpha ZR$, $PR\ll 1$, one then obtains the explicit expression[32] for the reduced $K$-shell conversion probability:

$$\Omega_K = 2[2\pi|\langle i|H'(L=0)|f\rangle|^2]/\rho^2 = \frac{\alpha^2}{36}\frac{1+\gamma}{\Gamma(2\gamma+1)}\frac{P(W+\gamma)}{\alpha Z}(2\alpha ZR)^{2\gamma+2}F(Z,P), \quad (12)$$

where $\gamma\equiv[1-(\alpha Z)^2]^{\frac{1}{2}}$, $W=[P^2+1]^{\frac{1}{2}}$ is the total energy of the ejected electron, and

$$F(Z,P)\equiv\frac{2(1+\gamma)}{[\Gamma(2\gamma+1)]^2}(2PR)^{2\gamma-2} \times e^{\pi\alpha ZW/P}|\Gamma(\gamma+i\alpha ZW/P)|^2, \quad (13)$$

[32] See, for example, H. A. Bethe *et al.*, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), second edition, Vol. 24, Part 1, p. 316, for the bound functions, and M. E. Rose, Phys. Rev. **51**, 484 (1937), for the continuum functions.

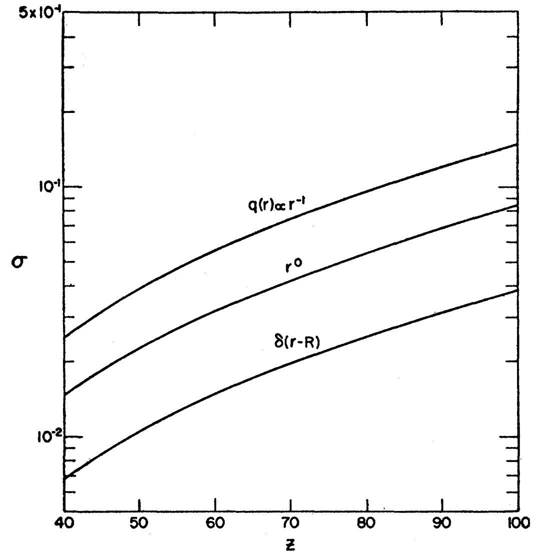


FIG. 5. The coefficient $\sigma$ as a function of atomic number for various nuclear charge distributions, $q(r)$—shell, uniform, and reciprocal-radius. Results are given for the case of the $K$-shell conversion of a transition of energy one $mc^2$, although they are insensitive to the choice of the converting shell and the nuclear transition energy.

is the Fermi function for negatron decay.[33] This result is in essential agreement with previous relativistic[34–36] and nonrelativistic[37,38] results. Analogous expressions for the conversion ratios are

$$K/L_{\rm I} = 2\frac{P_K(W_K+\gamma)F(Z,P_K)}{P_L(W_L+\gamma)F(Z,P_L)}\frac{X+1}{X+2}\frac{X^{2\gamma+2}}{2\gamma+1},$$

$$L_{\rm I}/L_{\rm II} = \frac{2+X}{2-X}\frac{X-1}{X+1}\frac{W_L+\gamma}{W_L-\gamma}, \quad (14)$$

$$L_{\rm I}/L_{\rm III} \sim \frac{54}{(\alpha Z)^2[W_L^2-\gamma^2]R^4}[1+O(\alpha Z)^2+\cdots],$$

[33] See, for example, *Tables for the Analysis of Beta Spectra*, National Bureau of Standards, Applied Mathematics Series 13 (U. S. Government Printing Office, Washington, D. C., 1952). The tables on pages 21–61 of this reference were used in the evaluation of the results in Figs. 1 and 3.

[34] H. Yukawa and S. Sakata, Proc. Phys.-Math. Soc. Japan **17**, 397 (1935).

[35] R. Thomas, Phys. Rev. **58**, 714 (1940). The analytic result quoted for the $K$-conversion probability has apparently been derived for a particular nuclear charge distribution, and differs slightly from (12). However, these expressions are numerically equivalent in all cases of practical interest. It is also pointed out in this reference that $E0$ pair production is negligible relative to the internal conversion of the 1.42-Mev $0\to 0$ transition in $Po^{214}$.

[36] S. D. Drell, reference 11. An integral sign in his Eq. (9) was omitted in transcription.

[37] J. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 620. Their expression is four times the nonrelativistic form of (12), as a consequence of their neglect of the term $1/r_e$ appearing in (6).

[38] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Cambridge, 1953), p. 268. The nonrelativistic expression quoted has not been doubled to take into account the presence of two $K$ electrons.

where $X\equiv[2(1+\gamma)]^{\frac{1}{2}}$, and $W_{K,L}=[P_{K,L}^2+1]^{\frac{1}{2}}$ are the total energies of the electrons ejected from the $K$ and $L$ shells, respectively. In the absence of screening, $W_K=k+\gamma$ and $W_L=k+\frac{1}{2}X$, where $k$ is the nuclear transition energy.

(D) The results in Figs. 1 to 4 include a number of corrections to the approximate expressions (12)–(14) arising from the use of more realistic expressions for the electron wave functions. These corrections to the "point-nucleus" electron functions involve the inclusion of the effects of (1) the finite nuclear size, (2) atomic screening, and (3) terms of the order $\alpha ZR$ and $PR$ in the Dirac wave functions.[32] The effect of the finite nuclear size on the electron wave functions increases the approximate result (12) by a small amount which increases with increasing atomic number. Since $W\ll|V|$ in general, the effects on the electron functions themselves are essentially the same for bound and free electrons and very nearly independent of the nuclear transition energy. The effects of screening on the bound wave functions, on the other hand, lowers the pure Coulomb results (12)–(14) slightly for the $K$ shell and by a larger amount for the $L$ shell. These effects decrease with increasing $Z$. The net result of all three corrections is to *increase* the "point-nucleus" values of the $K$ and $L$ shell reduced transition probabilities by $\sim$25 and $\sim$15%, respectively, for $Z=85$, and by lesser amounts for lower atomic numbers. At low $Z$, $L$ conversion is appreciably attenuated by bound-state screening.

Analysis of the effects of the finite nuclear size leads to two related corrections to the "point-nucleus" values of the constant $C$ appearing in the expansion of the electron wave functions (7). The values of the electron functions at the nuclear surface are decreased from the "point-Coulomb" values, and the electron wave functions within the nucleus are not "flat," but reach a maximum at the origin. As indicated above, this variation of the electron functions within the nucleus gives rise to higher-order nuclear matrix elements in the strength parameter (9). The "large" components of the wave functions are only slightly affected by the finite nuclear size, while the "small" components are more seriously attenuated. Since the $E0$ transition probability depends only on the large components of the electron functions to lowest order, the dependence on the details of the nuclear charge distribution is slight, and the "point-nucleus" approximation for the monopole conversion is a good one. In evaluating the dependence of the electron wave functions on the nuclear charge distribution, three very different distributions have been considered—a shell distribution, a uniform distribution, and a reciprocal-radius distribution. It is found that in these cases the reduced transition probability for $E0$ conversion is increased over the "point-nucleus" values by approximately 10, 30, and 50%, respectively, for $Z=85$. The conversion ratios are unaffected to first order. Since the effects of the finite nuclear charge distribution on the magnitudes of the electron wave functions does not differ greatly for the extreme distributions considered, the results given in Figs. 1 to 4 have been computed for the reasonable case of a uniform nuclear charge distribution.[39]

The effects of atomic screening may be considered as appearing in two related ways. The wave functions at the nuclear surface are lowered by the change in normalization of the entire wave function, and the binding energies are shifted from their "point-Coulomb" values. The attenuation of the bound functions at the nuclear radius was taken from the results of Brysk and Rose,[40] which are based on calculations for a Fermi-Thomas-Dirac atom. For $Z=85$ the attenuations of the $K$, $L_{\rm I}$, and $L_{\rm II}$ functions are 1, 7, and 9%, respectively. The shift in binding energies has been taken into account by the use of empirical binding energies.[41] The screening corrections to the continuum functions may be important in the immediate vicinity of threshold.[42] However, for transition energies greater than 100 kev above threshold, the wave functions are apparently altered by less than 3% for $Z=85$, and the effect decreases rapidly with decreasing atomic number. Since the effect is small, and since there is no comprehensive source of continuum screening corrections readily available, this correction has been omitted in the present calculations.

[39] The decrease of the bound-electron wave functions at the nuclear surface due to the finite nuclear size was determined by the method of Brysk and Rose.[40] In doing so, however, a more accurate value of the isotope shift was used, computed by an extension of their methods rather than the perturbation-theory value.

[40] N. Brysk and M. E. Rose, Oak Ridge National Laboratory Report ORNL-1830, 1955 (unpublished).

[41] Hill, Church, and Mihelich, Rev. Sci. Instr. **23**, 523 (1952).

[42] J. R. Reitz, Phys. Rev. **77**, 10 (1950); (private communication, 1955).