Surface Oscillations in Even-Even Nuclei*

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Surface oscillations in nuclei with deformation potentials independent of the shape parameter γ are discussed, and are found to describe qualitatively the regularities in even-even nuclei of the type discussed by Scharff-Goldhaber and Weneser.

I. INTRODUCTION

VARIOUS investigators[1] have pointed out the following regularities in even-even nuclei:

(1) The ground-state character is invariably 0+. The first excited state has character 2+ with very few exceptions. The second excited state is 0+, 2+, 4+ or odd spin, either parity.

(2) The energy of the first excited state is correlated with proton and (especially) neutron number, passing through distinct maxima at closed shells.

There exist particular classes of even-even nuclei which exhibit further characteristic regularities. Perhaps the most striking of these nuclei are those which display rotational spectra.[2,3] These are found with great regularity in the region $150<A<185$ and $A>225$. They are characterized by:

(3) The low-lying energy levels follow the energy spectrum

$$E=\text{const}I(I+1),$$

with the character

(4) 0+, 2+, 4+, $\cdots$.

(5) For many purposes these nuclei are found to be very accurately described in terms of a body possessing an axially symmetric deformation of order two. The wave functions for the rotational states are then given by

$$\psi=Y_{IM}(\theta,\varphi),$$

where θ and φ describe the orientation of the symmetry axis. The intrinsic quadrupole moment Q_0 of the system is large compared with single-particle moments, giving rise to enhanced $E2$ radiation.

(6) Q_0 is correlated with the energy of the first excited state, E_1, such that large Q_0 correspond to small E_1. The rotational energies are considerably larger than would be given by a model of solid-body nuclear rotation.

Scharff-Goldhaber and Weneser[4] have reported another large class of even-even nuclei found over a large range of nuclear mass numbers—but especially in the region $66<A<150$—exhibiting the following regularities:

(3′) The ratio of the energies of the second excited state (E_2) to that of the first (E_1) ranges between 2 and 2.5.

(4′) The character of the low-lying states are commonly in the sequence 0+, 2+, 2+. In the range $66<A<150$, there are five nuclei known to have spin 2 for the second excited state, four with spin 4, and one each with spin zero and three—all of positive parity.

(5′) The first and second excited states decay predominantly by $E2$ radiation. The $E2$ transition matrix elements are enhanced over the single-particle matrix elements, although perhaps not so much as for the rotational states.

(6′) When the sequence 0+, 2+, 2+ appears, the $E2$ crossover transition (second excited to ground state) occurs with much smaller probability than the upper transition (second excited to first excited state). The crossover transition ($2+\rightarrow 0+$) proceeds by $E2$ radiation while the upper transition ($2+\rightarrow 2+$) proceeds by $E2$ with a small admixture of $M1$. The ratios of the reduced $E2$ matrix elements (crossover: upper transition) are a few percent or less with no exceptions reported (Goldhaber-Kraushaar selection rule).

The enhanced quadrupole transitions suggest some form of collective motion. For this reason, in Fig. 1 we have plotted the ratio E_2/E_1 as a function of $E_1/\hbar\omega$, where ω is the characteristic phonon frequency of a statically undeformed nucleus. The values of $\omega(A)$, which are taken from Bohr and Mottelson[3] (Eqs. II. 6a and II. 6b, with $\lambda=2$; see also their Fig. 2), are based on estimates of surface and Coulomb energies and employ the hydrodynamic assumption of irrotational flow. While the numerical estimates of ω are not very significant (e.g., the function $\omega(A)$ is expected to have

* A preliminary report has been given by M. Jean and L. Wilets, Compt. rend. **241**, 1108 (1955).

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[1] M. Goldhaber and A. W. Sunyar, Phys. Rev. **83**, 906 (1951); G. Scharff-Goldhaber, Phys. Rev. **90**, 587 (1953).

[2] A. Bohr, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **26**, No. 14 (1952).

[3] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[4] G. Scharff-Goldhaber and J. Weneser, Phys. Rev. **98**, 212 (1955); See also M. Nagasaki and T. Tamura, Progr. Theoret. Phys. **12**, 248 (1954).

shell structure fluctuations superimposed), using $E_1/\hbar\omega$ rather than E_1 for the plot does have the advantage of removing much of the A dependence in E_1 in such a way that there is more pronounced grouping of the points. The points include nuclei with $A \geq 46$; odd-parity states (if known to be so) have been excluded from consideration—this involves only a few examples—and in some cases higher excitations are included (e.g., doublets).

For small values of $E_1/\hbar\omega$ we find a clustering of rotational spectra with $E_2/E_1 \approx 3\frac{1}{3}$. The break between the Bohr-Mottelson rotational spectra and the spectra of Scharff-Goldhaber and Weneser is marked, with the majority of the latter occurring for $0.2 < E_1/\hbar\omega < 0.5$.

Although the scatter of points is considerable, it is worthwhile noting that isotopic sequences such as Xe and Te show that (i) $E_1/\hbar\omega$ decreases in moving away from a closed shell, and (ii) E_2/E_1 is a generally decreasing function of $E_1/\hbar\omega$. Ba^{138} and Ce^{140}, each with 82 (magic) neutrons, exhibit very similar spectra and differ significantly from the general trends.

Scharff-Goldhaber and Weneser have described the nuclei satisfying (3′) to (6′) in terms of the coupling of individual nucleons (as an example they used four $f_{7/2}$ particles) to a core with free phonon (surface) vibrations. Whereas for the free core $E_1 = \hbar\omega$ and $E_2/E_1 = 2$, they showed that E_1 decreases and the ratio increases as a function of the coupling. Their results are in qualitative agreement with the experimental data. The range of validity of such a perturbation calculation may be expected, however, to be rather small.

Another approach is given in terms of collective surface oscillations where the individual nucleons are treated in first approximation as only contributing to an effective potential energy through their coupling to the surface (strong-coupling approximation). Plausibility arguments for the type of potential required to reproduce the data (γ unstable) are given in Sec. VI.A. Such a description more clearly displays the collective features of the nuclei. It cannot be hoped that the collective description can be complete, since the individual particles may be expected to play an important role besides contributing to the surface potential. While the calculations of Scharff-Goldhaber and Weneser represent an approach from "weak coupling," the present investigation represents the limit of "strong coupling" or "adiabatic" approximation.

It is hoped that the investigation may also help to round out the picture of various types of surfon oscillations.

II. THE COLLECTIVE MODEL

We begin by reviewing the analysis of surface oscillations given by Bohr.[2] The nuclear surface is described, for our purposes, by a second-order deformation such that

$$R = R_0\{1 + \sum_{\mu=-2}^{2} \alpha_\mu Y_{2\mu}(\theta,\phi)\}, \tag{1}$$

where, for reality,

$$\alpha_\mu = (-)^\mu \alpha_{-\mu}^*. \tag{2}$$

It is convenient to transform to a coordinate system which is "fixed" in the oscillating body. The coordinates in the body-fixed system are then related to the space-fixed system by the transformation

$$a_\nu = \sum_\mu \alpha_\mu D_{\mu\nu}(\theta_i), \tag{3}$$

where the $D_{\mu\nu}(\theta_i)$ are the transformation functions for the spherical harmonics of order 2, and θ_i represents the triad of Eulerian angles θ, ϕ, ψ describing the relative orientation of the axes. The body-fixed coordinate system may be conveniently chosen so that its axes coincide with the principal axes of the ellipsoid, in which case

$$a_1 = a_{-1} = 0, \quad a_2 = a_{-2}. \tag{4}$$

Thus the five variables α_μ are replaced by the three Eulerian angles θ_i and the two real "internal" coordinates a_0 and a_2.

Again for convenience we may replace a_0 and a_2 by β and γ, defined by

$$\begin{aligned} a_0 &= \beta \cos\gamma, \\ a_2 &= a_{-2} = (\beta/\sqrt{2}) \sin\gamma. \end{aligned} \tag{5}$$

The expression for the kinetic energy is given by [Bohr,[2] Eqs. (48), (50), and (27)]

$$T = -\frac{\hbar^2}{2B}\left\{\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\beta^4\frac{\partial}{\partial\beta} + \frac{1}{\beta^2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma} - \frac{1}{4}\sum_\kappa \frac{Q_\kappa^2}{\sin^2(\gamma - \frac{2}{3}\pi\kappa)}\right\}, \tag{6}$$

where the Q_κ are angular momentum operators in the variables θ_i.

In the hydrodynamic assumption of irrotational flow, B is a constant given by [Bohr, Eq. (4)]

$$B = \tfrac{1}{2}\rho_0 R_0^5 \text{ (irrotational flow).} \tag{7}$$

Generally, a detailed knowledge of nucleonic wave functions is needed to determine B. Inglis[5] has shown that for any deformation parameter α which enters into the classical expression for the kinetic energy as $\frac{1}{2}B_\alpha|\alpha|^2$, the mass parameter B_α is given by

$$B_\alpha = 2\hbar^2 \sum_i{}' \frac{|\langle 0|\partial/\partial\alpha|i\rangle|^2}{E_i - E_0}, \tag{8}$$

the prime indicating that the summation is taken over excited (particle) states of the nucleus. Assuming com-

[5] D. Inglis, Phys. Rev. 97, 701 (1955).

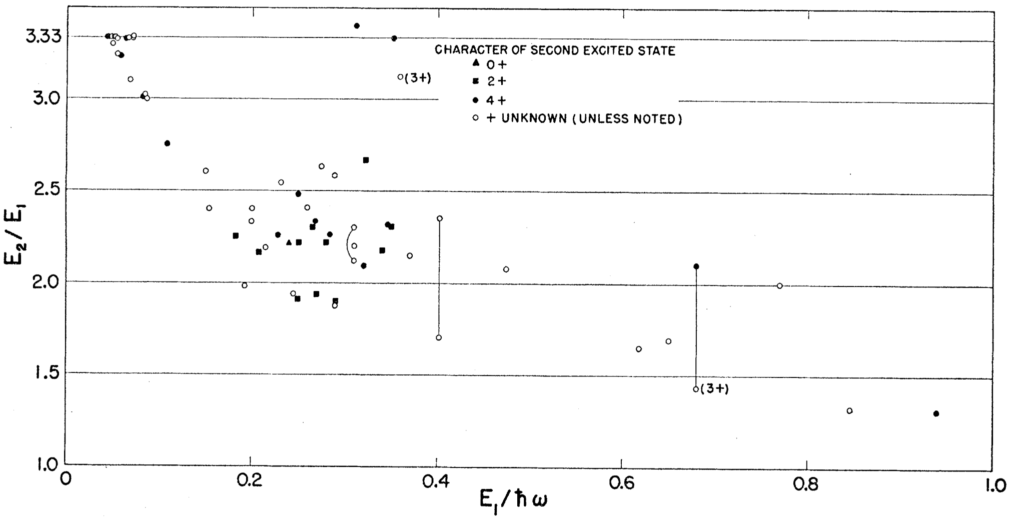

FIG. 1. Experimental values of the ratio $E_2:E_1$ as a function of $E_1/\hbar\omega$. The value of ω is the hydrodynamic estimate ($\lambda=2$) given by Bohr and Mottelson[3] (Eqs. II.6a and II.6b; see also their Fig. 2). Negative-parity states have been omitted from consideration—this involves only a few examples. In some cases higher excitations are included, whence the various points are connected by a line.

The sources of data are given below. In order are given: the nuclide; $E_1/\hbar\omega$; E_2/E_1; spin and parity of the second excited state; reference. If an asterisk occurs before the nuclide, the level scheme assumed is different from that given in the reference.

*Ti^{46}	0.23	2.26	4+	G. L. Keister and F. H. Schmidt, Phys. Rev. **93**, 140 (1954).
Ti^{48}	0.27	2.33	4+	Jastram, Hurley, and Whittle (private communication).
*Cr^{52}	0.15	2.60		T. Wiedling, Phys. Rev. **91**, 767 (1953).
Fe^{56}	0.25	2.48	4+	Hurley, Rudman, and Jastram, Phys. Rev. **98**, 1187(A) (1955).
Ni^{60}	0.40	1.6/2.4		R. H. Nussbaum *et al.*, Phys. Rev. **93**, 255 (1954).
Zn^{66}	0.35	2.31	2+	L. G. Mann *et al.*, Phys. Rev. **92**, 1481 (1953).
Ge^{72}	0.29	1.75		Brun, Meyerhof, and Kraushaar, Phys. Rev. **100**, 1795(A) (1955).
Se^{76}	0.21	2.16	2+	Kurbatov, Murray, and Sakai, Phys. Rev. **98**, 674 (1955).
Kr^{82}	0.29	1.90		R. C. Waddell, private communication.
Kr^{84}	0.36	2.15		J. Welker and M. Perlman, Phys. Rev. **100**, 74 (1955).
Zr^{92}	0.19	1.98		W. A. Cassatt, Jr., and W. W. Meinke, Phys. Rev. **99**, 760 (1955).
Mo^{94}	0.36	3.12	3+	H. Medicus *et al.*, Helv. Phys. Acta **22**, 603 (1949).
Mo^{96}	0.32	2.09	4+	M. Goldhaber and R. D. Hill, Revs. Modern Phys. **24**, 179 (1952).
Ru^{100}	0.23	2.54		L. Marquez, Phys. Rev. **92**, 151 (1953).
Pd^{104}	0.26	2.41		Bunker, Mize, and Starner, Phys. Rev. **99**, 659(A) (1955).
Pd^{106}	0.24	2.22	0+	Lu, Kelly, and Wiedenbeck, Phys. Rev. **97**, 139 (1955).
Pd^{108}	0.20	2.40		M. L. Perlman *et al.*, Phys. Rev. **92**, 1236 (1953).
*Cd^{110}	0.31	2.1/2.3		F. I. Boley, Phys. Rev. **94**, 1078 (1954).
Cd^{114}	0.26	2.30	2+	M. W. Johns *et al.*, Can. J. Phys. **32**, 35 (1954).
Sn^{116}	0.63	1.65		M. Goldhaber and R. D. Hill, Revs. Modern Phys. **24**, 179 (1952).
*Sn^{120}	0.65	1.69		G. L. McGinnis, Phys. Rev. **98**, 1172(A) (1955).
Te^{122}	0.29	2.24	2+	B. Farrelly *et al.*, Phys. Rev. **99**, 1440 (1955).
Te^{124}	0.31	2.20		Lu, Kelly, and Wiedenbeck, Phys. Rev **95**, 1533 (1954)
Te^{126}	0.34	2.18	2+	M. Perlman and J. Welker, Phys. Rev. **95**, 133 (1954).
Xe^{126}	0.20	2.25	2+	M. Perlman and J. Welker, Phys. Rev. **95**, 133 (1954).
Xe^{128}	0.23	2.22	2+	J. M. Hollander and M. I. Kalkstein, Phys. Rev. **98**, 260 (1955).
Xe^{130}	0.28	2.26	4+	R. S. Caird *et al.*, Phys. Rev. **94**, 412 (1954).
Xe^{132}	0.37	2.15		H. L. Finston and W. Bernstein, Phys. Rev. **96**, 71 (1954).
Xe^{134}	0.47	2.07		M. McKeown and S. Katcoff, Phys. Rev. **94**, 965 (1954).
Xe^{136}	0.78	2.0		M. McKeown and S. Katcoff, Phys. Rev. **94**, 965 (1954).
Ba^{134}	0.35	2.32	4+	Keister, Lee, and Schmidt, Phys. Rev. **97**, 451 (1955).
Ba^{138}	0.84	1.33		Duffield, Bunker, Mize, and Starner, Phys. Rev. **100**, 1236(A) (1955).
Ce^{140}	0.94	1.31	4+	G. R. Bishop and J. P. Perez y Jorba, Phys. Rev. **98**, 89 (1955).
Nd^{146}	0.28	2.63		W. Bernstein *et al.*, Phys. Rev. **93**, 1073 (1954).
Sm^{150}	0.20	2.33		C. T. Hibdon and C. O. Muehlhause, Phys. Rev. **88**, 943 (1952).
Sm^{152}	0.08	3.0		R. E. Slattery *et al.*, Phys. Rev. **96**, 465 (1954).
Gd^{152}	0.21	2.19(?)		R. E. Slattery *et al.*, Phys. Rev. **96**, 465 (1954).
Gd^{154}	0.08	3.02		E. L. Church and M. Goldhaber, Phys. Rev. **95**, 626(A) (1954).
Gd^{156}	0.06	3.24		E. L. Church and M. Goldhaber, Phys. Rev. **95**, 626(A) (1954).
Gd^{158}	0.05	3.30		E. L. Church and M. Goldhaber, Phys. Rev. **95**, 626(A) (1954).
Dy^{160}	0.05	3.27		Keshishian, Kruse, Klotz, and Fowler, Phys. Rev. **96**, 1050 (1954).
Er^{166}	0.05	3.33		Milton, Fraser, and Milton, Phys. Rev. **98**, 1173(A) (1955).
Hf^{176}	0.06	3.28	4+	J. A. Arnold, Phys. Rev. **93**, 743 (1954).
Hf^{180}	0.07	3.31	4+	Mihelich, Goldhaber, and McKeown, Phys. Rev. **94**, 794(A) (1954).
*Os^{192}	0.16	2.40		Pringle, Turchinetz, and Taylor, Phys. Rev. **95**, 115 (1954).

(*continued on page 791*)

pletely independent particles, Eq. (8) leads to a solid-body moment of inertia for rotational states, but Bohr and Mottelson[6] have shown that the inclusion of residual interparticle (*pairing*) forces leads to a lower moment of inertia.

If the potential energy is only a function of the internal coordinates β and γ, the angular momentum of the nucleus will be a constant of the motion, i.e., I and M will be good quantum numbers. If, furthermore, the potential depends only on β, the Hamiltonian is separable. For if in

$$[T+V(\beta)]\Psi(\beta,\gamma,\theta_i)=E\Psi(\beta,\gamma,\theta_i), \tag{9}$$

we set

$$\Psi(\beta,\gamma,\theta_i)=f(\beta)\Phi(\gamma,\theta_i), \tag{10}$$

we have the equations

$$E[\beta^2 f(\beta)]=\left\{\frac{h^2}{2B}\left(-\frac{\partial^2}{\partial\beta^2}+\frac{(\lambda+1)(\lambda+2)}{\beta^2}\right)+V(\beta)\right\}\times[\beta^2 f(\beta)] \tag{11}$$

and

$$\Lambda\Phi(\gamma,\theta_i)=\left\{-\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\sin 3\gamma\frac{\partial}{\partial\gamma}+\tfrac{1}{4}\sum_{\kappa}\frac{Q_\kappa^2}{\sin^2(\gamma-\frac{2}{3}\pi\kappa)}\right\}\Phi(\gamma,\theta_i)\equiv L\Phi(\gamma,\theta_i). \tag{12}$$

The separation parameter,

$$\Lambda=\lambda(\lambda+3),\quad \lambda=0,1,2,3,\cdots, \tag{13}$$

is degenerate with respect to the angular momentum for the first few states as follows:

$$\left.\begin{array}{ccc} \lambda & \Lambda & I \\ 0 & 0 & 0 \\ 1 & 4 & 2 \\ 2 & 10 & 2,4 \\ 3 & 18 & 0,3,4,6 \end{array}\right\}. \tag{14}$$

[6] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **30**, No. 1 (1955).

There are five quantum numbers in all: I, M, λ, n_β, and n_γ. Φ can be written

$$\Phi_{I,M}{}^{\lambda,n_\gamma}=\sum_{K=-I}^{I} g_K(\lambda,n_\gamma;\gamma)\mathfrak{D}_{KM}{}^I(\theta_i), \tag{15}$$

where $g_K=g_{-K}$.

The usual phonon spectrum is obtained when

$$V(\beta)=\frac{C}{2}\beta^2=\frac{C}{2}\sum_{\mu=-2}^{2}|\alpha_\mu|^2. \tag{16}$$

In the α_μ representation, it is immediately obvious that the energy levels are those of a five-dimensional harmonic oscillator,

$$E=(N+5/2)\hbar\omega$$
$$N=\sum_{\mu=-2}^{2} n_\mu=2n_\beta+\lambda,\quad n_\beta=0,1,2,\cdots \tag{17}$$

where $\omega=(C/B)^{\frac{1}{2}}$. The spin assignment of the first few states is then given by

$$\left.\begin{array}{cc} N & I \\ 0 & 0 \\ 1 & 2 \\ 2 & 0,2,4 \\ 3 & 0,2,3,4,6 \end{array}\right\}. \tag{18}$$

The rotational spectrum is obtained when the potential stabilizes γ (as well as β) about some equilibrium value: $\gamma=0, \pm\frac{2}{3}\pi$, for prolate nuclei and $\gamma=\pm\frac{1}{3}\pi, \pi$ for oblate nuclei; that is, when the potential assumes the form

$$V(\beta,\gamma)=\tfrac{1}{2}k_1(\beta-\beta_0)^2+\tfrac{1}{2}k_2(\gamma-\gamma_0)^2, \tag{19}$$

where the zero-point vibrations are small compared with the equilibrium values of both β and γ. The wave function for the lowest state then reduces to just one term,

$$\Phi\approx g_0(\gamma)\mathfrak{D}_{0M}{}^I\propto g_0(\gamma)Y_{IM}\quad(\gamma\text{ stable}). \tag{20}$$

III. γ-UNSTABLE SPECTRUM

The oscillator potential $V=\frac{1}{2}C\beta^2$ arises from general considerations of surface tension and Coulomb energy

Pt^{192}	0.24	1.94		M. W. Johns and S. V. Nablo, Phys. Rev. **96**, 1599 (1955).
Pt^{194}	0.25	1.91	2	Mandeville, Varma, and Saraf, Phys. Rev. **98**, 1185 (1955).
Pt^{196}	0.27	1.94 or 2.07		R. M. Steffen, Phys. Rev. **90**, 328 (1953).
Hg^{198}	0.32	2.65	2+	L. I. Schiff and F. R. Metzger, Phys. Rev. **90**, 849 (1953).
Hg^{200}	0.29	2.58		Bergstrom, Hill, and Pasquale, Phys. Rev. **92**, 918 (1953).
Pb^{202}	0.35	3.30	4+	Maeder *et al.*, Phys. Rev. **93**, 1433 (1954).
$^*Pb^{204}$	0.31	3.40	4+	V. E. Krohm and S. Raboy, Phys. Rev. **97**, 1017 (1955).
Pb^{206}	0.68	1.4/2.1	3+/4+	D. E. Alburger and M. A. L. Pryce, Phys. Rev. **95**, 1482 (1954).
Ra^{222}	0.11	2.75	4+	Stephens, Asaro, and Perlman, Phys. Rev. **96**, 1568 (1954).
Ra^{224}	0.08	3.01	4+	Stephens, Asaro, and Perlman, Phys. Rev. **96**, 1568 (1954).
Ra^{226}	0.07	3.10		F. Rasetti and E. C. Booth, Phys. Rev. **91**, 315 (1953).
Th^{226}	0.07	3.29	4+	Stephens, Asaro, and Perlman, Phys. Rev. **96**, 1568 (1954).
Th^{228}	0.06	3.23	4+	F. Asaro and I. Perlman, Phys. Rev. **99**, 37 (1955).
Th^{230}		3.33		F. Asaro and I. Perlman, Phys. Rev. **91**, 763 (1953).
U^{234}	0.04	3.33	4+	Newton and Rose (private communication to A. Bohr).
U^{236}		3.33		F. Asaro and I. Perlman, Phys. Rev. **91**, 763 (1953).
Pu^{238}	0.04	3.34	4+	Asaro, Thompson, and Perlman, Phys. Rev. **92**, 694 (1953).

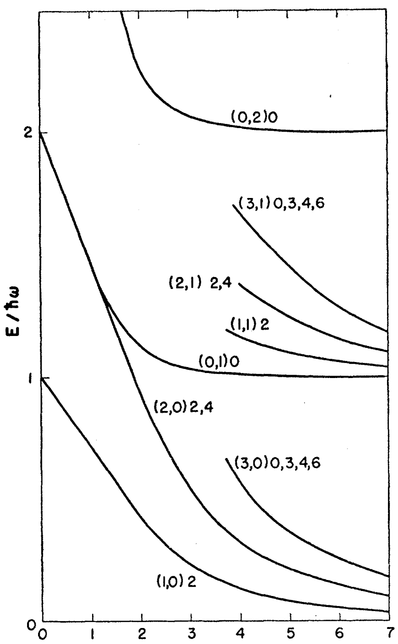

FIG. 2. Surface energies as a function of the potential minimum x_0. The states are designated by $(\lambda,n_\beta)I$. At $x_0=0$, the spectrum is given by $E/\hbar\omega=2n_\beta+\lambda$ and for $x_0\gg1$ by $E/\hbar\omega=n_\beta+\lambda(\lambda+3)/(2x_0^2)$.

(and small oscillations). The success of the individual-particle model indicates that effects of the nucleons, through their interaction with the surface, must also be considered. In the case of even-even nuclei, the nucleons may be regarded, in first approximation, as giving rise only to an effective surface potential (strong-coupling approximation) although particle structure may also play a role in determining the mass parameter B, as discussed above.

The form of the potential is a matter we will have more to say about later, but let us first consider a potential of the form $V(\beta)$; that is, there is no γ dependence in the potential, but it is otherwise arbitrary. Such an idealized system is referred to as γ unstable. The Hamiltonian is separable, as described by (9)–(15).

A. Transition Probabilities

The transition probability for $E2$ radiation is given by

$$T(2)=\frac{4\pi}{75\hbar}\left(\frac{\omega}{C}\right)^5 B(2), \tag{21}$$

$$B(2)=\sum_{\mu M_f}|\langle i|\mathfrak{M}(2,\mu)|f\rangle|^2, \tag{22}$$

$$\mathfrak{M}(2,\mu)=\frac{3}{4\pi}ZeR_0^2\alpha_\mu^* \tag{23}$$

$$\alpha_\mu^*=\beta\{D_{\mu0}^*\cos\gamma + (1/\sqrt{2})(D_{\mu2}^*+D_{\mu-2}^*)\sin\gamma\}. \tag{24}$$

The quantity in curly brackets is (up to a normalization constant) just the eigenfunction $\Phi(\gamma,\theta_i)$ for the state $\lambda=1, I=2$. By orthogonality of the Φ's, the only state connected to the ground state by $E2$ matrix elements is the $\lambda=1, I=2$ state itself. In particular, we have the selection rule

(6′) *The crossover transition is forbidden*

as a result of γ instability.

For the transition $\lambda=1, I=2\rightarrow\lambda=0, I=0$, we obtain

$$B(2)=\frac{1}{5}\left|\frac{3}{4\pi}ZeR_0^2\langle i\|\beta\|_0\rangle\right|^2, \tag{25}$$

which is identical with the expression for rotational states (Bohr and Mottelson, Eqs. V.7 and VIII.17).

$M1$ transitions are forbidden for purely collective transitions of second order in even-even nuclei because of the high degree of symmetry present. Although magnetic dipole moments may be generated by the collective motion, such are classically stationary and cannot radiate. (In the case of odd-A rotational nuclei, the dipole moment resulting from the collective motion precesses about the total angular momentum $\mathbf{I}=\mathbf{R}+\mathbf{\Omega}$ and so can give rise to magnetic dipole radiation.)

B. Examples of γ-Unstable Potentials

(1) *Anharmonic Oscillator*

The addition of anharmonic terms to the β potential is clearly capable of altering the ratio E_2/E_1 in either sense from two. Since experiment indicates a larger value, the anharmonic terms should lead to an increase in the force constant with increasing β. A limiting example of such a potential is the infinite square well:

$$V(\beta)=\begin{cases}\text{const}, & \beta<b\\ \infty\ , & \beta>b.\end{cases} \tag{26}$$

The eigenenergies of Eq. (11) are then given by the roots $x_{n,l}$ of the Bessel functions $J_l(x)$:

$$E_{n_\beta,\lambda}=\frac{\hbar^2}{2Bb^2}(x_{n_\beta,\lambda+\frac{3}{2}})^2. \tag{27}$$

Thus we obtain

$$E_2/E_1=2.20 \tag{28}$$

for the largest ratio obtainable from anharmonic terms.

Although this ratio is of the right order to describe the effects, it represents the extreme case, not realizable in nature. A larger value of the ratio can be obtained from a potential with a minimum for some finite value of β (see Sec. VI.A).

(2) Displaced Harmonic Oscillator

Consider a potential given by

$$V(\beta)=\tfrac{1}{2}C(\beta-\beta_0)^2. \qquad (29)$$

Equation (10) may then be more conveniently written

$$\epsilon\varphi=\frac{1}{2}\left\{-\frac{\partial^2}{\partial x^2}+\frac{(\lambda+1)(\lambda+2)}{x^2}+(x-x_0)^2\right\}\varphi, \qquad (30)$$

where

$$x=(B\omega/\hbar\beta)^{\frac{1}{2}},\quad \omega=(B/C)^{\frac{1}{2}},\quad \epsilon=E/\hbar\omega,\quad \varphi(x)=x^2 f(\beta).$$

An approximate solution of (30) is readily obtained by expanding the "effective" potential $v(x)$ about the minimum x':

$$v(x)\equiv\frac{1}{2}\left\{\frac{(\lambda+1)(\lambda+2)}{x^2}+(x-x_0)^2\right\}=v(x')+\frac{\omega'^2}{2}(x-x')^2 +O((x-x')^3). \qquad (31)$$

If we neglect terms of the order $(x-x')^3$, the energy spectrum is given by

$$\epsilon=(n_\beta+\tfrac{1}{2})\omega'+v(x'),\quad (n_\beta=0,1,2,\cdots) \qquad (32)$$

and the eigenfunctions by

$$\varphi(x)=h_{n_\beta}((x-x')\sqrt{\omega'})\exp[-\tfrac{1}{2}(x-x')^2\omega']. \qquad (33)$$

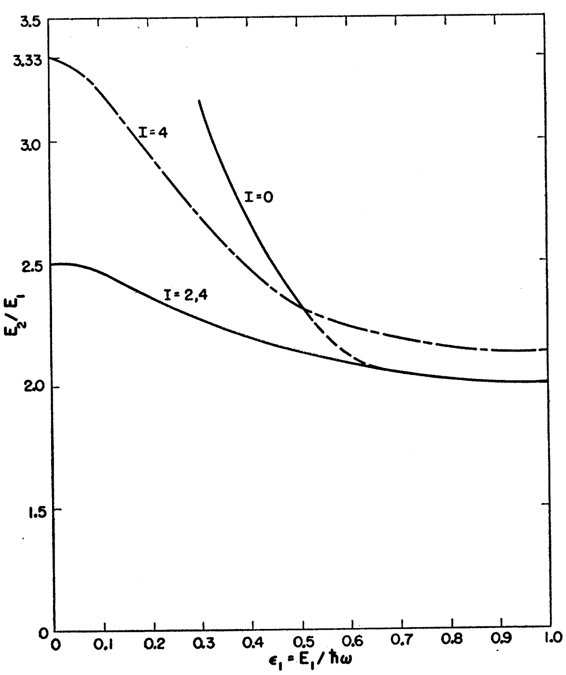

FIG. 3. Solid curves (γ unstable): energies relative to E_1 of the state $\lambda=0$, $n_\beta=1$, $I=0$ and the degenerate pair of states $\lambda=2$, $n_\beta=0$, $I=2$ and 4, as a function of $\epsilon_1=E_1/\hbar\omega$. Broken curve ($\gamma$ rigid): the ratio E_2/E_1 as a function of $\epsilon_1=E_1/h\omega$. The second excited state is of spin 2, while the states of spin 0 and 2, illustrated for the γ-unstable spectrum, lie infinitely high in the limit of γ rigidity. The ω for the two types of spectra are not defined in precisely the same manner.

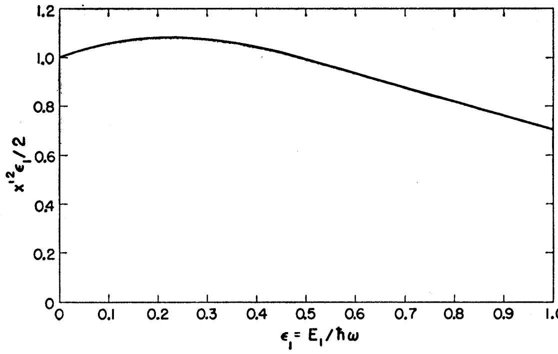

FIG. 4. A plot of $\frac{1}{2}x'\epsilon_1$ as a function of ϵ_1. The value does not deviate very much from unity.

The approximation is worst for the case $x_0=0$, where we obtain

$$\epsilon=2n_\beta+[(\lambda+1)(\lambda+2)]^{\frac{1}{2}}+1,\quad n_\beta,\lambda=0,1,2,\cdots \qquad (34a)$$

compared with the exact solution

$$\epsilon=2n_\beta+\lambda+5/2,\quad n_\beta,\lambda=0,1,2,\cdots. \qquad (34b)$$

The error in the energies is always less than 4%—which is sufficient for our purposes—and decreases rapidly with increasing x_0.

Figure 2 shows the dependence of the energy levels on x_0, while in Fig. 3 (solid curves) is plotted E/E_1 for the state $\lambda=0$, $n_\beta=1$, $I=0$ and for the degenerate pair $\lambda=2$, $n_\beta=0$, $I=2,4$ as a function $\epsilon_1=E_1/\hbar\omega$. The ratio for the degenerate pair $I=2,4$ varies from 2 for $\epsilon_1=1$ to 2.5 for $\epsilon_1=0$ in qualitative agreement with the data. The state with $I=0$ is generally higher. For a large x' (or x_0), the energy of the first excited state $\epsilon_1=E_1/\hbar\omega$ approaches the value $2/x'^2$. Figure 4 is a plot of $\frac{1}{2}x'^2\epsilon_1$ as a function of ϵ_1. These energy levels may be compared with the calculations of Scharff-Goldhaber and Weneser, who obtain the spectrum 0+, 2+, and then a close triplet 4+, 2+, 0+. While their results are similar to ours for small equilibrium deformations ($E_1/\hbar\omega$ near unity), we find the excited 0+ lies considerably higher than the 4+, 2+ pair for $E_1/\hbar\omega<0.5$.

For computation of radiation transitions (25) between states for which β' does not change appreciably, we may take

$$\langle i\|\beta\|f\rangle\approx\beta_i'\approx\beta_f'. \qquad (35)$$

IV DEVIATIONS FROM γ INSTABILITY

In order to discuss deviations from γ instability, we will introduce a γ-dependent potential. For the Hamiltonian to be separable, this would imply a potential of the form $V\sim v(\gamma)/\beta^2$. Physically, we expect this to be unrealistic, since γ stability should increase rather than decrease with β, but if β is relatively stable, we can treat β^2 as a constant and "approximately" separate the Hamiltonian. Thus the equation for determining Λ becomes

$$\{L+v(\gamma)\}\Phi=\Lambda\Phi. \qquad (36)$$

The points γ and $\gamma\pm\frac{2}{3}\pi$ correspond to the same shape of nucleus, but oriented differently. Thus if we wish a potential which depends only on the nuclear shape, we must choose a periodic function of 3γ.

A potential which tends to stabilize γ about 0 or π (axial symmetry) removes the degeneracy in the $\lambda=2$ states so that the $I=2$ state appears higher than the $I=4$ state.

The limit of strong γ stability can be most easily visualized by considering a droplet which can only execute axially symmetric vibrations. Such a system has only three degrees of freedom: β (which can take on negative values), θ, and ϕ. The expression for the energy can be written

$$Eu(\beta)=\left\{\frac{\hbar^2}{2B}\left(-\frac{\partial^2}{\partial\beta^2}+\frac{I(I+1)}{3\beta^2}\right)+V(\beta)\right\}u(\beta), \quad (37)$$

with the eigenfunctions

$$\Psi=\beta^{-1}u(\beta)Y_{I,M}(\theta,\varphi), \quad (d\tau=\beta^2\sin\theta d\beta d\theta d\varphi).$$

The ratio E_2/E_1 as a function of $E_1/\hbar\omega$ is shown in Fig. 3 (broken curve) for a potential of the form $\frac{1}{2}C(\beta-\beta_0)^2$.

The spectra arising from potentials deviating only slightly from both β and γ stability (see Eq. (19)) have been discussed by Bohr and Mottelson. The behavior is similar for both types of instability and is of the type shown in the left-hand part of the broken curve in Fig. 3. The energy is given by

$$E=\frac{\hbar^2}{2\mathcal{J}}I(I+1)-\frac{1}{2}\left(\frac{\hbar^2}{\mathcal{J}}\right)^3\left(\frac{3}{\hbar\omega_\beta}+\frac{1}{h\omega_\gamma}\right)[I(I+1)]^2, \quad (38)$$

where $\mathcal{J}=3B\beta^2$, $\omega_\beta=(k_1/B)^{\frac{1}{2}}$, $\omega_\gamma=(k_2/\beta_0{}^2B)^{\frac{1}{2}}$, and k_1 and k_2 are defined in Eq. (19). The first term on the right-hand side of (38) gives the usual rotational spectrum, while the second term is referred to as the rotation-vibration correction.

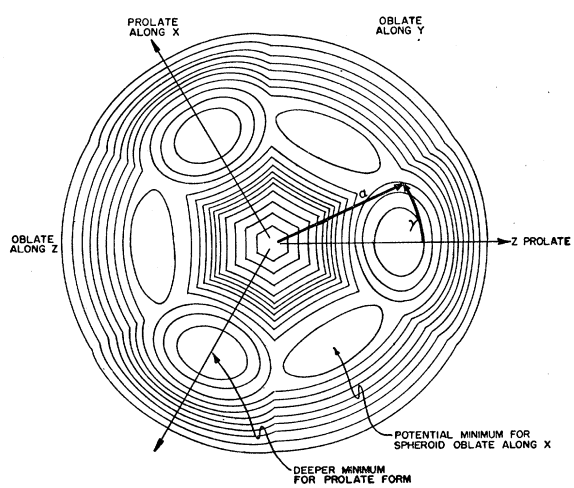

FIG. 5. *Schematic* energy contours for a nucleus intermediate between closed shells, after Hill and Wheeler (reference 8, Fig. 28). Their α and γ correspond to our β and γ.

The examination of small and intermediate deviations from γ instability is a more difficult problem. A perturbation or Tamm-Dancoff calculation has only a very limited range of convergence. Yet insight can be gained by considering first-order effects.

We begin with a representation in which the basis vectors are eigenfunctions of (12), designated by

$$|\lambda n_\gamma IM\rangle=\sum_{K=-I}^{I}g_K(\lambda,n_\gamma;\gamma)\mathfrak{D}_{KM}{}^I(\theta_i). \quad (39)$$

We introduce an expansion parameter k for the potential so that

$$v(\gamma)=kv'(\gamma). \quad (40)$$

In first-order perturbation theory, the perturbed wave functions are given by

$$|i,k\rangle=|i\rangle+k\sum_j{}'\frac{\langle j|v'(\gamma)|i\rangle|j\rangle}{\Lambda_i-\Lambda_j}, \quad (41)$$

where i (or j) represents the quartet of quantum numbers in (39).

It is of particular interest to examine the selection rules for $E2$ radiation. In the unperturbed system, we have the selection rule that the crossover transition is forbidden:

$$\langle 0000|\alpha^*{}_\mu|202M\rangle=0. \quad (42)$$

For the perturbed functions, we have

$$\begin{aligned}&\langle 0000,k|\alpha^*{}_\mu|202M,k\rangle\\ &=k\{\langle 0000|\alpha^*{}_\mu 102M\rangle\langle 102M|v'(\gamma)|202M\rangle/6\\ &+\langle 3000|\alpha^*{}_\mu|202M\rangle\langle 0000|v'(\gamma)|3000\rangle/18\}\\ &=-\tfrac{1}{6}k\int_{-\pi/3}^{\pi/3}v'(\gamma)\cos 3\gamma|\sin 3\gamma|d\gamma\delta_{\mu M}, \quad (43)\end{aligned}$$

which does not generally vanish in the first-order perturbation [since $v'(\gamma)$ should be an even function]. Thus the selection rule forbidding the crossover transition fails.

If the strong-coupling collective model description has any validity for the nuclei of Scharff-Goldhaber and Weneser, it would appear that the nuclei are highly "γ unstable."

Small deviations from γ instability will remove the degeneracy in the states with the same n_γ and λ. In particular, the pair $\lambda=2$, $n_\beta=0$ ($I=2$ and 4) will emerge as a doublet although, as noted above, the state with spin 4 will appear lower than the state with spin two, contrary to the frequency of spin 2 in the nuclei of Scharff-Goldhaber and Weneser.

The selection rule forbidding $M1$ radiation is not affected by deviations from γ instability; the rule remains even in the limit of the γ-stable rotational spectrum.

V. ODD NUCLEI

The treatment of an odd nucleon coupled to a γ-unstable core can be handled in a straightforward manner by techniques indicated by Bohr and Mottelson for weak coupling. The concept of a strong coupling for odd nuclei is not so easily generalizable to γ-unstable nuclei for the following reasons:

In the case of rotational nuclei, the field experienced by an individual nucleon approximately preserves its shape as a function of time, but rotates in space with a period long compared with the nucleonic period. The wave function of the nucleon can be solved for in the (momentarily) fixed field of the nucleus. Because of axial symmetry, the projection of the angular momentum along the symmetry axes, Ω, is a constant of the motion and may be added to angular momentum of the rest of the nucleus by the rules of the vector addition.

In the case of γ-unstable nuclei, however, the shape of the field changes with time and there remains no symmetry axis. There are no simple constants of the motion.

Certain qualitative features may be understood without solving particular problems.

(1) An odd nucleon might be expected to tend to stabilize the core about axial symmetry.

(2) The magnitudes of spectroscopic quadrupole moments may be expected to be considerably smaller than those entering into such considerations as quadrupole radiation, Coulomb excitation, or atomic isotope shifts. The spectroscopic quadrupole moments depend on the sign of the deformation. For γ-unstable cores, the sign is not constant and the expectation value of Q would vanish. There are insufficient data at present to compare quadrupole moments measured spectroscopically and those determined by Coulomb excitation. A possible example is Sm^{149}, which has a very small spectroscopic quadrupole moment. On the basis of atomic isotope-shift considerations, however, the neighboring even isotopes would be expected to have respectably large deformations. A consistent interpretation could be that Sm^{149} also has a deformed core which is γ unstable. It would be interesting to see if Coulomb excitations also yield a large deformation.

VI. DISCUSSION

A. Deformation Potentials

The sum of individual particle energies[7] as a function of the deformation parameters, in the strong-coupling approximation, represents the potential energy of deformation. A qualitative description of the potential energy surface as function of β and γ has been given by

[7] Care must be taken in such a sum not to count potential energy twice—if the primary forces are assumed to be two-body.

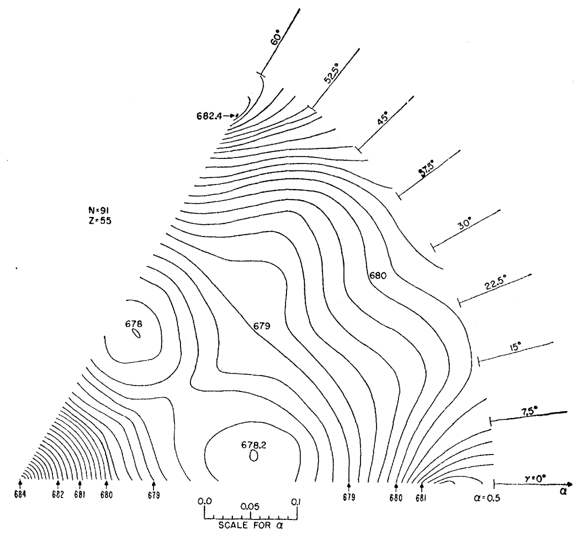

FIG. 6. Energy contours for an ellipsoidal anisotropic harmonic oscillator, midway between neutron and proton shells, after Gursky (reference 9). The numerical values of the energy contours are given in units of $\hbar\omega$, where ω is the oscillator frequency for zero deformation. Gursky's α and γ have approximately the same significance as our $(5/4\pi)^{\frac{1}{2}}\beta$ and γ. The figure has reflectional symmetry about the axes $\gamma=0$, 60°, 120°, etc. Cusps which occur at these symmetry lines and at $\beta=0$ are expected to disappear when direct interparticle interactions and other perturbations are considered.

Hill and Wheeler,[8] who first pointed out that the easiest path between oblate and prolate deformations is not through spherical symmetry, but through nonsymmetric deformations—hence the possibility of γ oscillations. This is illustrated in Fig. 5, which is taken from their paper. The figure schematizes the potential energy surface (for one type of particle) just over halfway between closed shells. Although axial symmetry is favored—and in this case, the prolate shape—there is a "valley" following the locus of points $\beta=\beta_0$, where β_0 is the "equilibrium" deformation. The valley has local minima in it, corresponding to preferred shapes, but the depths of the minima may be small compared with the peak at zero deformation. The γ-unstable approximation is to ignore the bumps in the valley.

Gursky[9] has examined the potential surface quantitatively for an anisotropic harmonic oscillator (Fig. 6), and between closed shells finds results similar to those predicted by Hill and Wheeler, except that the local minima in the valley are usually found for nonaxial shapes. The potential at $\beta=0$ does have a cusp (as we have assumed for simplicity) except when the minimum of the energy lies there. Bohr and Mottelson (private communication) suggest that more detailed calculations including direct interparticle interactions would necessarily lead to a flat potential at $\beta=0$. As

[8] D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).

[9] M. Gursky (private communication). Further details will be published in a forthcoming paper.

expected, Gursky finds that the spherical shape is preferred for closed shells, and that the positions of the minima increase in β as one moves away from closed shells.

In the case of axial shapes ($\gamma=0$ or π), detailed calculations of the single-particle energy levels and the deformation potential have been carried out by several authors,[10] including effects of spin orbit coupling. These calculations have proved extremely valuable in the regions of well developed rotational spectra, and similar calculations, as a function of both β and γ, would prove valuable in the region where the surfon-type spectrum is observed.

B. Validity of Strong Coupling

In order that the "strong coupling" or "adiabatic" approximation be valid, at least two conditions must be satisfied. The first and more obvious of these is that the nuclear potential change sufficiently slowly with time so that an individual particle can continuously "readjust" its wave function to the new potential without changing "state" (in the absence of crossing of levels). This condition will be satisfied if the energy of the collective modes is small compared with the energy level spacings of the single particle excitations. If this criterion is satisfied, the nucleons are obliged to follow the deformations. This criterion is well satisfied for the nuclei which exhibit well developed rotational spectra, where the rotational energies are of the order of 100 kev.

The surfon energies involved in the spectra of Scharff-Goldhaber and Weneser are higher than the rotational energies and may be an Mev or more. The particle energy level spacings are comparable. This means that one cannot expect a well-developed surfon spectrum with many members. Nevertheless, the appearance of first and second excited states is not unreasonable.

The other condition which must be satisfied is that when crossing of levels occurs, a nucleon must remain in the lowest level. Individual particle levels can only cross if there are special symmetries present. As Hill and Wheeler have pointed out,[11] irregularities in the nuclear surface and, indeed, deviations from the single-particle model prevent actual crossing of levels. Nevertheless, at the point where levels would cross in the absence of perturbations, a jump can occur unless the perturbation energy (which is approximately the separation of the levels) is large compared with the energy of the collective modes. The perturbation energy is not easy to evaluate, but may be of the order of an Mev. It is possible that level crossings may play a role in particular spectra.

C. Mass Parameter

The evaluation of the mass parameter B requires [see Eq. (8)] a detailed knowledge of nucleonic configurations. The experience with rotational spectra[6] leads to expect that the hydrodynamic approximation of irrotational flow (7) is not sufficient for the surfon oscillations either. Estimates of the mass parameter for surfon oscillations have been made by Moszkowski[12] and indicate a value not greatly different from that calculated (and observed) for rotational nuclei.[6]

VII. CONCLUSIONS

Surfon oscillations in a deformation potential independent of the shape parameter have been studied and found to exhibit certain regularities of the kind observed in the nuclei of the type considered by Scharff-Goldhaber and Weneser. The condition of γ instability alone leads to the selection rule forbidding the $2+ \rightarrow 0+$ crossover transition, while it appears necessary to assume a nonzero equilibrium deformation in order to account for the observed ratios $E_2:E_1$.

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[11] D. L. Hill and J. A. Wheeler (to be published).

[12] S. A. Moszkowski (private communication); the results will be discussed in a forthcoming paper.