THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 102, NO. 3 MAY 1, 1956

Model for Collision Processes in Gases: Small-Amplitude Oscillations of Charged Two-Component Systems*

E. P. GROSS, *Physics Department, Syracuse University, Syracuse, New York*

AND

M. KROOK, *Solar Department, Harvard University, Cambridge, Massachusetts*

(Received January 23, 1956)

The general features of the small-amplitude oscillations of a two-component ionized gas are discussed. The effects of the random thermal motions of ions and electrons are described by one-particle distribution functions. When collisions are neglected, there are two types of waves for a given wavelength. One is a high-frequency electron plasma oscillation, slightly modified by ionic motions. The other is the Tonks-Langmuir positive-ion oscillation which is shown to be undamped when the electron temperature is considerably greater than the ion temperature. The effects of collisions are treated by a kinetic model which satisfies the conservation laws, and provides for energy and momentum exchange between components. The low-pressure waves are damped primarily because of the decreasing electron temperature with increasing collision frequency. The validity of transport treatments is investigated. At high density and high frequency one finds the correct sound wave for a mixture of gases. Study of the frequency as a series of inverse powers of the collision frequency shows that the first-power term yields absorption independent of the electric charge. Higher powers give contributions to the absorption and dispersion which depend on electrical polarization as well as on diffusion, viscosity, and heat conductivity. The behavior at the more frequently realized case of low frequency and high pressure depends on the electric charge more directly.

1. INTRODUCTION

IN paper I of this series,[1] a theoretical scheme for the study of collision processes in gases was introduced. This approach is aimed at treating situations where, for example, a Boltzmann equation for a one-particle distribution function is an adequate description, but where the mathematical difficulties of handling that equation are very great. We introduce alternative kinetic equations with collision terms that satisfy the instantaneous conservation laws, but which have a simpler structure than the Boltzmann collision terms. In I the procedure was illustrated for a constant-collision-time model. Within the limits of small-amplitude theory, it was possible to solve definite initial value problems explicitly, for arbitrary values of the collision time. The present article deals with an extension of the methods and results of I to two-component systems. Attention is again focused on the nature of the small-amplitude vibrations of a plasma.

For non-Coulombic force laws, our kinetic equations provide a less accurate description than does the Boltzmann equation. However, because of the mathematical simplicity of the equations, it is possible to obtain an over-all survey of the dynamical properties of gases. For systems of charged particles, on the other hand, there exists at present no rigorous set of equations describing the nonequilibrium properties. One can use experimental information concerning some nonequilibrium processes (e.g., the stationary transport processes), to fit the parameters of our semiphenomenological theory. The equations are then used to discuss other nonequilibrium processes, e.g., small-amplitude oscillations. This procedure will be discussed in a forthcoming paper; it leads to results of semiquantitative validity. Even for collisions in which many bodies are simultaneously involved, as is certainly the case for Coulomb forces, the division into absorption and emission has

* The research in this paper has been supported in part by the Geophysics Research Directorate, Air Force Cambridge Research Center, Cambridge, Massachusetts, under contract with Harvard University. A preliminary account was given at the meeting of the American Physical Society, New York, January, 1954 [Phys. Rev. 94, 778(A) (1954)].

[1] Bhatnagar, Gross, and Krook, Phys. Rev. 94, 511 (1954), henceforth referred to as I.

perhaps some validity. Other descriptions of ionized gases by one-particle distribution functions are no less phenomenological, since they neglect the screening properties of a plasma. The main justification of the present method is its utility in investigating many dynamical properties of ionized gases. We use it here for a preliminary study of the oscillations of a two-component ionized gas.

For systems of charged particles, the work of I is chiefly of methodological interest. There the problem was simplified by replacing the positive ions by a positive continuum. This has two main consequences. First the ion motions have been neglected; these motions lead to a negligible change in the frequency of the electronic plasma oscillations. More important, under certain circumstances (electron temperature much greater than the ion temperature), a low-frequency "positive ion" oscillation can propagate. The second consequence of overlooking the discrete character of the positive charges is that electron-ion and ion-ion collisions have been disregarded. In I we found a dispersion relation $\omega^2=\omega_p{}^2+(5/3)(kT/m)p^2$ for sufficiently high electron-electron collision frequency. The frequency remains high because of the restoring forces arising from charge separation. In a real plasma the electron-ion and ion-ion collisions play a vital role at high pressures; they lead to a dispersion relation $\omega^2=p^2(5/3)[2kT/(m_1+m_2)]$. Here m_1 is the electronic mass, m_2 the positive-ion mass, T the common temperature of the two components. This is a sound oscillation, not involving the charge, and without dispersion. The sound speed is appropriate to particles of a mass equal to the mean of that of the ions and electrons. The intermediate region of densities is characterized by frequencies whose imaginary parts are in general larger than the real parts; no definite waves are then propagated. We shall be interested in ascertaining the role of charge separation for the absorptive and dispersive effects as the pressure is lowered.

In I an electron-electron collision frequency λ was introduced; no attempt was made to express λ in terms of the actual temperature, density or mass of the electrons of the system. For the two-component system one requires the relative magnitude of electron-electron, electron-ion, and ion-ion collisions. For three-component systems with neutral particles, the collisions of neutrals with charged particles may alter entirely the behavior of the ionized gas. A rough estimate of the magnitudes of the collision frequencies has been obtained by binary collision arguments.[2] The work of Jeans[3] and Cohen, Spitzer, and Routley[3] shows, however, that for most processes the collisions with impact parameters between the mean interparticle distance and the Debye length are of overwhelming importance. All the considerations indicate that in any case for common ion and electron temperatures the electron-electron and electron-ion collision frequencies differ by a factor of the order of unity, and that the ion-ion collision frequency is at least an order of magnitude lower. This suffices for our purposes. Investigation of the numerical constants shows that for most fully-ionized plasmas the collision diameter is much smaller than the mean distance between particles, the Debye length or the mean free paths. Collisions of charge particles with neutrals play an important role for a degree of ionization less than a few percent.

2. MATHEMATICAL FORMULATION OF BASIC EQUATIONS

To treat the kinetic theory of mixtures of particles, we define a distribution function for each type of particle. For convenience, discussion will be limited to the case where two types of particles are present. $f_i(\mathbf{x},\mathbf{v}_i,t)d\mathbf{x}_i d\mathbf{v}_i$ represents the number of particles of the ith type between $\mathbf{x}$, $\mathbf{v}_i$ and $\mathbf{x}+d\mathbf{x}$, $\mathbf{v}_i+d\mathbf{v}_i$. The kinetic equations governing the distribution functions are

$$\frac{\partial f_i}{\partial t}+\mathbf{v}_i\cdot\frac{\partial f}{\partial \mathbf{x}_i}+\mathbf{a}_i\cdot\frac{\partial f}{\partial \mathbf{v}_i}=\frac{\delta f_{i1}}{\delta t}+\frac{\delta f_{i2}}{\delta t},\quad i=1,2, \tag{1}$$

where $\mathbf{a}_i(\mathbf{x},\mathbf{v}_i,t)$ is the acceleration of a particle of the ith type, of velocity $\mathbf{v}_i$, at $\mathbf{x}$, t due to body forces. For the longitudinal oscillations of an ionized gas, one may put $\mathbf{a}_i=(e_i/m_i)\mathbf{E}(\mathbf{x},t)$ where m_i and e_i are the mass and charge of the ith type of particle and $\mathbf{E}(\mathbf{x},t)$ is the electric field at $\mathbf{x}$, t. For more general problems the acceleration is determined by full Lorentz force. $\delta f_{ij}/\delta t$ is the change per unit time at $(\mathbf{x},\mathbf{v}_i,t)$ of the number of ith type particles as a result of collisions with the jth type.

The present approach to the theory of oscillations of neutral and ionized gases is based on the following expressions for the collision terms:

$$\frac{\delta f_{ij}}{\delta t}=-\frac{n_j f_i}{\sigma_{ji}}+\frac{n_i n_j}{\sigma_{ji}}\Phi_{ji},\quad i,j=1,2. \tag{2}$$

The σ_{ji} are collision parameters; they are chosen to be independent of velocity, although they depend on the masses m_1 and m_2. The quantity n_1/σ_{11} has the dimensions of a frequency; it is the collision frequency of particle 1 with others of its own type. $n_i(\mathbf{x},t)$ is the density of the ith type of particle at $\mathbf{x}$, t; it is defined as

$$n_i(\mathbf{x},t)\equiv\int f_i(\mathbf{x},\mathbf{v}_i,t)d\mathbf{v}_i. \tag{3}$$

[2] R. Rompe and M. Steenbeck, Ergeb. exakt. Naturw. **18**, 257 (1939); S. Chapman and C. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, London, 1952); K. C. Westfold, Phil. Mag. **44**, 711 (1953).

[3] J. Jeans, *Astronomy and Comogony* (Cambridge University Press, London, 1929), second edition; Cohen, Spitzer, and Routley, Phys. Rev. **80**, 230 (1950).

The quantities Φ_{ji} are

$$\Phi_{ji}=\left(\frac{m_i}{2\pi kT_{ji}}\right)^{\frac{3}{2}}\exp\left[\frac{-m_i}{2kT_{ji}}(\mathbf{v}_i-\mathbf{q}_{ji})^2\right],$$
$$\int\Phi_{ji}d\mathbf{v}_i=1. \tag{4}$$

In the expressions for Φ_{jj} we have introduced the flow velocity vector $\mathbf{q}_{jj}$ and partial temperature T_{jj} of the jth component. The definitions are

$$\mathbf{q}_{jj}(\mathbf{x},t)\equiv\frac{1}{n_i}\int\mathbf{v}_jf_j(\mathbf{x},\mathbf{v}_j,t)d\mathbf{v}_j,$$
$$\frac{3kT_{jj}(\mathbf{x},t)}{m_j}\equiv\frac{1}{n_i}\int(\mathbf{v}_j-\mathbf{q}_{jj})^2f_jd\mathbf{v}_j. \tag{5}$$

The collision term of Eq. (2) postulates that the number of particles of the ith type absorbed from the velocity range $\mathbf{v}_i$, $\mathbf{v}_i+d\mathbf{v}_i$ at $\mathbf{x}$, t, per unit time, is proportional to f_i. It is also proportional to the total collision frequency, i.e., for type 1, $(n_1/\sigma_{11})+(n_2/\sigma_{21})$. The number of particles of ith type emitted into the velocity range $\mathbf{v}_i$, $\mathbf{v}_i+d\mathbf{v}_i$ is made up of contributions from collisions between like and unlike types. The total number of like collisions per unit time is n_i^2/σ_{ii} these particles are emitted with a Maxwellian distribution of velocities, centered about a flow velocity $\mathbf{q}_{ii}(\mathbf{x},t)$, and with a kinetic temperature T_{ii}. For collisions between unlike types the total number is n_1n_2/σ_{21}, which is also the total number of absorptions per unit time. The particles are emitted with a Maxwellian distribution of velocities centered about a flow velocity $\mathbf{q}_{ji}$ and with a kinetic temperature T_{ji}. These "mixed" quantities must be specified to complete the kinetic equations; the next section deals with general requirements which enable one to partially determine $\mathbf{q}_{ji}$, T_{ji}.

For a system of charged particles, we must add the electromagnetic equations which couple the fields with the particle motions. For the longitudinal oscillations of an ionized gas, it suffices to use Poisson's equation

$$\nabla\cdot\mathbf{E}=4\pi(e_2n_2+e_1n_1), \tag{6}$$

where e_1 and e_2 are the charges of the two components. The problem reduces to finding solutions of Eqs. (1) and (6) under appropriate boundary conditions.

3. CONSERVATION LAWS AND RELAXATION PROBLEM

To be acceptable the collision model must satisfy certain general requirements. First, the collision terms must be such as to conserve the number of particles of each type, the total number of particles, the total momentum and the total energy. Energy and momentum are not conserved instantaneously for each type of particle; they are exchanged by the particle types by collisions. Second, the collision terms must express the irreversibility of kinetic phenomena. This aspect can be investigated by examining the time behavior of a spatially homogeneous non-Maxwellian distribution. We require that the system relax to a Maxwellian distribution of velocities.

To investigate conservation of particle number one integrates over the entire velocity range for $\mathbf{v}_i$. Instantaneous conservation of particle number for each constituent results; this property has been built into the structure of the equations.

The collisions of like particles leave the momentum unaltered; the time rate of change of the ith component momentum as a result of collisions with the jth component is

$$m_i\int\mathbf{v}_i\frac{\delta f_{ij}}{\delta t}d\mathbf{v}_i=m_i\frac{n_in_j}{\sigma_{ji}}(\mathbf{q}_{ji}-\mathbf{q}_{ii}).$$

Since the number of collisions per unit time by particles of type 1 with particles of type 2 is the same as the number of collisions made by type 2 with type 1 we have $\sigma_{12}=\sigma_{21}$. Thus the conservation of total momentum is

$$m_1(\mathbf{q}_{21}-\mathbf{q}_{11})+m_2(\mathbf{q}_{12}-\mathbf{q}_{22})=0. \tag{7}$$

Similar calculations yield for the requirement of conservation of total energy

$$3k(T_{21}-T_{11})+3k(T_{12}-T_{22})+m_1(q_{21}^2-q_{11}^2)+m_2(q_{12}^2-q_{22}^2)=0. \tag{8}$$

Equation (7) is a vector relation yielding three conditions on the six components of $\mathbf{q}_{12}$, $\mathbf{q}_{21}$; Eq. (8) gives one relation for the two additional numbers T_{12}, T_{21}. We have therefore four relations between eight independent quantities. Further conditions will now be obtained from a study of relaxation problems.

In I the one-component system was described by Eqs. (15)–(20). For processes in which there are no space gradients the density, flow velocity, and temperature are constants with respect to time. However, the distribution function itself, and other velocity moments such as heat flow and stresses, can change with time. If at $t=0$, $f(\mathbf{v},\,t=0)=A(\mathbf{v})$, the solution of (15) is

$$f(\mathbf{v},t)=A(\mathbf{v})e^{-\lambda t}+n\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}\times\exp\left[\frac{-m(\mathbf{v}-\mathbf{q})^2}{2kT}\right](1-e^{-\lambda t}), \tag{9}$$

where

$$\lambda=n/\sigma,\qquad n=\int Ad\mathbf{v},$$
$$n\mathbf{q}=\int\mathbf{v}Ad\mathbf{v},\quad n\left(\frac{3kT}{m}\right)=\int(\mathbf{v}-\mathbf{q})^2Ad\mathbf{v}.$$

This yields the correct Maxwellian distribution as $t\to\infty$. The distribution function decays to equilibrium exponentially, at a rate which is independent of the size and nature of the initial deviation from equilibrium. This is a special property of our oversimplified relaxation model; nevertheless an account is given of the irreversible nature of collision processes.

For mixtures of particles the relaxation problem is defined by the equations:

$$\frac{\partial f_i}{\partial t}=-\left(\frac{n_i}{\sigma_{ii}}+\frac{n_j}{\sigma_{ji}}\right)f_i+\frac{n_i^2}{\sigma_{ii}}\Phi_{ii}+\frac{n_in_j}{\sigma_{ji}}\Phi_{ji}. \quad (10)$$

Since collisions conserve the number of particles of each component, n_1 and n_2 are constants. The quantities $\mathbf{q}_{ii}$, T_{ii}, $\mathbf{q}_{ij}$, T_{ij} are in general functions of time. By forming the moments of Eq. (10) one finds that the time dependence of these moments is governed by the equations:

$$\begin{aligned}\frac{\partial q_{ii}}{\partial t}&=\frac{n_j}{\sigma_{ji}}(\mathbf{q}_{ji}-\mathbf{q}_{ii}),\\ \frac{\partial T_{ii}}{\partial t}&=\frac{n_j}{\sigma_{ji}}\left[T_{ji}-T_{ii}+\frac{m_i}{3k}(\mathbf{q}_{ji}-\mathbf{q}_{ii})^2\right].\end{aligned} \quad (11)$$

If the $\mathbf{q}_{11}\cdots T_{21}$ are determined, straightforward integration of Eq. (10) yields the time dependence of the distribution functions. The requirement that the system tends to equilibrium implies that a common flow velocity and temperature for the two components is reached. Thus

$$\mathbf{q}_{11}(t\to\infty)=\mathbf{q}_{22}(\infty)=\mathbf{q}_\infty,$$
$$T_{11}(\infty)=T_{22}(\infty)=T_\infty.$$

In addition, one must have $(\partial\mathbf{q}_{ii}/\partial t)(\infty)=(\partial T_{ii}/\partial t)(\infty)=0$, which results in

$$\mathbf{q}_{ij}(\infty)=\mathbf{q}_{ii}(\infty)=\mathbf{q}_\infty \text{ and } T_{ji}(\infty)=T_{ii}(\infty)=T_\infty. \quad (12)$$

We now introduce a further basic assumption in order to complete the theoretical scheme (1)–(6) in the simplest manner. We take $\mathbf{q}_{21}$ and $\mathbf{q}_{12}$ at any point $\mathbf{x}$, t to be linear combinations of $\mathbf{q}_1$ and $\mathbf{q}_2$ at $\mathbf{x}$, t:

$$\mathbf{q}_{ji}=\alpha_{jj}\mathbf{q}_{jj}+\alpha_{ji}\mathbf{q}_{ii}, \quad (13)$$

where α_{ij} are constants which depend only on the masses m_1 and m_2. These constants measure the average fraction of momentum transferred from one component to the other by collisions. The assumption of a linear combination is restrictive, since the mean velocity of particles coming off is given in terms of the first moments $\mathbf{q}_1$ and $\mathbf{q}_2$ alone. It is however in keeping with the general type of investigation of this series of papers. (The limitations of these assumptions will be discussed in paper III.)

If the expressions for $\mathbf{q}_{ji}$ are inserted into the momentum conservation Eq. (7), one finds

$$m_1\alpha_{22}=m_2(1-\alpha_{12});\quad m_2\alpha_{11}=m_1(1-\alpha_{21}). \quad (14)$$

Use has been made of the fact that $\mathbf{q}_1$, $\mathbf{q}_2$ can be prescribed separately and arbitrarily at $t=0$.

The relaxation requirement, Eqs. (12), together with Eq. (13), yields, (since $\mathbf{q}_\infty$ is arbitrary),

$$\alpha_{22}=1-\alpha_{21},\quad \alpha_{12}=1-\alpha_{11}. \quad (15)$$

Because of the structure of Eqs. (14) the single additional condition

$$\alpha_{22}=(m_2/m_1)\alpha_{11} \quad (16)$$

is obtained. There is thus only one free quantity; this measures the average fraction of momentum transferred per collision. For the Boltzmann equation this parameter would be connected with the law of force between the two types of particles. In our approach we do not take account of the detailed geometry of an encounter. For the electron ion case, to a sufficient approximation, $\alpha_{21}=m_1/(m_1+m_2)$.[4]

The momentum change per unit time as a result of collisions between components, now becomes

$$(m_1n_1n_2/\sigma_{21})\alpha_{22}(\mathbf{q}_{22}-\mathbf{q}_{11}).$$

This agrees with the equations of Chapman and Cowling[2] and Westfold[2] with the appropriate value for $1/\sigma_{21}[1/\sigma_{21}=(1/m_1+1/m_2)^{\frac{1}{2}}]$.

For the temperatures T_{21} and T_{12}, we assume

$$\begin{aligned}T_{21}&=\alpha_{22}'T_{22}+\alpha_{21}'T_{11}+Aq_{22}^2+B\mathbf{q}_{11}\cdot\mathbf{q}_{22}+Cq_{11}^2,\\ T_{12}&=\alpha_{12}'T_{22}+\alpha_{11}'T_{11}+Dq_{22}^2+E\mathbf{q}_{11}\cdot\mathbf{q}_{22}+Fq_{11}^2.\end{aligned} \quad (17)$$

The ten constants are of course not independent. Insert (18) into the energy conservation law (8), and set the coefficients of T_{11}, T_{22}, q_{11}^2, q_{22}^2, and $\mathbf{q}_{11}\cdot\mathbf{q}_{22}$ separately equal to zero. The coefficients of T_{11} and T_{22} yield the relations

$$\alpha_{22}'=1-\alpha_{12}',\quad \alpha_{11}'=1-\alpha_{21}'. \quad (18)$$

The coefficients of q_1^2, q_2^2 and $\mathbf{q}_1\cdot\mathbf{q}_2$ give three relations for the six quantities A, B, C, D, E, and F.

Further relations are obtained by considering spatially homogeneous relaxation processes. For a process in which $\mathbf{q}_1$ and $\mathbf{q}_2$ are zero, $T_{21}(\infty)=T_1(\infty)$ and $T_{12}(\infty)=T_2(\infty)$. Thus,

$$\alpha_{22}'T_2(\infty)+(\alpha_{21}'-1)T_1(\infty)=0,$$
$$(\alpha_{12}'-1)T_2(\infty)+\alpha_{11}'T_1(\infty)=0.$$

If furthermore the plasma is "isothermal," the common temperature $T_\infty=T_1(\infty)=T_2(\infty)$ is reached and

$$\alpha_{22}'=1-\alpha_{21}',\quad \alpha_{12}'=1-\alpha_{11}'. \quad (20)$$

These equations imply

$$\alpha_{22}'=\alpha_{11}'. \quad (21)$$

[4] J. Jeans, *Introduction to Kinetic Theory of Gases* (Cambridge University Press, London, 1946).

Thus one of the four α' remains free. If one considers a more general process in which $\mathbf{q}_{11}$ and $\mathbf{q}_{22}$ tend to the common value $\mathbf{q}_\infty$, the conditions $T_{21}(\infty)=T_1(\infty)$ and $T_{12}(\infty)=T_2(\infty)$ yield the additional relations

$$A+B+C=0,\quad D+E+F=0. \tag{22}$$

Only one of the six quantities remains free. We are left with three undetermined constants, say α_{11}, α_{11}', and A. For the small-amplitude processes dealt with here, the constant A will not appear since it occurs as a coefficient of the second-order term q_{22}^2. The quantities α_{11}, α_{11}' measure the average momentum and energy exchange between components as a result of collisions.

A correct mathematical treatment of the plasma oscillation problem should include the processes maintaining the plasma in a steady state. Our procedure is to assume that the steady-state distribution is known and to consider the effects of elastic collisions and space charge forces in determining the character of small oscillations. The effects of inelastic collisions are only taken into account insofar as they determine the steady-state distribution. This is probably a good approximation provided one takes into account the non-Maxwellian character of the distribution. For low-pressure discharge-tube plasmas, the electron temperature may be considerably higher than the ion temperature and there may exist mutual diffusion of the constituents. It is possible to deal with some aspects of these more general cases very schematically within the framework of the present theory. One throws the entire burden of maintaining the non-Maxwellian steady state onto the elastic collisions. The relations between $\alpha_{ij}\cdots F$ are modified so that the momentum and energy conservation laws hold, and such that there is a prescribed steady-state ratio of electron to ion temperature. This will not be done here, since these considerations are important only for the low-pressure ionized gas, the main features of which are deducible from the limit of zero collision frequencies.

4. LOW COLLISION FREQUENCIES-POSITIVE ION OSCILLATIONS

The characteristics of the oscillations of a two-component ionized gas are very varied, since the system is described by many independent parameters. In the present section, we treat the low-pressure plasma for which the frequency of close collisions is small compared to other characteristic frequencies. We study the possible waves as a function of the parameters m_1/m_2, T_{11}/T_{22}, etc.

A feature not present in the one-component plasma is that under certain circumstances two types of modes can propagate for a given wavelength. One wave is a high-frequency electronic plasma oscillation, slightly modified by the motions of the positive ions; the second mode is a low-frequency positive-ion oscillation. These ionic waves have been discussed previously by Tonks and Langmuir[5] and Rompe and Steenbeck.[2] In the Tonks-Langmuir derivation the random temperature motions of the ions are neglected; the ions have, however, an organized component of motion. The electrons partially follow any displacement of the ions so as to neutralize the space charge created, and take positions in accordance with a Boltzmann distribution. Mathematical analysis based on this idea leads to the dispersion relation:

$$\omega^2=p^2\omega_2^2/(p^2+k_1^2),$$

where k_1 is the Debye wave number for electrons, $k_1^2=4\pi ne^2/kT_{11}$; n is the mean density of electrons (or singly ionized atoms), and $\omega_2=(4\pi ne^2/m_2)^{\frac{1}{2}}$ is the positive ion plasma frequency. For wavelengths shorter than the electronic Debye length, the frequency tends to $\omega=\omega_2$ so that the mode becomes highly dispersive. The reason is that the electrons cannot respond effectively to periodic charge displacements over distances less than the electronic Debye length. The electrons then form a practically uniform negative background for the positive ions; this is the inverse situation to the electronic plasma oscillation. For long waves, i.e., $p\ll k_1$, the dispersion relation becomes $\omega^2\rightarrow p^2(kT_{11}/m_2)$. This shows no dispersion; the spectrum is that of a sound wave with a speed determined by the electronic temperature and the ionic mass. It does not contain the charge because the electron motions tend to compensate the positive space charge so that there is no electrical restoring force.

The theory to be presented substantiates this picture in broad outline, with, however, some qualifications. We find that the low-frequency modes are damped at a rate which depends on the ratio of the electron temperature T_{11} to the ion temperature T_{22}. For waves long compared to the Debye length and $T_{11}/T_{22}\gg 1$, the ratio of imaginary and real parts of the frequency, i.e., the damping per cycle, is negligible. For T_{11} comparable to T_{22}, the damping is appreciable. In many plasmas (e.g., in discharge tubes) where the collision frequency is small, T_{11}/T_{22} is actually large. In considering the damping per cycle as a function of wavelength, one must consider three regions. For example, for $T_{11}/T_{22}=9$ the electronic Debye length is three times the ionic Debye length. The region $pv_2/\omega_2<\frac{1}{3}$ represents waves longer than both electronic and ionic lengths; the damping per cycle is negligible. The second region $\frac{1}{3}<pv_2/\omega_2<1$ covers waves of lengths between the two Debye lengths; the damping per cycle has already become appreciable. For $pv_2/\omega_2>1$, i.e., waves shorter than both Debye lengths, the damping is so great that no organized oscillations can be said to exist.

To treat the problem mathematically, set the right-hand sides of Eqs. (1) equal to zero. In the zero-collision frequency case a Laplace transformation treatment is necessary since one deals with "drift" damping.[6]

[5] L. Tonks and I. Langmuir, Phys. Rev. 33, 195 (1929).
[6] See reference 1, p. 520.

Introduce

$$f_i = nG_i(1+\phi_i),$$
$$G_i = \left(\frac{m_i}{2\pi kT_{ii}}\right)^{\frac{3}{2}} \exp\left(\frac{-m_i v_i^2}{2kT_{ii}}\right). \tag{23}$$

The ϕ_i are dimensionless quantities, which are of first order in the electric fields which occur. We will consider deviations from a steady-state Maxwellian distribution. (The fact that the distribution is in general non-Maxwellian may be important for the excitation of oscillations.) The linearized equations are

$$\frac{\partial\phi_i}{\partial t} + \mathbf{v}_i \cdot \frac{\partial\phi_i}{\partial \mathbf{x}_i} - e_i \frac{\mathbf{E}\cdot\mathbf{v}_i}{kT_{ii}} = 0,$$
$$\nabla\cdot\mathbf{E} = 4\pi n\left(e_2\int\phi_2 G_2 d\mathbf{v}_2 + e_1\int\phi_1 G_1 d\mathbf{v}_1\right). \tag{24}$$

As in paper I, initial-value problems will be studied. We therefore perform a Fourier transformation with respect to the space variables and a Laplace transformation with respect to the time variables. Introduce the notation:

$$\phi(\mathbf{v},\mathbf{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int \phi_{\mathbf{p}}(\mathbf{v},t) e^{i\mathbf{p}\cdot\mathbf{x}} d\mathbf{p},$$
$$\phi_{\mathbf{p}\sigma} = \int_0^\infty \phi_{\mathbf{p}}(\mathbf{v},t) e^{-\sigma t} dt.$$

The usual inversion formulas hold. For disturbances along the x-direction, the transformed equations are

$$(\sigma + ipu_i)\phi_{i\mathbf{p}\sigma} = \frac{e_i E_{\mathbf{p}\sigma x} u_i}{kT_{ii}} + \phi_{i\mathbf{p}}(\mathbf{v}_i,0),$$
$$ipE_{\mathbf{p}\sigma x} = 4\pi n\left(e_2\int\phi_{2\mathbf{p}\sigma}G_2 d\mathbf{v}_2 + e_1\int\phi_{1\mathbf{p}\sigma}G_1 d\mathbf{v}_1\right). \tag{25}$$

Here u_i are the x-components of the velocities $\mathbf{v}_i$; $\phi_{i\mathbf{p}}(\mathbf{v}_i,0)$ are the Fourier components of the distribution functions at $t=0$.

The solution of Eq. (27) is obtained in the manner indicated in Eqs. (37)–(44) of paper I. One finds a set of linear inhomogeneous coupled equations for the density perturbations

$$\nu_{i\mathbf{p}\sigma} = \int \phi_{i\mathbf{p}\sigma} G_i d\mathbf{v}_i.$$

The integrals occurring have been studied in I [Eqs. (79) and (80)]. We define $\Omega_i = (\sigma/p)(m/2kT_i)^{\frac{1}{2}}$ and use the derivative of the error function, $F_1(\Omega) = 2\Omega F(\Omega) - 1$ [where $F(\Omega)$ is given by Eq. (59) of I]. The solution of the set of equations is

$$\Delta\cdot\nu_{1\mathbf{p}\sigma} = \left(1 - \frac{F_1(\Omega_1)}{p^2k_2^2}\right)M_1 - \frac{F_1(\Omega_1)}{p^2k_1^2}M_2,$$
$$\Delta\cdot\nu_{2\mathbf{p}\sigma} = \frac{F_1(\Omega_1)}{p^2k_2^2}M_1 - \left(1 - \frac{F_1(\Omega_1)}{p^2k_1^2}\right)M_2, \tag{26}$$

where

$$M_i = \int \frac{\phi_{i\mathbf{p}}(\mathbf{v},0)G_i d\mathbf{v}}{\sigma + ipu_i}.$$

The denominator Δ is

$$\Delta = 1 - \frac{1}{p^2}\left(\frac{F_1(\Omega_1)}{k_1^2} + \frac{F_1(\Omega_2)}{k_2^2}\right). \tag{27}$$

The expressions for $E_{\mathbf{p}\sigma}$ and $\phi_{i\mathbf{p}\sigma}$ are obtained by inserting the expressions for $\nu_{i\mathbf{p}\sigma}$ into Eq. (25). The asymptotic time behavior of the densities is determined by the root of Δ with the largest real part. For the case of zero collision frequency, the asymptotic dependence of the distribution functions differs from that of the densities or electric field [see discussion of I following Eqs. (43) and (51)].

We now study the dispersion relation

$$\Delta = 0. \tag{28}$$

For a given value of the wave vector $\mathbf{p}$ there are infinitely many roots; one is mainly interested in finding the cases where there is at least one approximately undamped root.

The simplest special case of interest is that for which the temperature T_{22} of the positive ions is zero. The ions, however, have ordered motions appropriate to the oscillation in question. It will be seen that $|\Omega_2| = |\sigma/pa_2\sqrt{2}| \gg 1$ for both the high- and low-frequency modes. Ω_2 lies in the second or third quadrant of the complex Ω plane. We therefore use the asymptotic development

$$F(\Omega_2) = \pi^{\frac{1}{2}}\exp(\Omega_2^2) + \frac{1}{2\Omega_2}\left(1 - \frac{1}{2\Omega_2^2} + \frac{3}{(2\Omega_2^2)^2} - \cdots\right).$$

In most cases the exponential term is unimportant. In the limit of zero ion temperature we retain only the second term; this yields a contribution to Δ of $(1/p^2)(F_1/k_1^2) \rightarrow \omega_2^2/\omega^2$.

For the electrons the velocity $a_1 = (kT_{11}/m_1)^{\frac{1}{2}}$ is high, both because of the small electronic mass and the high electron temperature. For the high-frequency electronic plasma waves the condition $|\Omega_1| \gg 1$ is satisfied when one considers waves longer than the electronic Debye length. When $|\Omega_1| \approx 1$ we find, as in I, a heavy damping for these waves. Using the asymptotic expression for the electrons, and neglecting the exponential term,

$$\omega^2 = \omega_2^2 + \omega_1^2\left[1 + \frac{3p^2a_1^2}{\omega^2} - 15\left(\frac{p^2a_1^2}{\omega^2}\right)^2 + \cdots\right]. \tag{30}$$

This is our earlier result for the electronic plasma oscillations. The modification of the frequency arising from the presence of the positive ions is negligible since $\omega_2^2 \ll \omega_1^2$. For wavelengths long compared to the electronic Debye length we verify the conditions $pa_i/\omega \ll 1$, insuring the validity of the expansions. Since $\Omega_i = -i\omega/pa_i\sqrt{2}$, the Ω_i^2 are negative and large; the exponential terms are therefore indeed small. For waves comparable to the Debye length these terms are important and lead to heavy drift damping. The corrections to the frequency arising from the finite ion temperature are very small; they are obtained from the higher terms of Eq. (29). The high-frequency oscillations thus occur for arbitrary values of T_1/T_2 and m_1/m_2. This type of mode propagates freely in a plasma consisting of ions of equal mass as well as for the electron-ion gas.

For the low-frequency positive-ion oscillations again $|\Omega_2| \gg 1$. For the electrons, however, $|\Omega_1| \ll 1$ so that we must use the convergent series for $F(\Omega_1)$. The first few terms are

$$F(\Omega_1) = \tfrac{1}{2}\pi^{\frac{1}{2}} - \Omega_1 + \tfrac{1}{2}\pi^{\frac{1}{2}}\Omega_1^2 - \cdots,$$
$$F_1(\Omega_1) = -1 + \pi^{\frac{1}{2}}\Omega_1 - \cdots. \tag{31}$$

If the leading term for both positive ions and the electrons is retained, one finds the Tonks-Langmuir result

$$\omega^2 = p^2\omega_2^2/(p^2 + k_1^2). \tag{32}$$

The condition for the validity of the positive ion expansion, $|\Omega_2| \gg 1$ is satisfied in virtue of the assumed zero ion temperature. For the electrons we evaluate $|\Omega_1|$ by inserting from Eq. (32). This yields the condition $\omega_2 \ll \sqrt{2}\omega_1$ which is well satisfied in virtue of the small electron-ion mass ratio.

Including higher terms for the asymptotic expansion of the positive ions

$$\omega^2 = \frac{p^2\omega_2^2}{p^2 + k_1^2}\left(1 + \frac{3p^2a_2^2}{\omega^2}\right) - (2\pi)^{\frac{1}{2}}\frac{i\omega}{pa_2}\exp(\Omega_2^2)\frac{\omega^2}{p^2 + k_1^2}. \tag{33}$$

The solution by a method of successive approximations reveals that the damping arising from the exponential term is extremely small provided $T_{11} \gg T_{22}$.

If the ion temperature is set equal to zero but an additional term is retained in the convergent series for the electrons, one finds

$$\omega^2 = \frac{p^2\omega_2^2}{p^2 + k_1^2}\left(1 - i\pi^{\frac{1}{2}}\frac{k_1^2\omega_2}{a_1\sqrt{2}}\frac{1}{(p^2 + k_1^2)^{\frac{3}{2}}}\right). \tag{34}$$

The additional term gives damping which is of order ω_2/ω_1 times the real part of the frequency. Higher terms give corrections which are smaller than the leading term by powers of ω_2/ω_1. Our conclusions are that relatively undamped positive-ion modes can propagate when both $\omega_2/\omega_1 \ll 1$ and $T_{11}/T_{22} \gg 1$. This is the case for many low-pressure electron-ion plasmas. At higher pressures (even with neglect of collisions) the waves are heavily damped since $T_{11} \rightarrow T_{22}$.

Finally we consider the effects of collisions on the positive ion oscillations. These can only lead to damping, for the procedure of this section is based on the use of the convergent series for $F(\Omega_1)$. The higher terms in this series increase in magnitude with increasing collision frequency; they have been shown to give damping corrections. The situation is particularly simple if we restrict ourselves to the case of zero ion temperature and infinite electron temperature. Then $a_2 \rightarrow 0$, $a_1 \rightarrow \infty$, and

$$\omega \approx \frac{p\omega_2}{k_1} \rightarrow \infty, \quad \Omega_2 \rightarrow \infty, \quad |\Omega_1| = \frac{\omega_2}{\omega_1\sqrt{2}} \ll 1. \tag{35}$$

From Eqs. (45) and (46), we obtain

$$L_{11} = 1 - \frac{k_1^2}{p^2}F_1(\Omega_1), \qquad L_{22} = \frac{k_1^2}{p^2}F_1(\Omega_1),$$

$$L_{22} \rightarrow \omega_2^2 - \omega^2 + \frac{i\omega n}{\sigma_{21}}\alpha_{11}, \quad L_{21} = \omega_2^2 - \frac{in\omega}{\sigma_{21}}\alpha_{11}.$$

The dispersion relation is

$$\omega^2 = \frac{\omega_2^2 - i(\omega n/\sigma_{21})\alpha_{11}}{1 - (k_1^2/p^2)F_1(\Omega_1)}.$$

The higher terms of the convergent expansion of $F_1(\Omega_1)$ give small corrections of the order of $(m_1/m_2)^{\frac{1}{2}}$ times the leading term. Neglecting these, we have

$$\omega^2 \sim \left(\omega_2^2 - i\omega\frac{n}{\sigma_{21}}\alpha_{11}\right)\frac{p^2}{k_1^2}.$$

The imaginary part of ω is then $(n/\sigma_{21}\omega_1)(p/2k_1)(m_1/m_2)^{\frac{1}{2}}$ times the real part of the frequency. The damping is thus appreciable only for collision frequencies greater than the electronic plasma frequency. It would appear that positive ion modes can propagate at relatively high densities. In reality, the electron and ion temperatures approach equality as the density is raised; as we have seen, this acts to destroy these modes. It would be possible to give detailed numerical discussion of the nature of the low-pressure modes with the aid of our general formulas; our goal is only to exhibit the general features.

The ratio of the density perturbations is

$$\nu_2/\nu_1 \approx 1 + (p^2/k_1^2),$$

where we have used Eq. (32). The positive and negative density variations are in phase and are approximately equal for $p \ll k_1$. This expresses the adjustment of the electron density to that of the positive ions. For $p \gg k_1$ the positive ions are mainly perturbed. In higher approximations one finds small out of phase components dependent on m_1/m_2, T_{22}/T_{11} and on the collision frequency. Examination of the distribution functions shows that the largest perturbations are experienced by electrons and ions moving near the wave speed.

The preceding conclusions have been obtained from a formalism involving single-particle distribution functions. It is possible that a more accurate treatment of screening effects will modify the conclusions.

5. LINEAR APPROXIMATION—THE DISPERSION RELATION

We now write the general Eqs. (1)–(6) in linear approximation. To the definitions (25) add

$$n_j=\bar{n}_j(1+\nu_j),\quad T_{jk}=\bar{T}_{jk}(1+\tau_{jk}),\quad j,k=1,2, \tag{36}$$

$\bar{n}_j$ and $\nu_j\bar{n}_j$ are the mean and fluctuating components of the densities, τ_{jj} the fluctuating components of the temperature. In the linear approximation,

$$\nu_j=\int G_j\phi_j d\mathbf{v}_j,$$
$$\frac{3k\bar{T}_{jj}}{m_j}(\nu_{jj}+\tau_{jj})=\int G_j\phi_j v_j^2 d\mathbf{v}_j, \tag{37}$$
$$\mathbf{q}_{jj}=\int G_j\phi_j d\mathbf{v}_j.$$

Expand all quantities contained in Eqs. (1)–(6), and retain only first-order fluctuating terms. The kinetic equations are (dropping the bars)

$$\frac{\partial\phi_j}{\partial t}+\mathbf{v}_j\frac{\partial\phi_j}{\partial\mathbf{x}}+\frac{e_j}{kT_{jj}}(\mathbf{E}\cdot\mathbf{v}_j)$$
$$=-\left(\frac{n_j}{\sigma_{jj}}+\frac{n_k}{\sigma_{kj}}\right)(\phi_j+\nu_j)+\frac{m_j\mathbf{v}_j}{kT_{jj}}\left(\frac{n_j\mathbf{q}_{jj}}{\sigma_{jj}}+\frac{n_k\mathbf{q}_{kj}}{\sigma_{kj}}\right)$$
$$+\left(\frac{m_jv_j^2}{2kT_{jj}}-\tfrac{3}{2}\right)\left(\tau_{jj}\frac{n_j}{\sigma_{jj}}+\tau_{kj}\frac{n_k}{\sigma_{kj}}\right). \tag{38}$$

For the two-component neutral gas the charge $e\to 0$ and the densities n_1 and n_2 are arbitrary. We shall be concerned with the ionized gas where the condition of quasi-neutrality requires that $n_1=n_2=n$. For the study of longitudinal oscillations one uses Poisson's equation,

$$\nabla\cdot\mathbf{E}=4\pi en(\nu_2-\nu_1) \tag{39}$$

and the linearized equations of continuity

$$(\partial\nu_j/\partial t)+\nabla\cdot\mathbf{q}_{jj}=0. \tag{40}$$

We shall assume that all oscillating quantities have a dependence on space and time of the form $e^{i(px-\omega t)}$. The relationship of this procedure to the exact method, which solves an initial-value problem by a Laplace transformation with respect to the time variable was discussed in paper I.

Equations (39) and (40) then yield

$$q_{jx}=\frac{\omega}{p}\nu_j,\quad E_x=\frac{4\pi en}{ip}(\nu_2-\nu_1). \tag{41}$$

Using (13) and (17), we find

$$\Delta_j\phi_j=\nu_j\left[\frac{n_j}{\sigma_{jj}}+\frac{n_k}{\sigma_{kj}}+\frac{u_jk_j^2}{ip}+\frac{u_j}{a_j^2}\frac{\omega}{p}\left(\frac{n_j}{\sigma_{jj}}+\frac{n_k}{\sigma_{kj}}\alpha_{kj}\right)\right]$$
$$+\nu_k\left[\frac{u_j}{a_j^2}\frac{n_k}{\sigma_{kj}}\frac{\omega}{p}\alpha_{kk}-\frac{u_jk_j^2}{ip}\right]$$
$$+\tau_{jj}\left[\left(\frac{v_j^2}{2a_j^2}-\tfrac{3}{2}\right)\frac{n_j}{\sigma_{jj}}+\frac{n_k}{\sigma_{kj}}\alpha_{kj}'\right]$$
$$+\tau_{kk}\left[\left(\frac{v_j^2}{2a_j^2}-\tfrac{3}{2}\right)\alpha_{kk}'\frac{n_k}{\sigma_{kj}}\right], \tag{42}$$

where

$$\Delta_j=\frac{n_j}{\sigma_{jj}}+\frac{n_k}{\sigma_{kj}}+i(\mathbf{p}\cdot\mathbf{v}_j-\omega).$$

In the present paper we work with the isothermal approximation discussed in Sec. 4 of I. We set the fluctuating temperatures τ_{jj} equal to zero and introduce the notation

$$\lambda_j=\left(\frac{n_j}{\sigma_{jj}}+\frac{n_k}{\sigma_{kj}}\right),\quad k_j^2=\frac{4\pi n_je^2}{kT_{jj}},$$
$$a_j^2=\frac{kT_{jj}}{m_j},\quad \omega_j^2=k_j^2a_j^2,\quad \Omega_j=\frac{\lambda_j-i\omega}{pa_j\sqrt{2}}. \tag{43}$$

To find the dispersion relation, we make use of the definitions $\nu_j=\int\phi_jG_jd\mathbf{v}_j$ and insert ϕ_j from Eqs. (42). The result is a set of coupled equations for ν_1 and ν_2:

$$\nu_1L_{11}+\nu_2L_{12}=0,$$
$$\nu_1L_{21}+\nu_2L_{22}=0, \tag{44}$$

where

$$L_{jj}=1-\left\{\frac{\lambda_j\sqrt{2}}{pa_j}F(\Omega_j)+\left[\omega_j^2+i\omega n\left(\frac{1}{\sigma_{jj}}+\frac{\alpha_{kj}}{\sigma_{kj}}\right)\right]\frac{F_1(\Omega_j)}{p^2a_j^2}\right\} \tag{45}$$

and

$$L_{jk}=\frac{F_1(\Omega_j)}{p^2a_j^2}\left(\omega_j^2-i\frac{n\omega}{\sigma_{jk}}\alpha_{kk}\right).$$

The dispersion is obtained from the condition that nonzero solutions exist. Setting the appropriate determinant equal to zero,

$$L_{11}L_{22}=L_{12}L_{21}. \tag{46}$$

When the solutions $\omega(p)$ or $p(\omega)$ have been found, the ratio ν_1/ν_2 is given by $\nu_1/\nu_2=-L_{12}/L_{11}$. The distribution functions ϕ_j may be expressed in terms of a single quantity, say ν_1. These steps are important for interpreting the physical nature of the modes which are permitted.

6. LIMITING CASES

1. Transition to One-Component System

Set $\sigma_{12}=\sigma_{21}\rightarrow\infty$, so that there are no collisions between particles of the two components. Put $m_2\rightarrow\infty$ so that the second component is immobile, and contributes only its stationary space charge. Then

$$a_2\rightarrow 0,\quad |\Omega_2|\rightarrow\infty,\quad F(\Omega_2)\rightarrow 0,\quad F_1(\Omega_2)\rightarrow 0.$$

L_{12} and L_{21} tend to zero and L_{22} to unity. Then $L_{11}=0$; this is the dispersion relation for the one-component system [I, Eq. (58)]. The ratio ν_2/ν_1 tends to zero.

2. The Case of Equal Masses

Situations are not often found where most of the negative and positive constituents of a plasma have comparable masses. However, this case is relatively simple from the mathematical point of view and permits insight into some of the properties of the intricate dispersion relation (46). For equal masses, all σ's and α's are equal and $a_1=a_2=a$. Then $L_{11}=L_{12}$, $L_{12}=L_{21}$, and

$$1-\frac{\lambda\sqrt{2}}{pa}F(\Omega)-\left[\binom{2}{0}\omega_1^2+\binom{\frac{1}{2}}{1}i\omega\lambda\right]\frac{F_1(\Omega)}{p^2a^2}=0,\quad \binom{47\text{a}}{47\text{b}}$$

where the upper and lower numbers arise from taking the positive and negative square root of Eq. (46). The lower equation does not involve the charge in any manner, while the upper equation does through the ω_1^2 term.

In order to study Eqs. (47a) and (47b), with particular emphasis on the undamped roots, it is necessary to use representations of $F(\Omega)$. The asymptotic series is particularly useful since the case $|\Omega|\gg 1$ includes the high-pressure limit. Even if $\lambda=0$ (low pressures), from the work of Sec. 4 we expect to find a solution if there are waves of frequency $\omega\gg pa$.

Equations (47a) and (47b) are, respectively [after multiplying by $(\lambda-i\omega)^2$],

$$-\omega^2+\frac{p^2a^2}{(\lambda-i\omega)^2}(\lambda^2-4i\omega\lambda)+\frac{3p^4a^4}{(\lambda-i\omega)^4}(-\lambda^2+6i\omega\lambda)+\cdots=0,\quad (48\text{a})$$

$$\left(-\omega^2-\frac{i\omega\lambda}{2}+2\omega_1^2\right)+\frac{p^2a^2}{(\lambda-i\omega)^2}\left(\lambda^2-\frac{5}{2}i\omega\lambda-6\omega_1^2\right)+\frac{3p^4a^4}{(\lambda-i\omega)^4}\left(-\lambda^2+10\omega_1^2+\frac{7}{2}i\omega\lambda\right)=0.\quad (48\text{b})$$

Let us first examine Eq. (48a). As $\lambda\rightarrow\infty$, $\omega^2\rightarrow p^2a^2$, which is the correct isothermal dispersion relation for sound waves. The complete dispersion relation (48a) has of course infinitely many roots. We are interested in those roots which are approximately undamped, since in the absence of undamped roots a detailed consideration of all the roots and the boundary conditions is necessary. It is clear that as $\lambda\rightarrow\infty$ there are slightly damped sound waves and that the frequency may be expressed as a power series in (pa/λ). We find, for $pa/\lambda\ll 1$,

$$\omega=pa\left[1-i\frac{pa}{\lambda}+\tfrac{1}{2}\left(\frac{pa}{\lambda}\right)^2+\cdots\right].\quad (49)$$

Examination of the full dispersion relation (47a) shows that the damping increases as pa/λ becomes comparable to unity and that other roots play a role. There are no undamped roots for low collision frequencies. Since $L_{11}=\pm L_{21}$ the ratio ν_2/ν_1 is ± 1 for the equal-mass case. For sound waves $\nu_2/\nu_1=+1$ showing that the two components are fully locked in phase. We have

$$\phi_1=\nu_1\lambda\left[1+\frac{mu}{kT}\frac{\omega}{p}\right]\Big/[\lambda+i(pu-\omega)].$$

Now examine Eq. (47b). For $\lambda\rightarrow 0$, we obtain[7]

$$\omega^2=2\omega_1^2\left(1+\frac{3p^2a^2}{\omega^2}+\cdots\right).$$

This is the correct dispersion relation for the case of equal masses in absence of collisions. If the collision frequency is small one can obtain corrections to Eq. (47b) as a power series in λ/ω_1. We consider the matter from a slightly different point of view. Undamped roots are obtained only for $pa\ll\omega_1$; one first sets $pa=0$. The solution of (48b) is

$$2\omega=-\frac{i\lambda}{2}\pm 2\omega_1\sqrt{2}\left[1-\frac{\lambda^2}{32\omega_1^2}\right]^{\frac{1}{2}}.$$

The roots with positive and negative sign will be discussed separately.

For the negative root we have $(\lambda\ll\omega_1)$

$$2\omega=-\frac{i\lambda}{2}-2\omega_1\sqrt{2}[1-(\lambda^2/64\omega_1^2)+\cdots].$$

[7] Note that $\omega\rightarrow\sqrt{2}\omega_1$ so that $\lambda/\omega_1\sim\sqrt{n}\rightarrow 0$ at low densities.

The real part of the frequency is negative; the damping is proportional to the collision frequency. There are also corrections to the real part starting with the second power of the collision frequency. As λ/ω_1 increases so does the damping. For $\lambda^2 \geqslant 32\omega_1^2$ the root is purely imaginary; the imaginary part of ω ranges between $-\lambda/4$ and $-\lambda/2$ as λ/ω_1 varies. The complete Eq. (48b) for $pa\neq 0$ has a solution which is approximately the one we have been considering. Thus if $pa \ll \omega_1$ one finds that $p^2a^2/(\lambda - i\omega)^2 \ll 1$ for all values of λ so that the second and third terms of (48b) give only small corrections.

The positive root has a behavior similar to that of the negative root for $\lambda \ll 4\sqrt{2}\omega_1$. For $\lambda \gg 4\sqrt{2}\omega_1$, however, one has

$$\omega = -i4\omega_1^2/\lambda.$$

This is purely imaginary, and the damping decreases with increasing collision frequency. Nevertheless there is no wave propagated since the real part of ω is zero. In the case of a neutral gas ($\omega_1 = 0$) the root is identically zero. For $pa\neq 0$, we again verify that $|pa/\lambda - i\omega| \ll 1$ for all λ when $pa \ll \omega_1$. For $\lambda \gg \omega_1$,

$$\omega \to -i(4\omega_1^2 + 2p^2a^2)/\lambda.$$

There is thus only a small correction to the imaginary part. In higher approximations there will be contributions to the real part of ω. These are smaller than the imaginary part so that the damping per cycle is overwhelming and waves are not propagated.

In view of the fact that the low-pressure "ion" modes are strongly dampled for equal masses, the additional damping due to collisions will not be discussed.

3. Transport Approximations

In this section we discuss an approximation in which the plasma is viewed as consisting of an electron- and an ion fluid coupled by collisions and by electrical restoring forces. Thus the ion-ion and electron-electron collision frequencies are infinite while the electron-ion collision frequency is arbitrary. We are particularly interested in the manner in which the high-pressure sound waves are modified due to the diffusion and polarization effects. The additional modifications arising from the viscosities and thermal conductivities of the individual components, can be treated with the full set of Eqs. (36)–(42). Since the transport approximation leads to tractable dispersion relations it is useful as orientation.

The transport approximation is essentially that made in the treatments of Thomson and Thomson.[8] For this case λ_1 and λ_2 are infinite so that only the lowest approximation in the asymptotic series of $F(\Omega)$ remains. Then

$$\left(\omega^2 - p^2a_1^2 - \omega_1^2 + \frac{Ani\omega}{m_1}\right)\left(\omega^2 - p^2a_2^2 - \omega_2^2 + \frac{Ani\omega}{m_2}\right) = \left(\omega_1^2 - \frac{Ani\omega}{m_1}\right)\left(\omega_2^2 - \frac{Ani\omega}{m_2}\right), \quad (50)$$

with

$$\frac{A}{m_1} = \frac{\alpha_{12}}{\sigma_{21}} = \left(\frac{m_2}{m_1+m_2}\right)\left(\frac{1}{\sigma_{21}}\right).$$

Thus An/m_1 is the electron ion collision frequency. This is the form obtained if one uses the equations of Thomson and Thomson, retaining the collision terms which they discard in their subsequent analysis.

For an initial value problem, Eq. (50) is to be regarded as an equation for ω as a function of p. It is then a quartic equation and is best investigated generally by numerical methods. We wish, however, to study the general features so as to obtain insight into the mathematical treatment of the complete dispersion relation.

Certain special cases are simple. If the masses are equal,

$$\omega^2 - \omega_1^2 - p^2a_1^2 + (Ani\omega/m_1) = \pm[\omega_1^2 - (Ani\omega/m_1)], \quad (51)$$

or

$$\omega^2 = p^2a^2,$$
$$\omega^2 - p^2a_1^2 - 2\omega_1^2 + (2A/m_1)ni\omega = 0.$$

The first equation represents the sound mode. For the present "two-fluid" model there is no effect of collisions, i.e., damping or dispersion. By way of contrast, in Sec. 4 we found that for the rigorous kinetic equations there was no positive ion mode for equal masses. The fact that the sound mode does not involve the electric charge is special to the case of equal masses. The second equation gives a plasma dispersion relation involving the electric charge. When $An/m_1 > \omega_1$, one finds highly damped roots as discussed earlier.

For unequal masses we order the terms according to the power of the collision frequency.

$$Ani\omega\left(\frac{\omega^2 - p^2a_2^2}{m_1} + \frac{\omega^2 - p^2a_1^2}{m_2}\right) = \omega_1^2\omega_2^2 - (\omega^2 - p^2a_2^2 - \omega_2^2)(\omega^2 - p^2a_1^2 - \omega_1^2).$$

As $An \to 0$, one has $(\omega^2 - p^2a_2^2 - \omega_2^2)(\omega^2 - p^2a_1^2 - \omega_1^2) = \omega_1^2\omega_2^2$. This is a dispersion relation for coupled electron and ion modes. When An is small the left-hand side may be treated by a method of successive approximations starting with one of the solutions for $An = 0$. The effect of collisions is to cause damping. However, in this range the dispersion relation has little relation to reality since it is not permissible to let $An \to 0$ while retaining the assumption of infinite collision frequencies λ_1, λ_2.

As $An \to \infty$ the expression inside the brackets on the

[8] J. J. Thomson and G. P. Thomson, *Conduction of Electricity through Gases* (Cambridge University Press, New York, 1933), Vol. 2, p. 353.

left-hand side must tend to zero. This gives

$$\omega^2 \to p^2 \cdot 2kT/(m_1+m_2),$$

which is the correct sound speed for a mixture in the isothermal case.

When An is large but not infinite, it is possible to obtain a solution in which the frequency ω is expanded directly in inverse powers of the collision frequency. In order to bring out the relative roles of the plasma frequencies and of the quantities pa_i, we proceed in a different manner.

Note that Eq. (50) is a quadratic equation for $p^2(\omega)$. This point of view is appropriate when one considers a boundary value problem. The solution of the quadratic equation is

$$p^2 = \frac{\omega^2(m_1+m_2)+2(Ani\omega-4\pi ne^2)}{2kT} \pm \frac{[\omega^4(m_1-m_2)^2+4(Ani\omega-4\pi ne^2)^2]^{\frac{1}{2}}}{2kT}.$$

To study the behavior further one expands the terms in the square root. the expansion parameter is $\zeta=\omega^2(m_1-m_2)/2(Ani\omega-4\pi ne^2)$. The negative root is

$$2p_-{}^2 = \frac{\omega^2(m_1+m_2)}{kT} + \frac{\omega^2(m_1-m_2)}{2kT}\zeta(1+\zeta^2/2).$$

Since $4\pi ne^2$ is real and $Ani\omega$ is imaginary the expansion in powers of ζ converges if (a) $\omega^2 < 2An\omega/|m_1-m_2|$, i.e., frequency less than m_1/m_2 times the collision frequency, or (b) $\omega^2|m_1-m_2| < 4\pi ne^2$, i.e., frequency less than ion plasma frequency. If further, $4\pi e^2 < 2A\omega$, an expansion in terms of the inverse collision frequency is possible. This limit requires both very high frequency and high density. It is hardly ever realized in practice. The expansion is

$$2p_-{}^2 = \frac{\omega^2(m_1+m_2)}{kT} + \frac{i\omega^4(m_1-m_2)^3}{4kTAn\omega} \times \left(1-\frac{i4\pi e^2}{A\omega}+\cdots\right) - \frac{i\omega^8(m_1-m_2)^4}{32kT(An\omega)^3}.$$

The first term yields damping, the second a correction to the real part of p. The effects of the charge enter in the second power of the inverse collision frequency, as a dispersive correction. The term involving the first power of the inverse collision frequency contains the mass difference. Thus at high pressures the absorption is mainly due to diffusion. The higher corrections, of course, yield contributions of both diffusion and polarization to the absorption and dispersion.

When $4\pi e^2 > |2A\omega|$, one can obtain the dispersion relation as a series of inverse powers of $4\pi e^2$. This low-frequency and moderate- or high-density case occurs more often. One finds

$$p_-{}^2 = \frac{\omega^2(m_1+m_2)}{2kT} - \frac{\omega^4(m_1-m_2)^2}{4\cdot 4\pi ne^2} \times\left[1+\frac{iA\omega}{4\pi e^2}-\left(\frac{A\omega}{4\pi e^2}\right)^2+\cdots\right]. \quad (53a)$$

The first term in the absorption is of the order of $(\omega^2/\omega_2{}^2)(A\omega/4\pi e^2)$ times the real part of p^2. The qualitative behavior of an ionized gas thus depends on both the frequency of the waves considered and on the density. The region where frequency, collision frequency, and plasma ion frequency are comparable is one of heavy damping.

We now return to the dispersion relation (50) considered from the $\omega(p)$ point of view. Introduce the symbol

$$\xi = 4\pi ne^2 - Ani\omega,$$

and consider Eq. (50) as a quadratic for ω^2 (in spite of the fact that it contains ω). Then

$$2\omega^2 = (\xi+p^2kT)\left(\frac{1}{m_1}+\frac{1}{m_2}\right) \pm\left[(\xi+p^2kT)^2\left(\frac{1}{m_1}-\frac{1}{m_2}\right)^2+\frac{4\xi^2}{m_1m_2}\right]^{\frac{1}{2}}. \quad (54)$$

The positive roots yield plasma oscillations in the low-pressure region but are heavily damped at high pressures; therefore consider the negative sign.

At high pressures, $|p^2kT+\xi| \approx |\xi|$ for long wavelengths. Note that with $m_1 \ll m_2$ the second term under the square root is of the order of m_1/m_2 times the first. The development of the square root yields

$$\omega^2 = \frac{\xi+p^2kT}{m_2} - \frac{2\xi^2}{(\xi+p^2kT)(m_2-m_1)} + \cdots. \quad (55)$$

We may develop in powers of m_1/m_2. To terms of order $(m_1/m_2)^0$,

$$\omega^2 = \frac{2p^2kT}{m_2} - \frac{(p^2kT)^2}{m_2\xi} + \frac{(p^2kT)^3}{m_2\xi^2}. \quad (56)$$

The discussion from this point on is similar to that given earlier for when $4\pi e^2 \ll A\omega$, we expand in inverse powers of the collision frequency:

$$\omega^2 = \frac{2p^2kT}{m_2} - i\frac{p^3(kT)^{\frac{3}{2}}}{m_2{}^{\frac{1}{2}}An\sqrt{2}} - \frac{2\pi ne^2p^2kT}{A^2n^2} - \frac{3}{8}\frac{p^4(kT)^2}{A^2n^2}. \quad (57)$$

The first inverse power of the collision frequency again does not involve the charge; the second power does. Note that the expansion in powers of $1/An$ is not the same as an expansion of powers of p^2kT. Thus there are corrections to the p^2kT term which depend on $4\pi e^2/A^2n$. If the charge is zero, the expansions are the same.

The case in which $4\pi e^2 \gg A\omega$ can be handled by suitable expansions starting from Eq. (55). In like manner one may find the corrections coming from higher powers of m_1/m_2.

7. DISPERSION RELATION FOR THE ISOTHERMAL CASE

We now discuss briefly the complete dispersion relation for the isothermal case, Eq. (46). Our treatment will make use of the asymptotic expansions of both $F(\Omega_1)$ and $F(\Omega_2)$; these expansions are valid when $|(\lambda_j - i\omega)/pa_j\sqrt{2}| \gg 1$. The case of high collision frequencies is included in the following development, and we may examine the effects of finite mean free path and electric charge on the sound propagation. In addition, at low collision frequencies the conditions for the asymptotic expansion can be satisfied if $|pa_j| \ll |\omega|$. This is the case for the electron plasma modes where $\omega \approx \omega_1$ provided one deals with waves longer than the electronic Debye length. It is interesting that such different types of waves come from the same dispersion relation. The frequencies of the electron plasma modes are practically independent of wavelength while the sound frequencies are proportional to the magnitude of the propagation vector. The region where frequency, collision frequencies, and plasma frequency are comparable is characterized by heavy damping, so that an ionized gas does not propagate organized motions in this region.

The expressions for L_{jj} and L_{jk} are given in Eq. (45).

We form the appropriate products in Eq. (46) and group terms according to powers of $p^2a_j^2$. Retaining terms of order p^2, one finds

$$\omega^2\left(\omega^2 + \frac{i\omega n}{\sigma_{21}} - \omega_1^2 - \omega_2^2\right) + \sum_j \frac{p^2 a_j^2}{(\lambda_j - i\omega)^2} \times \left\{3\omega_j^2\omega^2 - \frac{3i\omega^3 n}{\sigma_{kj}}\alpha_{kk} + [\lambda_j^2 - 4i\omega\lambda_j] \times \left[\omega_k^2 - \omega^2 - \frac{i\omega n}{\sigma_{kj}}\alpha_{jj}\right]\right\}. \quad (58)$$

Consider first the waves of infinite length, $(pa_j = 0)$. The solutions are $\omega^2 = 0$ and $\omega^2 + i\omega(n/\sigma_{21}) - \omega_1^2 - \omega_2^2 = 0$. For zero collision frequency, the second type is an electronic plasma wave of frequency ω_1. The term ω_2^2 is a small correction arising from the presence of the positive ions. For nonzero collision frequency the discussion follows that of Sec. 6.2 for the equal mass case. The waves are appreciably damped when the collision frequency is comparable to the electronic plasma frequency. For both signs of the square root the damping per cycle is large, for collision frequencies greater than the electronic plasma frequency. It is easy to see the character of these modes for finite wavelengths. We have, at zero collision frequency,

$$\omega^2 = \omega_1^2 + \omega_2^2 + \frac{3p^2}{\omega^2}(a_2^2\omega_2^2 + a_1^2\omega_1^2) + \cdots. \quad (59)$$

The second term introduces a small wavelength- and temperature-dependent effect which is small for waves longer than the Debye length. The contribution of the positive ions is again small. The damping arising from collisions is still given in good approximation by the term independent of wavelength.

The opposite case is that of high pressures. From our treatment of the transport approximation, we know that the grouping of terms in powers of p^2 differs from that in terms of powers of the collision frequency. For high collision frequencies, one must set equal to zero the coefficients of the terms proportional to the collision frequency. There are such terms contained in the constant and the p^2 group of terms of Eq. (58), but none in the groups for higher powers of p^2. We find

$$\omega^2 = p^2(a_2^2\alpha_{12} + a_1^2\alpha_{21}) = 2p^2kT/(m_1 + m_2),$$

since

$$\alpha_{21} = m_1/(m_1 + m_2), \quad \alpha_{12} = m_2/(m_1 + m_2);$$

i.e., we have the correct dispersion relation for a mixture of particles of equal concentrations under the isothermal assumptions. Here one obtains sound waves; the charge of nature of the particles does not enter since the two components are "locked together." In this limit of infinite collision frequencies,

$$\nu_2/\nu_1 = 1, \quad \phi_i = \nu_i[1 + (u_i c/a_i^2)],$$

where c is the isothermal sound speed. This is the linearized locally Maxwellian distribution. To gain insight into the decoupling of the two charged components as the collision frequency decreases, one can find the frequency as a power series in the inverse collision frequency. In order to do this consistently, we must expand ω wherever it occurs. To second order, it is necessary to include contributions from terms not written in Eq. (58). As in the transport approximation, the first power does not contain the charge, the second power gives a charged correction proportional to p^2. From the work of Sec. 6.3 we know that there is another high-pressure, low-frequency limit in which charge effects play a more dominant role. This is the case $4\pi e^2 > A\omega$, now modified by the viscosity of the individual components. The transport approximation is reached when $\lambda_1, \lambda_2 \rightarrow \infty$ in Eq. (58). The detailed and intricate discussion of the relative roles of diffusion, polarization, heat conductivity, and viscosity for the absorption and dispersion of the "high-pressure" ionized gas will be given elsewhere.