Synthesis of Covariant Particle Equations*

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(Received January 9, 1956)

By combining two irreducible representations of the proper inhomogeneous Lorentz group, certain irreducible unitary representations of the complete Lorentz group including space and time inversion are obtained, together with a Schrödinger equation whose solutions constitute the representation space for these representations. The representations thus define a "canonical" form for covariant particle theories, in which not only the wave equations but the manner in which the wave functions transform under Lorentz transformations is prescribed. It is shown that by a suitable choice of representation, the Dirac, Klein-Gordon, and Proca equations can all be reduced to this canonical form. It is further shown that in the representation space provided, several possibilities exist for the identification of the transformations to be associated with space inversion, time inversion, and charge conjugation, thus suggesting the existence of several distinct relativistic theories for particles of any given spin. Conjectures are made as to the physical significance of these different possibilities when the equations are second-quantized. It is shown that each of the conventional theories employs only one of the available possibilities for these transformations, the choices being different for integral and half-integral spin theories.

INTRODUCTION

THE one-particle equations[1] which, when appropriately second-quantized, seem most promising of providing a permanent basis for describing actual elementary particles are the Klein-Gordon, the Dirac, and the Proca equations,[2] the associated descriptions being relevant to particles of spin 0, $\frac{1}{2}$, and 1, respectively. Whether the same can be said of equations to describe particles of higher spin such as those of Dirac, Fierz, and Pauli,[3] is not so clear. All of these equations have the common property that they are invariant under the transformations of the inhomogeneous Lorentz group, or more explicitly, that their solutions constitute a representation space for this group. On the other hand, it is not at all clear as to the extent to which the condition of invariance under this group of transformations singles out the above equations for their special role. It is to this question, and some of its ramifications, that the present paper is devoted.

One of the considerations which led to the present investigation was the question as to what analogies could be established between the three fundamental equations referred to above. While there have been a number of investigations relevant to this question, we believe that the approach presented here differs sensibly from what has been done in the past. The recent work of Case[4] in showing that the Klein-Gordon and Proca equations could be cast into a form in close analogy with the Foldy-Wouthuysen[5] representation, together with the known relation of the latter to the irreducible representations of the proper inhomogeneous Lorentz group determined by Wigner and Bargmann[6] provided the clue for establishing a very close analogy indeed between these various equations. We shall demonstrate below that this analogy is even closer than an identity in form of the equations; it extends to the manner in which the wave functions in each of these theories transforms under the transformations of the Lorentz group. The common form to which the three equations can be reduced is characterized by the fact that it parallels exactly the reduction of the representations of the Lorentz group provided by these equations into the two irreducible Wigner-Bargmann representations of which they are compounded. Thus, this canonical form of these equations, as we shall call it, would appear to be one of the most promising approaches to any fundamental analysis of the familiar covariant particle equations.

The present paper will be limited to the discussion of one-particle equations describing primarily charged particles of finite mass and noninfinite spin. Much of what we do can be taken over completely for the case of neutral particles, but certain theories of neutral particles (such as the Majorana theory) are not encompassed by our analysis. The subject of the second quantization

* Supported in part by the U. S. Atomic Energy Commission and by the National Science Foundation.

[1] The literature on covariant particle equations is very extensive, and much of it is not directly pertinent to the contents of the present paper. A rather complete bibliography on this subject is contained in the volume by E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave Equations* (Blackie and Son, Ltd., London and Glasgow, 1953). Further recent papers which may be of particular value to the reader, apart from those specifically referred to in the text are: S. Watanabe, Revs. Modern Phys. **27**, 26, 40, 179 (1955); R. H. Good, Revs. Modern Phys. **27**, 187 (1955); R. Haag, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **29**, No. 6 (1955); A. S. Wightman and S. S. Schweber, Phys. Rev. **98**, 812 (1955).

[2] W. Pauli, Revs. Modern Phys. **13**, 203 (1941).

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **A155**, 447 (1936); M. Fierz, Helv. Phys. Acta **12**, 297 (1939); M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, 211 (1939).

[4] K. M. Case, Phys. Rev. **95**, 1323 (1954). The author understands that similar results were obtained by E. J. Kelly in his doctoral dissertation at Massachusetts Institute of Technology.

[5] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950); S. Tani, Progr. Theoret. Phys. (Japan) **6**, 267 (1951); R. Becker, Nachr. Akad. Wiss. Göttingen, Math.-physik., Kl., No. 1, 20 (1945).

[6] E. P. Wigner, Ann. of Math. **40**, 149 (1939); V. Bargmann, Ann. of Math. **48**, 568 (1947); V. Bargmann and E. P. Wigner, Proc. Nat. Acad. Sci. U. S. **34**, 211 (1948). See also E. P. Wigner, Z. Physik **124**, 665 (1947).

of the equations treated and their consequent extension into field theories will not be considered here but will be treated in a later publication.

The outline of this paper is as follows: We consider first certain of the irreducible representations of the proper inhomogeneous Lorentz group derived by Wigner and Bargmann in the context of a Schrödinger coordinate representation (as contrasted with the Heisenberg momentum representation which is essentially employed by these authors). We show that the corresponding representation space admits a representation of the space inversion transformation by a unitary transformation. On the other hand, the time inversion transformation can be represented only by an antiunitary[7] transformation of the Wigner type. The covariant one-particle equations associated with these representations do not include the three familiar equations mentioned above. To obtain these last equations, without obtaining the former, we may impose the requirement that they provide one with an irreducible representation space for a completely *unitary* representation of the complete Lorentz group, including time inversion. We derive such equations by combining two of the Wigner-Bargmann irreducible representations. In this way a "canonical form" is established for theories of this character which includes not only a canonical form for the equation but for the associated transformations under the Lorentz group. This canonical form is essentially unique except for the matrix to be associated with space inversion. It is shown that an antiunitary transformation satisfying the relations required of a time inversion transformation continues to exist in the representation space and is identified as the Wigner time inversion transformation as contrasted with the unitary transformation which we call the Pauli time inversion transformation. Similarly, there exists in this representation space another antiunitary transformation which is identified with the charge conjugation transformation. The formal requirements also do not determine uniquely certain matrices associated with these latter transformations. For the three matrices involved in space inversion, Wigner time inversion, and charge conjugation, apart from a unitary equivalence and an arbitrary phase factor, the arbitrariness in the choice of each is twofold. Thus, apart from a unitary equivalence and arbitrary phase factors, there are found to exist essentially eight distinct theories for a particle of definite mass and spin satisfying the requirements we have imposed. Following this it is shown that the conventional Dirac, Klein-Gordon, and Proca theories can all be put into the above canonical form by an appropriate choice of representation. Each of these theories corresponds to a realization of only one of the eight possibilities which appear to be acceptable on the grounds of covariance alone. The physical difference between the conventional theories and the possible alternatives, and the possibility of realizing the alternative theories are discussed briefly but without reaching any definite conclusion. It is suggested that if some of these alternate theories can indeed be crystallized into a physically consistent theory, then quantization according to either Bose-Einstein or Fermi-Dirac statistics is possible for particles of any spin.

[7] An antiunitary transformation is one which transforms the scaler product into its complex conjugate and carries a linear combination of two vectors into the same linear combination of their transforms but with the complex conjugate coefficients. Its form is that of a unitary transformation applied to the complex conjugate vector.

In what follows we shall consistently employ units in which $\hbar$ and c are unity. Certain groups of equations which may often be referred to collectively are designated by a capital letter as a prefix to the number of each of the equations of the group, but all equations are nevertheless numbered in consecutive order.

THE LORENTZ GROUP

We shall use the term *inhomogeneous proper Lorentz group* to describe the group formed by the space and time translations and those Lorentz transformations which are continuously connected to the identity. (The adjective *inhomogeneous* will usually be omitted since we shall have little occasion to refer to the homogeneous Lorentz group.) The above group is a ten-parameter continuous group whose infinitesimal generators may be taken to be the following[8]: the generators of infinitesimal translations along the three coordinate axes, $\mathbf{P}=(P_1,P_2,P_3)$; the generator of an infinitesimal time translation, H; the generators of infinitesimal rotations about the three coordinate axes, $\mathbf{J}=(J_1,J_2,J_3)$; and the generators of infinitesimal Lornetz transformations along the three coordinate axes, $\mathbf{K}=(K_1,K_2,K_3)$. We shall use the symbol L to refer to any of these ten generators. The infinitesimal transformation Λ associated with any of the ten generators can then be written $\Lambda=(1+i\epsilon L)$, where ϵ is infinitesimal.

The ten generators satisfy the following commutation relations which can be thought of as abstractly defining the inhomogeneous proper group:

$$[P_i,P_j]=0, \tag{A-1}$$

$$[P_i,H]=0, \tag{A-2}$$

$$[J_i,P_j]=i\epsilon_{ijk}P_k, \tag{A-3}$$

$$[J_i,H]=0, \tag{A-4}$$

$$[J_i,J_j]=i\epsilon_{ijk}J_k, \tag{A-5}$$

$$[P_i,K_j]=-i\delta_{ij}H, \tag{A-6}$$

$$[H,K_i]=-iP_i, \tag{A-7}$$

$$[J_i,K_j]=i\epsilon_{ijk}K_k, \tag{A-8}$$

$$[K_i,K_j]=-i\epsilon_{ijk}J_k. \tag{A-9}$$

[8] See reference 6; also, P. A. M. Dirac, Revs. Modern Phys. **21**, 392 (1949).

In the above, $[A,B]=AB-BA$, δ_{ij} is the Kronecker symbol, ϵ_{ijk} is the Levi-Civita three-index symbol, and the summation convention is employed for repeated indices.

When the space inversion transformation is adjoined as a generating element to the proper Lorentz group, we obtain a group which we shall simply call the (inhomogeneous) *Lorentz group*. The transformation S satisfies the following relations:

$$S(1+i\epsilon P_i)=(1-i\epsilon P_i)S, \tag{A'-10}$$

$$S(1+i\epsilon H)=(1+i\epsilon H)S, \tag{A'-11}$$

$$S(1+i\epsilon J_i)=(1+i\epsilon J_i)S, \tag{A'-12}$$

$$S(1+i\epsilon K_i)=(1-i\epsilon K_i)S, \tag{A'-13}$$

$$S^2=1. \tag{A'-14}$$

When the time inversion transformation T is adjoined as a generator to the Lorentz group we obtain a group which we shall refer to as the *full Lorentz group*. The transformation T satisfies the following relations:

$$T(1+i\epsilon P_i)=(1+i\epsilon P_i)T, \tag{A''-15}$$

$$T(1+i\epsilon H)=(1-i\epsilon H)T, \tag{A''-16}$$

$$T(1+i\epsilon J_i)=(1+i\epsilon J_i)T, \tag{A''-17}$$

$$T(1+i\epsilon K_i)=(1-i\epsilon K_i)T, \tag{A''-18}$$

$$TS=ST, \tag{A''-19}$$

$$T^2=1. \tag{A''-20}$$

IRREDUCIBLE REPRESENTATIONS OF THE INHOMOGENEOUS LORENTZ GROUP

The irreducible representations of the inhomogeneous Lorentz group have been studied by Wigner and Bargmann.[6] Our present discussion will be confined to those representations which are suitable for describing a particle of nonvanishing mass and noninfinite spin. We shall discuss these representations from the viewpoint of a covariant Schrödinger wave equation whose solutions form a representation space for these irreducible representations. The representations obtained by Wigner and Bargmann are essentially in a Heisenberg momentum representation.

Each of the representations in which we are interested is infinite dimensional and unitary (insofar as the proper group is concerned), and each inequivalent irreducible representation is specified by a *mass* m which may take any real positive value and a *spin* s which may take any positive integral or half-integral value including zero. A representation space for the representation (m,s) is provided by the solutions of the following Schrödinger equation:

$$i\partial\chi(\mathbf{r},t)/\partial t=\omega\chi(\mathbf{r},t). \tag{21}$$

In this equation $\chi(\mathbf{r},t)$ is a $(2s+1)$-component wave function and the operator ω is defined by

$$\omega\equiv(m^2+p^2)^{\frac{1}{2}},\quad p\equiv-i\nabla. \tag{22}$$

This operator is a linear integral operator whose precise definition is

$$\omega\chi(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int(m^2+k^2)^{\frac{1}{2}}e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')}\chi(\mathbf{r}',t)d\mathbf{r}'d\mathbf{k}. \tag{23}$$

The solutions of Eq. (21) form a Hilbert space with a scalar product defined by

$$(\chi_a,\chi_b)=\int\chi_a^*(\mathbf{r},t)\chi_b(\mathbf{r},t)d\mathbf{r}. \tag{24}$$

A unitary representation of the inhomogeneous proper Lorentz group is provided by this space through the following identification of the generators of the infinitesimal transformations of the group:

$$\mathbf{P}=\mathbf{p}, \tag{B-25}$$

$$H=\omega, \tag{B-26}$$

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{s}, \tag{B-27}$$

$$\mathbf{K}=\tfrac{1}{2}(\mathbf{r}\omega+\omega\mathbf{r})-\frac{\mathbf{s}\times\mathbf{p}}{m+\omega}-t\mathbf{p}. \tag{B-28}$$

Here $\mathbf{s}\equiv(s_1,s_2,s_3)$ are three $(2s+1)\times(2s+1)$ irreducible Hermitian matrices satisfying the commutation relations

$$[s_i,s_j]=i\epsilon_{ijk}s_k, \tag{29}$$

so that they are simply the infinitesimal generators of an irreducible $(2s+1)$-dimensional representation of the three-dimensional rotation group. It is well known that there is one and only one irreducible representation of these matrices of each dimensionality,[9] so that these matrices are just the spin matrices corresponding to the spin s.

From (29) and the commutation relation

$$[p_i,r_j]=-i\delta_{ij}, \tag{30}$$

one can easily verify that the identifications (B) do indeed satisfy the commutation relations (A) as required. For the representation space to be indeed the space spanned by solutions of Eq. (21) one must further verify that this equation is left invariant under the transformations

$$\chi\rightarrow\chi'=\Lambda\chi=(1+i\epsilon L)\chi, \tag{31}$$

where ϵ is a real infinitesimal and L is any of the operators (B). This means that χ' as given in (31)

[9] H. Weyl, *The Theory of Groups and Quantum Mechanics* (E. P. Dutton and Company, New York, 1932); E. P. Wigner, *Gruppentheorie und ihre Anwendung* (Friedrich Vieweg und Sohn, Braunschweig, 1931). The two-valued representations of the rotation group (half-integer spin) are here admitted, of course.

must also satisfy Eq. (21); it is easily verified that this is indeed the case for each of the ten generators.

We shall simply assert without proof that the representations given above are indeed irreducible and hence equivalent to certain of the representations obtained by Wigner and Bargmann. These authors have also shown that for each (m,s) there is one and only one such representation to within a unitary or antiunitary equivalence transformation. We note, however, for later reference that there exists an antiunitary transformation[10] which transforms Eq. (21) to

$$i\partial\chi(\mathbf{r},t)/\partial t=-\omega\chi(\mathbf{r},t), \tag{32}$$

and H and $\mathbf{K}$ to

$$H=-\omega, \tag{33}$$

$$\mathbf{K}=-\tfrac{1}{2}(\mathbf{r}\omega+\omega\mathbf{r})+\frac{\mathbf{s}\times\mathbf{p}}{m+\omega}-t\mathbf{p}, \tag{34}$$

while leaving $\mathbf{P}$ and $\mathbf{J}$ unchanged.

We shall now show that there exists a unitary transformation in the representation space we have defined above which can be employed to represent the space inversion transformation S. We must note first, however, that in quantum mechanics one requires only a *ray* representation, not necessarily a *vector* representation, of a group. This follows from the fact that two vectors χ and $e^{i\alpha}\chi$ represent the same state. For those elements of the group which are not continuously connected with the identity, such as S, we are then required to impose only the weaker condition

$$S^2\sim 1, \tag{35}$$

in place of (A′-14), where the symbol $\sim$ means "equal to within a multiplicative factor of unit modulus." The possibility of employing ray rather than vector representations makes no difference in the representation of the elements of the *proper* group except for the fact that it gives us the freedom to use the two-valued representations of the three-dimensional rotation group for $\mathbf{s}$ as we have already assumed above.

A unitary transformation to represent S in our representation space is provided by the following identification:

$$S\chi(\mathbf{r},t)=e^{i\alpha_s}\chi(-\mathbf{r},t), \tag{36}$$

where α_s is an arbitrary real number which is undetermined in view of the weak condition (35). One can readily show that this transformation satisfies the required relations (A′) and that it leaves Eq. (21) invariant.

If we try to include a transformation to represent the time inversion transformation T in our representation space, we encounter a new feature in that there exists no *unitary* transformation which satisfies the required relations. To see this we note that if T is unitary then Eq. (A″-16) requires that T anticommute with H. This then means that if ϕ is an eigenvector of H belonging to the eigenvalue E, then $T\phi$ is an eigenvector of H belonging to the eigenvalue $-E$. But this is impossible, since from (B-26) we see that all the eigenvalues of H are positive. While at first sight it thus appears that the Hilbert space of solutions of Eq. (21) cannot provide us with a representation of the *full* Lorentz group, this is not really the case. For, as Wigner has pointed out, the physical interpretation of quantum mechanics is such that there is no reason why a disjoint transformation like T cannot be represented by an *antiunitary* transformation. Indeed, we shall now show that the representation space provided by Eq. (21) does admit an antiunitary transformation which satisfies the required relations (A″).

If T is represented by an antiunitary transformation, then the set of relations (A″) require

$$TP_i=-P_iT, \tag{C-37}$$

$$TH=HT, \tag{C-38}$$

$$TJ_i=-J_iT, \tag{C-39}$$

$$TK_i=K_iT, \tag{C-40}$$

$$TS\sim ST, \tag{C-41}$$

$$T^2\sim 1. \tag{C-42}$$

The first four of these follow from the fact that the complex conjugation involved in an antiunitary transformation leads to $T(1+i\epsilon L)=(T-i\epsilon TL)$. The last two weak relations again follow from the fact that we require only a ray representation of the group. The above conditions can be fulfilled through the following antiunitary transformation:

$$T\chi(\mathbf{r},t)=\tau\chi^*(\mathbf{r},-t), \tag{B''-43}$$

provided that there exists a $(2s+1)$-dimensional *unitary* matrix τ satisfying the following conditions:

$$\tau s_i^*\tau^{-1}=-s_i. \tag{44}$$

But it is easy to prove that such a matrix always exists as follows: Since the matrices $-s_i^*$ satisfy the same commutation relations as the s_i, and since there is only one irreducible representation of these commutation relations of each dimensionality, it follows that s_i and $-s_i^*$ are related by an equivalence transformation of the form (44). Combining (44) and the Hermitian conjugate equation, one finds using Schur's lemma that $\tau\tau^+$ is a real positive multiple of the identity, so that τ may always be chosen to be unitary. By a similar consideration of (44) and the transpose equation, one can show that τ is either symmetric or antisymmetric, which implies that

$$\tau\tau^*=\pm 1, \tag{45}$$

which allows (C-42) to be satisfied. One can show

[10] This transformation is $\chi\rightarrow\tau\chi^*$ where τ is the matrix defined by Eq. (44).

further that τ is unique to within a multiplicative factor of modulus unity and that it is symmetric for integral spin and antisymmetric for half-integral spin.

One can now verify that the transformation (B″-43) does indeed satisfy all of the required relations (C). The fact that (B″-43) leaves Eq. (21) invariant may be demonstrated by taking the complex conjugate of this equation, reversing the sign of t, and multiplying by the matrix τ. Thus so long as we do not require a completely unitary representation of the full Lorentz group, Eq. (21) does indeed provide us with a representation space for an irreducible representation of the full group.

While Eq. (21) thus provides one with completely Lorentz covariant theories of a free particle of any spin value, it does not encompass the familiar theories like the Dirac, Klein-Gordon, and Proca theories. In particular, the representation space does not provide one with a transformation corresponding to the charge-conjugation transformation in the familiar theories, and thus, presumably, even after second-quantization it corresponds to a theory in which particles are not necessarily accompanied by antiparticles. To what extent a completely satisfactory particle theory can be based on this equation is not known at the present time, but this represents a problem which is well worthy of further investigation. We shall not enter into this problem here but shall defer it to a later publication. It may not be amiss to mention here, nevertheless, that we have found that it is possible to build a completely relativistic many-particle theory with interaction based on this equation.

For the present, however, we shall return to our prime objective of discovering a basis for the synthesis of the familiar Dirac, Klein-Gordon, and Proca equations.

PAULI TIME INVERSION AND CHARGE CONJUGATION

By our earlier analysis we have been led to certain unfamiliar Lorentz covariant equations rather than to the familiar equations in which we are primarily interested. We shall now show that we can indeed obtain the equations of interest in a special representation by the imposition of the following requirement:

The solutions of the equation shall constitute a representation space for an irreducible *unitary* representation of the *full* Lorentz group.

This means then that we reject the antiunitary transformation that we have admitted previously to represent the time inversion transformation, and we must extend the representation space sufficiently that there exists a unitary transformation in the space which satisfies the relations (A″) for a time inversion transformation. We shall say that a theory satisfying the above requirement is *strongly* Lorentz covariant, while we shall use the term *weak* Lorentz covarance to describe the transformation properties of the theories we have considered earlier.

Since we shall find that an antiunitary transformation satisfying the condition (A″) will continue to exist even in our strongly covariant theories, it will be convenient to introduce nomenclature and notation to distinguish the two types of time inversion transformations we encounter. We shall continue to designate an *antiunitary* transformation satisfying the conditions (A″) and hence (C) by the letter T and shall call such a transformation a *Wigner* time inversion transformation, since it is closely related to the time inversion transformation introduced by Wigner in nonrelativistic theories.[11] On the other hand, we shall designate a *unitary* transformation satisfying the conditions (A″) by the latter Z, and shall call such a transformation a *Pauli* time inversion transformation since for the Dirac equation it is exactly the transformation introduced by Pauli in his investigation of the Lorentz covariance of this equation.[12] The transformation Z must then satisfy the following relations with the infinitesimal generators of the proper Lorentz group and with the space inversion transformation:

$$ZP_i = P_i Z, \qquad \text{(C′-46)}$$

$$ZH = -HZ, \qquad \text{(C′-47)}$$

$$ZJ_i = J_i Z, \qquad \text{(C′-48)}$$

$$ZK_i = -K_i Z, \qquad \text{(C′-49)}$$

$$ZS \sim SZ, \qquad \text{(C′-50)}$$

$$Z^2 \sim 1. \qquad \text{(C′-51)}$$

If, as we have asserted, the representation space provided by an equation satisfying the strong covariance requirements nevertheless contains a Wigner time inversion transformation T, then the transformation

$$C = ZT, \qquad (52)$$

is an antiunitary transformation which commutes with all infinitesimal Lorentz transformations $\Lambda = (1 + i\epsilon L)$, and thus satisfies the following relations:

$$CP_i = -P_i C, \qquad \text{(C″-53)}$$

$$CH = -HC, \qquad \text{(C″-54)}$$

$$CJ_i = -J_i C, \qquad \text{(C″-55)}$$

$$CK_i = -K_i C, \qquad \text{(C″-56)}$$

Furthermore, if T and Z commute[13] in the sense that

$$TZ \sim ZT, \qquad (57)$$

[11] E. P. Wigner, Gött Nachr. 546 (1932); see also, R. H. Good, reference 1.

[12] W. Pauli, *Handbuch der physik* (Verlag Julius Springer, Berlin, 1933), second edition, Vol. 24, part 1.

[13] Although we know of no compelling reason for assuming that the two types of time reversal shall commute in the sense of (57), it seems reasonable to make this assumption. It will be assumed in all our later work.

and we shall see that there always exists a transformation T having this property, then

$$CS\sim SC, \tag{C''-58}$$

$$CT\sim TC, \tag{C''-59}$$

$$CZ\sim ZC, \tag{C''-60}$$

$$C^2\sim 1. \tag{C''-61}$$

We shall call the transformation C having the properties (C″) the *charge conjugation transformation*. The justification for this identification is that the charge conjugation transformation as ordinarily defined for the one-particle Dirac, Klein-Gordon, and Proca equations does indeed have these properties. (When one goes to second-quantized theories one must carefully re-examine these definitions of time inversion and charge conjugation. This will not be done in the present paper.)

THE CANONICAL FORM

We shall now consider the problem of constructing equations satisfying the extended Lorentz covariance requirements by a method which involves combining two antiunitarily equivalent irreducible representations of the type we have obtained previously. We do not know whether this method will lead us to all equations of this type which are suitable for describing charged particles of finite mass and spin, but we strongly suspect that this is indeed the case.

The clue to how we must proceed is provided by the fact that the additional requirements of strong covariance postulates the existence of a transformation Z which anticommutes with H. As we have noted earlier, this implies that the eigenvalues of H must occur in pairs, equal in magnitude but opposite in sign. This suggests then combining Eq. (21) and the representation of the proper Lorentz group which it provides with its antiunitary equivalent Eq. (32) and the representation it provides as given in Eqs. (33) and (34). To this end, for the description of a particle of mass m and spin s we assume a $2(2s+1)$-component wave function $\chi(\mathbf{r},t)$, now satisfying the equation

$$i\partial\chi(\mathbf{r},t)/\partial t=\beta\omega\chi(\mathbf{r},t), \tag{62}$$

where β is a $2(2s+1)\times 2(2s+1)$ diagonal Hermitian matrix of the form

$$\beta=\begin{pmatrix}1' & 0'\\ 0' & -1'\end{pmatrix}, \tag{63}$$

where $0'$ and $1'$ are respectively, the null and unit matrices of rank $2s+1$. Then in view of our earlier results, the solutions of this equation provide us with the following (reducible) representation of the infinitesimal generators of the proper Lorentz group:

$$\mathbf{P}=\mathbf{p}, \tag{D-64}$$

$$H=\beta\omega, \tag{D-65}$$

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{s}, \tag{D-66}$$

$$\mathbf{K}=\frac{\beta}{2}(\mathbf{r}\omega+\omega\mathbf{r})-\frac{\beta\mathbf{s}\times\mathbf{p}}{m+\omega}-t\mathbf{p}. \tag{D-67}$$

In the foregoing, the s_i are now matrices of rank $2(2s+1)$ of the form

$$s_i=\begin{pmatrix}s_i' & 0'\\ 0' & s_i'\end{pmatrix}, \tag{68}$$

where the s_i' are the matrices of rank $2s+1$ which we previously designated by s_i. The new matrices s_i continue to satisfy the commutation relations (29), of course. (Note that we could have employed the freedom we have available to use two different but unitarily equivalent representations of s_i' for the two matrices occurring in (68); however, since there exists a unitary transformation in that case which would restore (68) to the form above without changing the form of β, we actually lose no generality by our special choice.)

We now turn our attention to the representation of the space inversion transformation S. One easily finds that if it is represented by a unitary transformation as we require, then its representation must be of the form

$$S\chi(\mathbf{r},t)=\sigma\chi(-\mathbf{r},t), \tag{69}$$

where σ is a unitary matrix which commutes with β and with the s_i and which satisfies the condition $\sigma^2\sim 1$. With β and s_i having the forms given above, these conditions require that σ be of the form

$$\sigma=e^{i\theta_S}\begin{pmatrix}1' & 0'\\ 0' & \pm 1'\end{pmatrix}. \tag{70}$$

Thus apart from the phase factor σ can be represented by either the unit matrix or the matrix β.

We must now search for a *unitary* transformation to represent the Pauli time inversion transformation Z. Taking Z to be of the form:

$$Z\chi(\mathbf{r},t)=\zeta\chi(-\mathbf{r},t), \tag{71}$$

one finds that the conditions (C′) then require that the unitary matrix satisfy the following conditions:

$$\zeta\beta=-\beta\zeta, \tag{72}$$

$$\zeta s_i=s_i\zeta, \tag{73}$$

$$\zeta\sigma\sim\sigma\zeta, \tag{74}$$

$$\zeta^2\sim 1. \tag{75}$$

These conditions can only be satisfied if ζ is of the form

$$\zeta=\begin{pmatrix}0' & e^{i\theta_1}1'\\ e^{i\theta_2}1' & 0'\end{pmatrix}. \tag{76}$$

However, we can always find a unitary transformation on our matrices which leaves the matrices β and s_i unchanged in form, but reduces ζ to the form

$$\zeta = e^{i\theta_z}\begin{pmatrix} 0' & 1' \\ 1' & 0' \end{pmatrix}. \tag{77}$$

Thus, apart from equivalence transformations (which we shall discuss in greater detail later) and the phase factor in (77), the Pauli time reversal transformation is uniquely determined in the representation space provided by Eq. (62).

Next we shall show that the representation space always admits a Wigner time inversion transformation T satisfying the conditions (C). This transformation we write as

$$T\chi(\mathbf{r},t) = \tau\chi^*(\mathbf{r},-t), \tag{78}$$

with τ a unitary matrix satisfying the conditions:

$$\tau\beta^*\tau^{-1} = \beta, \tag{79}$$

$$\tau s_i^*\tau^{-1} = -s_i, \tag{80}$$

$$\tau\sigma^*\tau^{-1} \sim \sigma, \tag{81}$$

$$\tau\zeta^*\tau^{-1} \sim \zeta, \tag{82}$$

$$\tau\tau^* \sim 1. \tag{83}$$

To determine if a matrix τ exists satisfying these conditions, we note first that since in our present representation β is real, (79) requires that τ commute with β. This means that τ must be of the form

$$\tau = \begin{pmatrix} \tau' & 0' \\ 0' & \tau'' \end{pmatrix}. \tag{84}$$

The condition (80) then requires that both τ' and τ'' satisfy the conditions given in Eq. (44). We have already proved that a matrix satisfying these conditions always exists, and furthermore that it is unique to within a phase factor. Calling any such matrix simply τ', we then have as possibilities for τ:

$$\tau = \begin{pmatrix} \tau' & 0' \\ 0' & e^{i\theta}\tau' \end{pmatrix}. \tag{85}$$

The condition (82) then requires that $\theta = 0$ or π, and hence apart from the arbitrary phase factor already implicit in τ', we must have

$$\tau = \begin{pmatrix} \tau' & 0' \\ 0' & \pm\tau' \end{pmatrix}. \tag{86}$$

The conditions (81) and (83) are then automatically satisfied. For the charge conjugation transformation we then have

$$C\chi(\mathbf{r},t) = ZT\chi(\mathbf{r},t) = \kappa\chi^*(\mathbf{r},t) \tag{87}$$

with

$$\kappa = \zeta\tau = e^{i\theta_z}\begin{pmatrix} 0' & \pm\tau' \\ \tau' & 0' \end{pmatrix}. \tag{88}$$

Actually, we may prefer to define C not as TZ but as a transformation satisfying the conditions (C″); then one finds that the most general form of this transformation is still (87) but with

$$\kappa = e^{i\theta_c}\begin{pmatrix} 0' & \pm\tau' \\ \tau' & 0' \end{pmatrix}, \tag{89}$$

with the ambiguous sign in (89) independent of the ambiguous sign in (86) and θ_c unrelated to θ_z.

We have now shown that by combining two of our irreducible representations of the Lorentz group, we obtain indeed irreducible representations of the *full* Lorentz group satisfying the conditions of strong covariance. Whether by this process we obtain all strongly covariant equations suitable for representing a particle of finite mass and noninfinite spin we do not know, but we conjecture that this is indeed the case. This means that our procedure above has also led us to what we may call "a canonical form" for strongly covariant equations. This canonical form consists of the Schrödinger equation (62), the identifications of the unitary transformations which are a representation of the Lorentz group as given in the set of equations (D), (69), and (71), and a set of abstract relations satisfied by the matrices β, s_i, σ, and ζ which occur in the above equations. These relations, which we summarize below, may actually be thought of as defining the matrices β, s_i, σ, and ζ, independent of representation:

$$\beta = \beta^+, \quad \beta^2 = 1, \tag{E-90}$$

$$s_i = s_i^+, \tag{E-91}$$

$$[s_i, s_j] = i\epsilon_{ijk}s_k, \tag{E-92}$$

$$s_i\beta = \beta s_i, \tag{E-93}$$

$$\sigma\sigma^+ = 1, \quad \sigma^2 \sim 1, \tag{E-94}$$

$$\sigma\beta = \beta\sigma \tag{E-95}$$

$$\sigma s_i = s_i\sigma, \tag{E-96}$$

$$\zeta\zeta^+ = 1, \quad \zeta^2 \sim 1, \tag{E-97}$$

$$\zeta\beta = -\beta\zeta, \tag{E-98}$$

$$\zeta s_i = s_i\zeta, \tag{E-99}$$

$$\zeta\sigma \sim \sigma\zeta. \tag{E-100}$$

Note, for example, that these relations already imply that β has eigenvalues $+1$ and -1 in pairs; for (E-90) requires that its eigenvalues can only be these values, while (E-98) tells us that if $\phi_1, \phi_2, \cdots$ are distinct eigenvectors of β, belonging to one of these eigenvalues, then $\zeta\phi_1, \zeta\phi_2, \cdots$ are distinct eigenvectors belonging to the other. As a corollary we have that any representation of these matrices must be even-dimensional.

So far we have not required that a representation of a Wigner time inversion transformation or the charge conjugation transformation exist in the representation space, though we have seen that transformations having the correct properties always do exist. If we wish to add the requirement of the existence of a representation of either or both of these transformations we can supplement the above canonical form with the Eqs. (78) and (87) and define the matrices τ and κ occurring therein through the following equations supplementing the set (E):

$$\tau\tau^{+}=1, \quad \tau\tau^{*}\sim 1, \tag{E$'$-101}$$

$$\tau\beta^{*}\tau^{-1}=\beta, \tag{E$'$-102}$$

$$\tau s_i^{*}\tau^{-1}=-s_i, \tag{E$'$-103}$$

$$\tau\sigma^{*}\tau^{-1}\sim\sigma, \tag{E$'$-104}$$

$$\tau\zeta^{*}\tau^{-1}\sim\zeta, \tag{E$'$-105}$$

$$\kappa\kappa^{+}=1, \quad \kappa\kappa^{*}\sim 1, \tag{E$'$-106}$$

$$\kappa\beta^{*}\kappa^{-1}=-\beta, \tag{E$'$-107}$$

$$\kappa s_i^{*}\kappa^{-1}=-s_i, \tag{E$'$-108}$$

$$\kappa\sigma^{*}\kappa^{-1}\sim\sigma, \tag{E$'$-109}$$

$$\kappa\zeta^{*}\kappa^{-1}\sim\zeta, \tag{E$'$-110}$$

$$\kappa\tau^{*}\sim\tau\kappa^{*}. \tag{E$'$-111}$$

The canonical form which we have described above is itself invariant under a set of unitary transformations, namely those transformations U which commute with $\mathbf{p}$ and $\mathbf{r}$. Under such unitary transformations, Eqs. (62), (D), (69), and (71) retain their form but with the matrices β, s_i, σ, and ζ, transformed according to

$$\xi \rightarrow U\xi U^{-1}, \tag{112}$$

where ξ is any of these six matrices. These matrices then continue to satisfy the relations (E). Furthermore Eqs. (78) and (87) may be retained if the matrices τ and κ are assumed to transform under U as

$$\tau \rightarrow U\tau(U^{*})^{-1}=U\tau U^{T}, \tag{113}$$

$$\kappa \rightarrow U\kappa(U^{*})^{-1}=U\kappa U^{T}. \tag{114}$$

The set of relations (E$'$) and (E$''$) are then also left invariant. We shall regard representations connected in this way as equivalent and shall not distinguish between them. It will be convenient to call a representation in which β and the s_i have the reduced form as given in Eqs. (63) and (68) and in which σ, ζ, τ, and κ have the forms (70), (77), (86), and (89) as the *normal* canonical form.

It will be noted that the canonical form, and in particular the relations (E), (E$'$), and (E$''$), are also invariant under changes in the phase factors associated with the matrices σ, ζ, τ, and κ:

$$\sigma \rightarrow e^{i\alpha_s}\sigma, \tag{115}$$

$$\zeta \rightarrow e^{i\alpha_z}\zeta, \tag{116}$$

$$\tau \rightarrow e^{i\alpha_t}\tau, \tag{117}$$

$$\kappa \rightarrow e^{i\alpha_c}\kappa. \tag{118}$$

The work of Yang and Tiomno[14] has indicated that such transformations may not be devoid of physical significance, and this raises a question as to whether representations connected in this way should be regarded as equivalent. For our present purposes however, there seems to be no purpose served in distinguishing between representations connected in this way, and hence we shall not emphasize the associated ambiguity in the choice of these matrices. This allows us, if we wish, to impose on σ and ζ the stronger conditions

$$\sigma^2=1, \tag{119}$$

$$\zeta^2=1, \tag{120}$$

and we shall assume that this is done. Then ζ is determined uniquely apart from sign in the normal canonical form to be

$$\zeta=\begin{pmatrix} 0' & 1' \\ 1' & 0' \end{pmatrix}, \tag{121}$$

while σ must be represented by either the unit matrix or by β (apart from a sign).

The phase factors of τ and κ may jointly be changed by a unitary transformation $U=e^{i\theta}$. For we then have

$$\tau \rightarrow U\tau(U^{*})^{-1}=e^{i\theta}\tau e^{i\theta}=e^{2i\theta}\tau, \tag{122}$$

$$\kappa \rightarrow U\kappa(U^{*})^{-1}=e^{2i\theta}\kappa, \tag{123}$$

though the relative phase factor of the two cannot be changed in this way. However, if we disregard the phase factors associated with these matrices we see that we have a choice of two distinct matrices by which each may be represented as a consequence of the ambiguity in sign in (86) and (89).

To summarize, even disregarding a phase factor in the matrices associated with the disjoint transformations, there persists nevertheless a twofold choice for these matrices in the case of space inversion, Wigner time inversion, and charge conjugation. This ambiguity is an essential one and each choice leads to a *distinct theory* to be associated with Eq. (62). Thus if we do not require the existence of a Wigner time inversion transformation or a charge conjugation transformation, we still have two theories for each mass and spin, one in which the unit matrix is associated with space inversion, the other in which the matrix β is associated with space inversion. This distinction may be expected

[14] C. N. Yang and J. Tiomno, Phys. Rev. **79**, 495 (1950); Wick, Wightman, and Wigner, Phys. Rev. **88**, 101 (1952).

to lead on second quantization to the two situations according to which there either occurs a change in space parity or there does not when a particle and an antiparticle annihilate one another in a state of definite orbital momentum. We shall also see shortly that both possibilities are employed in the usual theories.

The next part of this paper will be devoted to demonstrating that the three familiar theories, the Dirac, Klein-Gordon, and Proca, can all, by a suitable choice of representation, be put into the normal canonical form which we have constructed above.

REDUCTION OF THE DIRAC EQUATION TO CANONICAL FORM

We begin by writing the Dirac equation in the familiar form

$$i\partial\psi(\mathbf{r},t)/\partial t=(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})\psi(\mathbf{r},t), \tag{124}$$

where, as usual, $\beta, \alpha_1, \alpha_2, \alpha_3$, are four Hermitian mutually anticommuting matrices of four rows and columns. The solutions of this equation form a representation space for the inhomogeneous Lorentz group with the scalar product defined by

$$(\psi_a,\psi_b)=\int\psi_a{}^*(\mathbf{r},t)\psi_b(\mathbf{r},t)d\mathbf{r}. \tag{125}$$

The representation is fixed by the following identification of the infinitesimal generators of the inhomogeneous Lorentz group:

$$\mathbf{P}=\mathbf{p}, \tag{126}$$

$$H=\beta m+\boldsymbol{\alpha}\cdot\mathbf{p}, \tag{127}$$

$$\mathbf{J}=\mathbf{r}\times\mathbf{p}+\mathbf{s}, \tag{128}$$

$$\mathbf{K}=\tfrac{1}{2}[\mathbf{r}(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})+(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{r}]-t\mathbf{p}, \tag{129}$$

where

$$s_i=-i\epsilon_{ijk}\alpha_j\alpha_k, \tag{130}$$

and hence

$$[s_i,s_j]=i\epsilon_{ijk}s_k. \tag{131}$$

The only one of these identifications which is perhaps unfamiliar is (129), so we shall briefly sketch its origin. For an infinitesimal Lorentz transformation, the transformation of the wave function is generally written[15]

$$\psi(\mathbf{r},t)\rightarrow\psi'(\mathbf{r},t)=[1+\tfrac{1}{2}\boldsymbol{\xi}\cdot\boldsymbol{\alpha}]\psi(\mathbf{r}-\boldsymbol{\xi}t,\,t-\boldsymbol{\xi}\cdot\mathbf{r}). \tag{132}$$

Hence we have

$$\begin{aligned}\psi'(\mathbf{r},t)&=[1+\tfrac{1}{2}\boldsymbol{\xi}\cdot\boldsymbol{\alpha}][\psi(\mathbf{r},t)-\boldsymbol{\xi}t\cdot\nabla\psi-\boldsymbol{\xi}\cdot\mathbf{r}\partial\psi/\partial t]\\&=[1+\tfrac{1}{2}\boldsymbol{\xi}\cdot\boldsymbol{\alpha}][1-i\boldsymbol{\xi}t\cdot\mathbf{p}+i\boldsymbol{\xi}\cdot\mathbf{r}(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})]\psi(\mathbf{r},t)\\&=[1+i\boldsymbol{\xi}\cdot\{\tfrac{1}{2}[\mathbf{r}(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})+(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})\mathbf{r}]\\&\qquad-t\mathbf{p}\}]\psi(\mathbf{r},t).\end{aligned} \tag{133}$$

[15] W. Pauli, reference 12. Note that (132) is the specialization of the general case to the specific case of a Lorentz transformation as contrasted with a rotation, and with the transformation of the space and time coordinates explicitly indicated.

The representation of the disjoint transformations is given by

$$S\psi(\mathbf{r},t)=\beta\psi(-\mathbf{r},t), \tag{134}$$

$$Z\psi(\mathbf{r},t)=\beta\alpha_1\alpha_2\alpha_3\psi(\mathbf{r},-t), \tag{135}$$

$$T\psi(\mathbf{r},t)=\tau\psi^*(\mathbf{r},-t), \tag{136}$$

$$C\psi(\mathbf{r},t)=\kappa\psi^*(\mathbf{r},t). \tag{137}$$

The matrices τ and κ are defined by the following relations:

$$\tau=-\tau^T,\quad \tau\tau^+=-\tau\tau^*=-1, \tag{138}$$

$$\tau\beta^*\tau^{-1}=\beta, \tag{139}$$

$$\tau\alpha_i{}^*\tau^{-1}=-\alpha_i, \tag{140}$$

$$\kappa=\kappa^T,\quad \kappa\kappa^+=\kappa\kappa^*=1, \tag{141}$$

$$\kappa\beta^*\kappa^{-1}=-\beta, \tag{142}$$

$$\kappa\alpha_i{}^*\kappa^{-1}=-\alpha_i. \tag{143}$$

The existence of such matrices can be established in any representation of the Dirac matrices. One can show, of course, that the Dirac equation (124) is left invariant under all these transformations.

The reduction of the Dirac equation to the canonical form is accomplished by the Foldy-Wouthuysen transformation[5]:

$$\chi(\mathbf{r},t)=U\psi(\mathbf{r},t), \tag{144}$$

where

$$\begin{aligned}U&=\exp\left\{\frac{\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{2p}\tan^{-1}\left(\frac{p}{m}\right)\right\}\\&=\left(\frac{\omega+m}{2\omega}\right)^{\frac{1}{2}}+\frac{\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{p}\left(\frac{\omega-m}{2\omega}\right)^{\frac{1}{2}}\\&=\frac{m+\omega+\beta\boldsymbol{\alpha}\cdot\mathbf{p}}{[2\omega(\omega+m)]^{\frac{1}{2}}}.\end{aligned} \tag{145}$$

Then

$$i\partial\chi(\mathbf{r},t)/\partial t=\beta\omega\chi(\mathbf{r},t), \tag{146}$$

as required. One can easily transform the representation of the Lorentz group and charge conjugation transformations into this new representation. The results are just the identifications (D), (69), (71), and (87) with the fundamental matrices identified as

$$\sigma=\beta,\quad \zeta=\beta\alpha_1\alpha_2\alpha_3,\quad \tau=\tau,\quad \kappa=\kappa. \tag{147}$$

These then do indeed satisfy the canonical relations (E), (E′).

For the fundamental matrices to take the normal form of the last section we need only choose the initial representation of the Dirac matrices as

$$\beta=\begin{pmatrix}1' & 0'\\ 0' & -1'\end{pmatrix},\quad \alpha_i=\begin{pmatrix}0' & -i\sigma_i{}'\\ i\sigma_i{}' & 0'\end{pmatrix}, \tag{148}$$

where the $\sigma_i{}'$ are the usual Pauli matrices. In this case the matrices τ and κ have the form

$$\tau=\begin{pmatrix}\tau' & 0\\ 0 & -\tau'\end{pmatrix},\quad \kappa=\begin{pmatrix}0' & \tau'\\ -\tau' & 0'\end{pmatrix}. \tag{149}$$

Since τ' is an antisymmetric matrix in this case, τ is also antisymmetric while κ is symmetric. Hence $\tau\tau^*=-1$ while $\kappa\kappa^*=1$.

REDUCTION OF THE KLEIN-GORDON EQUATION TO CANONICAL FORM[16]

We begin with the Klein-Gordon equation[2] in the form

$$-\nabla^2\varphi+\frac{\partial^2\varphi}{\partial t^2}+m^2\varphi=\omega^2\varphi-\frac{\partial^2\varphi}{\partial t^2}=0. \tag{150}$$

Introducing a two-component wave function defined by

$$\chi(\mathbf{r},t)=\begin{bmatrix}\dfrac{1}{\sqrt{2}}\left[\omega^{-\frac{1}{2}}\dfrac{\partial\varphi}{\partial t}-i\omega^{\frac{1}{2}}\varphi\right]\\[2ex] \dfrac{1}{\sqrt{2}}\left[\omega^{-\frac{1}{2}}\dfrac{\partial\varphi}{\partial t}+i\omega^{\frac{1}{2}}\varphi\right]\end{bmatrix}, \tag{151}$$

one has

$$i\partial\chi/\partial t=\beta\omega\chi, \tag{152}$$

with β the matrix

$$\beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{153}$$

The solutions of (152) constitute a unitary representation space with scalar product

$$(\chi_a,\chi_b)=\int\chi_a^*(\mathbf{r},t)\chi_b(\mathbf{r},t)d\mathbf{r}. \tag{154}$$

The transformation properties of φ under a Lorentz transformation then determine the transformation of χ. One finds the infinitesimal generators to have the form (D) but with the s_i now represented by *null* 2×2 matrices. To illustrate the derivation of these, we again take the infinitesimal Lorentz transformation as an example. We have

$$\begin{aligned}\varphi'(\mathbf{r},t)&=\varphi(\mathbf{r}-\boldsymbol{\xi}t,\, t-\boldsymbol{\xi}\cdot\mathbf{r})\\ &=\varphi(\mathbf{r},t)-\boldsymbol{\xi}t\cdot\nabla\varphi-\boldsymbol{\xi}\cdot\mathbf{r}\partial\varphi/\partial t\\ &=[1-i\boldsymbol{\xi}\cdot t\mathbf{p}]\varphi(\mathbf{r},t)-\boldsymbol{\xi}\cdot\mathbf{r}\partial\varphi(\mathbf{r},t)/\partial t,\end{aligned} \tag{155}$$

[16] The reduction to canonical form of the Klein-Gordon and Proca equations in this and the following sections is essentially a transcription of that given by Case, reference 4. We have omitted some intermediate steps which are important in the case of external electromagnetic fields but are not necessary for our purposes. The simplified calculation given here facilitates the calculation of the behavior of the functions under Lorentz transformations.

and

$$\begin{aligned}\partial\varphi'(\mathbf{r},t)/\partial t&=[1-i\boldsymbol{\xi}\cdot t\mathbf{p}]\partial\varphi/\partial t-i\boldsymbol{\xi}\cdot\mathbf{p}\varphi-\boldsymbol{\xi}\cdot\mathbf{r}\partial^2\varphi/\partial t^2\\ &=[1-i\boldsymbol{\xi}\cdot t\mathbf{p}]\partial\varphi/\partial t\\ &\qquad+[-i\boldsymbol{\xi}\cdot\mathbf{p}-\boldsymbol{\xi}\cdot\mathbf{r}\omega^2]\varphi,\end{aligned} \tag{156}$$

whence

$$\begin{aligned}\chi_1'(\mathbf{r},t)&=(1/\sqrt{2})[\omega^{-\frac{1}{2}}\partial\varphi'/\partial t-i\omega^{\frac{1}{2}}\varphi']\\ &=(1/\sqrt{2})\{\omega^{-\frac{1}{2}}[(1-i\boldsymbol{\xi}\cdot t\mathbf{p})\partial\varphi/\partial t\\ &\qquad+(-i\boldsymbol{\xi}\cdot\mathbf{p}-\boldsymbol{\xi}\cdot\mathbf{r}\omega^2)\varphi]\\ &\qquad-i\omega^{\frac{1}{2}}[(1-i\boldsymbol{\xi}\cdot t\mathbf{p})\varphi-\boldsymbol{\xi}\cdot\mathbf{r}\partial\varphi/\partial t]\}\\ &=\{1+i\boldsymbol{\xi}\cdot[\tfrac{1}{2}(\mathbf{r}\omega+\omega\mathbf{r})-t\mathbf{p}]\}\chi_1(\mathbf{r},t).\end{aligned} \tag{157}$$

Similarly,

$$\chi_2'(\mathbf{r},t)=\{1+i\boldsymbol{\xi}\cdot[-\tfrac{1}{2}(\mathbf{r}\omega+\omega\mathbf{r})-t\mathbf{p}]\}\chi_2(\mathbf{r},t), \tag{158}$$

from which follows

$$\chi'(\mathbf{r},t)=\{1+i\boldsymbol{\xi}\cdot[\tfrac{1}{2}\beta(\mathbf{r}\omega+\omega\mathbf{r})-t\mathbf{p}]\}\chi(\mathbf{r},t). \tag{159}$$

Under space inversion we usually have $\varphi(\mathbf{r},t)\rightarrow\pm\varphi(\mathbf{r},t)$ according as the field is scalar or pseudoscalar. Then $\chi(\mathbf{r},t)\rightarrow\pm\chi(\mathbf{r},t)$ so we have (69) with $\sigma=\pm 1$. Actually the phase is not fixed even to this degree, but in view of our expressed disinterest in phase factors we need only consider the choice $\sigma=1$. For the Pauli time inversion we take $\varphi(\mathbf{r},t)\rightarrow-\varphi(\mathbf{r},-t)$, whence $\partial\varphi(\mathbf{r},t)/\partial t\rightarrow\partial\varphi(\mathbf{r},-t)/\partial t$ and therefore

$$Z\chi(\mathbf{r},t)=\zeta\chi(\mathbf{r},-t), \tag{160}$$

with

$$\zeta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}. \tag{161}$$

For the Wigner time inversion we take $\varphi(\mathbf{r},t)\rightarrow\varphi^*(\mathbf{r},-t)$ whence

$$T\chi(\mathbf{r},t)=\tau\chi^*(\mathbf{r},-t) \tag{162}$$

with τ the unit matrix. These results yield for charge conjugation $[\varphi(\mathbf{r},t)\rightarrow\varphi^*(\mathbf{r},t)]$ the form

$$C\chi(\mathbf{r},t)=\kappa\chi^*(\mathbf{r},t) \tag{163}$$

with $\kappa\equiv\zeta$. Thus we have put the theory into the canonical form but with τ', τ, and κ represented in this case by symmetric matrices in contrast to the Dirac case. Therefore, $\tau\tau^*=1$, $\kappa\kappa^*=1$.

REDUCTION OF THE PROCA EQUATIONS TO CANONICAL FORM

Of the various alternative forms for the Proca equations,[2] we shall adopt the form

$$\mathbf{B}=\text{curl }\mathbf{A}, \tag{164}$$

$$\varphi=-m^{-2}\text{ div }\mathbf{E}, \tag{165}$$

$$\partial\mathbf{A}/\partial t=-\mathbf{E}-\text{grad }\varphi, \tag{166}$$

$$\partial\mathbf{E}/\partial t=m^2\mathbf{A}+\text{curl }\mathbf{B}, \tag{167}$$

where φ, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{E}$ are complex functions. Equations

(164) and (165) may be regarded simply as definitions of $\mathbf{B}$ and φ and if these are eliminated from the latter two equations we have

$$\partial\mathbf{A}/\partial t=-\mathbf{E}+(1/2m^2)\nabla^2\mathbf{E}+(1/2m^2)[\text{curl curl}+\text{grad div}]\mathbf{E}, \quad (168)$$

$$\partial\mathbf{E}/\partial t=m^2\mathbf{A}-\tfrac{1}{2}\nabla^2 A+\tfrac{1}{2}[\text{curl curl}+\text{grad div}]\mathbf{A}. \quad (169)$$

We introduce the operator

$$q=\text{curl curl}+\text{grad div}, \quad (170)$$

and note that

$$q^2=p^4=(\omega-m)^2(\omega+m)^2. \quad (171)$$

Then (168) and (169) can be written

$$\partial\mathbf{A}/\partial t=(1/2m^2)[-\omega^2-m^2+q]\mathbf{E}, \quad (172)$$

$$\partial\mathbf{E}/\partial t=\tfrac{1}{2}[\omega^2+m^2+q]\mathbf{A}. \quad (173)$$

We now introduce two new vector functions $\mathbf{u}$ and $\mathbf{v}$ defined by

$$\mathbf{u}=(8\omega)^{-\frac{1}{2}}\left\{\left[\omega+m-\frac{q}{\omega+m}\right]\frac{\mathbf{E}}{m}+i\left[\omega+m+\frac{q}{\omega+m}\right]\mathbf{A}\right\}, \quad (174)$$

$$\mathbf{v}=(8\omega)^{-\frac{1}{2}}\left\{\left[\omega+m-\frac{q}{\omega+m}\right]\frac{\mathbf{E}}{m}-i\left[\omega+m+\frac{q}{\omega+m}\right]\mathbf{A}\right\}, \quad (175)$$

and then find from (172) and (173) that

$$i\partial\mathbf{u}/\partial t=\omega\mathbf{u}, \quad (176)$$

$$i\partial\mathbf{v}/\partial t=-\omega\mathbf{v}. \quad (177)$$

It is important to note that (174) and (175) can be inverted so that $\mathbf{u}$ and $\mathbf{v}$ can serve as wave functions. Finally introducing a six-component wave function χ through

$$\chi=\begin{pmatrix}u_1\\u_2\\u_3\\v_1\\v_2\\v_3\end{pmatrix}, \quad (178)$$

we have

$$i\partial\chi/\partial t=\beta\omega\chi, \quad (179)$$

with the matrix β given by

$$\beta=\begin{pmatrix}1' & 0\\ 0 & -1'\end{pmatrix}, \quad (180)$$

where $1'$ is the unit 3×3 matrix.

From the known transformation relations of $\mathbf{A}$ and $\mathbf{E}$, those of χ may be derived. These turn out to be of the canonical form[17] (D), (69), (71), (78), (87) with the following fundamental matrices:

$$s_i=\begin{pmatrix}s_i' & 0\\ 0 & s_i'\end{pmatrix}, \quad (181)$$

$$\sigma=\begin{pmatrix}1' & 0\\ 0 & 1'\end{pmatrix}, \quad (182)$$

$$\tau=\begin{pmatrix}\tau' & 0\\ 0 & \tau'\end{pmatrix}=\begin{pmatrix}1' & 0\\ 0 & 1'\end{pmatrix}, \quad (183)$$

$$\zeta=\begin{pmatrix}0 & 1'\\ 1' & 0\end{pmatrix}, \quad (184)$$

$$\kappa=\begin{pmatrix}0 & \tau'\\ \tau' & 0\end{pmatrix}=\begin{pmatrix}0 & 1'\\ 1' & 0\end{pmatrix}. \quad (185)$$

Here the s_i' are the matrices

$$s_1'=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\quad s_2'=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},\quad s_3'=\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}, \quad (186)$$

constituting a three-dimensional representation of the rotation group. In this special representation of the s_i' the time inversion matrix τ' is simply the unit matrix. Thus we again have the canonical form, in this case with τ', τ, and κ all symmetric matrices as in the Klein-Gordon case so that one has $\tau\tau^*=1$, $\kappa\kappa^*=1$.

COMPARISON OF THE CONVENTIONAL THEORIES

The work of the last three sections has shown that a very close analogy may be exhibited between the Dirac, Klein-Gordon, and Proca equations by writing these equations in the particular representation which corresponds to the canonical form that we have synthesized earlier from the requirement that a covariant wave equation shall provide us with a unitary representation of the complete Lorentz group. This analogy extends beyond the equations themselves and includes the manner in which the wave functions transform under the transformations of the inhomogeneous Lorentz group. We conjecture that by an appropriate choice of representation with the elimination of redundant components, the theories of the Dirac-Pauli-Fierz type to represent particles of higher spin can also be brought into this same canonical form.

By the above procedure, we have, however, also brought out some interesting differences between the equations. The most striking of these is the trans-

[17] The only difficult transformation to obtain is that under Lorentz transformation. This calculation was carried out by Dr. B. P. Nigam at the author's request. We are indebted to Dr. Nigam for this favor.

formation of the wave function under space inversion in which our original synthesis allowed us two distinct possibilities corresponding to the choice of the unit matrix or the matrix β to represent the space inversion matrix σ. We now find that both of these possibilities are realized in the familiar theories; we have $\sigma=1$ in the integral-spin theories (Klein-Gordon, Proca, and presumably the integral-spin Dirac-Pauli-Fierz theories) while we have $\sigma=\beta$ in the half-integral spin theories (Dirac, and presumably the half-integral spin Dirac-Pauli-Fierz theories). Further differences are apparent with respect to the transformations employed to represent Wigner time inversion and charge conjugation. Again two distinct possibilities for each of these is allowed by covariance alone. However, in the conventional half-integral spin theories, one has

$$\tau=\begin{pmatrix}\tau' & 0' \\ 0' & -\tau'\end{pmatrix},\quad \kappa=\begin{pmatrix}0 & \tau' \\ -\tau' & 0'\end{pmatrix}, \tag{187}$$

while in the conventional integral-spin theories, one has

$$\tau=\begin{pmatrix}\tau' & 0' \\ 0' & \tau'\end{pmatrix},\quad \kappa=\begin{pmatrix}0' & \tau' \\ \tau' & 0'\end{pmatrix}. \tag{188}$$

No clue has been forthcoming in our analysis as to why these particular choices should take precedence over the other alternative made available by covariance arguments.

We have already made a conjecture as to the physical significance of the two choices 1 and β for the space inversion matrix σ. Namely, we expect that in the second-quantized form of these theories, the two choices correspond to whether or not there is a change in space parity on annihilation of a particle and antiparticle in a state of given orbital angular momentum. Similarly, we may conjecture that the two choices for the charge-conjugation matrix will correspond in the second-quantized theory to whether or not there is a change of charge conjugation parity on the annihilation of a particle and antiparticle in a state of given spin and orbital angular momentum. Thus the choice of the matrices used to represent these transformations is expected to be of great physical significance in determining the selection rules associated with certain reactions and decay schemes for the particles involved. It is thus of great importance to know whether there exist further reasons which impose the special choices associated with the conventional theories, and hence fix certain of these selection rules as soon as the spin of the particle is assigned; or whether on the other hand, there exist reasonable and consistent alternate theories of these particles which employ the "abnormal" or "unconventional" choice of the matrices to represent these transformations. A specific example of such a question is: Does there exist another theory of spin-$\frac{1}{2}$ particles than the Dirac theory in which the annihilation of a particle and antiparticle in an S state involves no charge in space parity?

We are in no position to answer this question at the present time, except to say that in the one-particle theory, the requirements of strong covariance do not impose a specific choice. Our analysis has only provided us with certain raw material for the construction of a complete physical theory. For these one-particle equations to develop into a complete and consistent theory, other requirements must presumably still be met. Thus one may require: (1) that it be possible to second-quantize the equations in a consistent way; (2) that it be possible to introduce interaction in a covariant and consistent manner and, in the case of interaction with an electromagnetic field, in a gauge invariant manner; (3) that the resulting theories give no contradiction to an appropriate causality condition, such as, that physically observable effects are not propagated with a velocity greater than the velocity of light, at least on a macroscopic scale; (4) that one can define densities like the energy-momentum density tensor in these theories, that satisfy appropriate physical conditions. A discussion of the necessity for these requirements or of the feasibility of meeting them in theories employing the "unconventional" choice of representation matrices, lies beyond the scope of the present paper. Thus we shall be satisfied here with simply pointing out the possibility of such alternate theories which satisfy the requirements of relativistic covariance. It is worth emphasizing again however, that our results above show that the "relative intrinsic space and charge conjugation parity" of a particle and antiparticle of given spin is not a consequence of relativistic requirements alone. If, in fact, the spin does determine the relative intrinsic space parity, it is for reasons which go beyond the requirements of covariance.

It is perhaps not out of place to make a further remark here about second quantization of these equations. It is well known that in the familiar Dirac, Klein-Gordon, and Proca theories, one can give certain reasons to justify the fact that the integral spin equations are quantized according to Bose-Einstein statistics while the half-integral spin theory is quantized according to Fermi-Dirac statistics. These reasons are not entirely dissociated from the particular choice of matrices involved in the transformations associated with space inversion, Wigner time reversal, and charge conjugation. Thus, at least some of these arguments concerning the connection between spin and statistics may lose their cogency if it should indeed be possible to develop theories based on the alternate or "abnormal" choice of these matrices. There is perhaps a slim possibility that the connection between spin and statistics is not a rigid one. In view of the recent discovery of many new "elementary" particles about which almost nothing is known directly concerning their statistics, it would be well to understand clearly in advance just how binding or how flexible is the connection between spin and statistics from the theoretical point of view.

INTERPRETATION OF THE COVARIANT EQUATIONS

We have said little so far concerning the physical interpretation of the one-particle equation (62) which we have developed. Of course, any direct physical interpretation outside the scheme of second quantization has little meaning because of the presence of negative eigenvalues for the Hamiltonian, but we can at least attempt to make a comparison between this equation and, say, the Dirac equation which suffers from the same difficulty. To the same extent that the latter equation has a hypothetical one-particle interpretation, we can make an analogous interpretation for our equation.

We may note first that we can introduce *observables* into the theory in the usual way and identify $\mathbf{P}$, H, and $\mathbf{J}$, and $\mathbf{s}$ with the observables of momentum, energy, total angular momentum, and spin angular momentum as usual. The observable represented by $\mathbf{r}$, however, needs some explanation. Since this operator is defined in the manifold of positive and negative energy solutions of Eq. (62) separately, it does not coincide with the usual position operator even in the Dirac theory. It is in fact the position operator defined by Newton and Wigner[18] which in the case of the Dirac equation was called the "mean position operator" by Foldy and Wouthuysen. The first-named authors have shown that it is only position operator which satisfies certain reasonable conditions and is defined on the manifold of positive- and negative-energy solutions of the equation separately, and thus is the only position operator defined within one of the irreducible representations of the Lorentz group as obtained by Wigner and Bargmann. In the case of spin $\frac{1}{2}$ it is related nonlocally to the conventional position operator in the Dirac theory, and it has no simple Lorentz transformation properties. It does have the property, however, that its time rate of change is given by the eminently reasonable expression $\beta\mathbf{p}/\omega$. We may note in addition that $\mathbf{s}$ in this theory does not coincide with the conventional spin operator of the Dirac theory in the case of spin $\frac{1}{2}$ but represents what Foldy and Wouthuysen call the "mean spin angular momentum." It has the property that it is a constant of the motion for a free particle which the conventional Dirac spin angular momentum has not.

If we accept the hypothesis that by some experimental arrangement it is possible to determine the position observable $\mathbf{r}$, then we can consistently regard $\chi^*(\mathbf{r},t)\chi(\mathbf{r},t)$ as the relative probability of finding the particle at the point $\mathbf{r}$ at the time t. Note, however, that because of the nonlocal character of the representation (D-67) of Lorentz transformations, if in one Lorentz frame the particle is localized at the point $\mathbf{r}=0$ at $t=0$, then in another Lorentz frame whose origin coincides with the first at $t=0$, the same state will not be localized at the origin at time $t=0$. In fact, the state will be represented by a wave function which is distributed over distances of the order of the Compton wavelength of the particle from the origin. This is a consequence of the complicated transformation character of $\mathbf{r}$ under Lorentz transformations, but it implies no lack of relativistic covariance in the theory.

With $\chi^*(\mathbf{r},t)\chi(\mathbf{r},t)$ representing a probability density, one can always define a probability current density or flux, such that one has a differential conservation theorem. This follows from the fact that the equation

$$\mathrm{div}\mathfrak{S} = -\partial(\chi^*\chi)/\partial t, \tag{189}$$

has generally one, and actually many solutions. However, one has no guarantee that any of these solutions will define a probability current density *locally* in terms of the wave function χ. This need not interfere with the interpretation of the theory, however.

To introduce the concepts of charge density and current density, however, one must exercise more care. If one wishes to assign to the particle a point charge located at the point $\mathbf{r}$, then the charge density will be represented as usual by $\chi^*(\mathbf{r},t)\chi(\mathbf{r},t)$. However, it may not then be possible to find a current density expression from (189) such that the charge and current density transform under Lorentz transformations as a four-vector. Hence it may well turn out not to be possible to attribute to the particle a point charge located at $\mathbf{r}$. Nevertheless, it may be possible to define a charge and current density which are both nonlocally defined in terms of $\chi(\mathbf{r},t)$, corresponding to attributing to the particle a spatially extended charge distribution in the space spanned by $\mathbf{r}$, such that the densities transform as a four-vector under Lorentz transformations and charge is differentially conserved. This is exactly what one finds when one takes the conventional charge and current density expressions in the Dirac, Klein-Gordon, and Proca theories and transforms them into our canonical representation. Thus, for example, the conventional expressions for the charge and current density in the Klein-Gordon theory when written in canonical form are

$$\rho(\mathbf{r},t) = \{\omega^{\frac{1}{2}}\chi\}^*\beta\left(\frac{1-\zeta}{2}\right)\{\omega^{-\frac{1}{2}}\chi\} + \{\omega^{-\frac{1}{2}}\chi\}^*\left(\frac{1-\zeta}{2}\right)\beta\{\omega^{\frac{1}{2}}\chi\}, \tag{190}$$

$$\mathbf{j}(\mathbf{r},t) = \{\omega^{-\frac{1}{2}}\mathbf{p}\chi\}^*\left(\frac{1-\zeta}{2}\right)\{\omega^{-\frac{1}{2}}\chi\} + \{\omega^{-\frac{1}{2}}\chi\}^*\left(\frac{1-\zeta}{2}\right)\{\omega^{-\frac{1}{2}}\mathbf{p}\chi\}, \tag{191}$$

[18] T. D. Newton and E. P. Wigner, Revs. Modern Phys. 21, 400 (1949).

and are thus nonlocally defined in the space of the variable **r**. One can verify, nevertheless, that they do transform as a four-vector under Lorentz transformations and satisfy a differential charge conservation law. They are not, however, the only expressions which have this property. One can show that the same is true for the following identification of charge and current density expressions in this theory:

$$\rho(\mathbf{r},t)=\{\omega^{\frac{1}{2}}\chi\}^*\{\omega^{-\frac{1}{2}}\chi\}+\{\omega^{-\frac{1}{2}}\chi\}^*\{\omega^{\frac{1}{2}}\chi\}, \tag{192}$$

$$\mathbf{j}(\mathbf{r},t)=\{\omega^{-\frac{1}{2}}\mathbf{p}\chi\}^*\beta\{\omega^{-\frac{1}{2}}\chi\}+\{\omega^{\frac{1}{2}}\chi\}^*\beta\{\omega^{-\frac{1}{2}}\mathbf{p}\chi\}, \tag{193}$$

which simply emphasizes the point that the interaction of a particle with an electromagnetic field is not determined by the free-particle equation alone. Whether from (192) and (193) one can build up a completely gauge invariant theory, however, is not yet known. An interesting difference between the expressions (190, 191) and (192, 193) is that the former transform correctly under space inversion only with the choice $\sigma=1$ for the space inversion matrix, while the latter transform correctly with either $\sigma=1$ or $\sigma=\beta$. In fact, the latter are defined in the manifold of positive- and negative-energy solutions of the wave equation separately, and hence may be considered as defined even in one of the Wigner-Bargmann irreducible representations. It is these facts that make the author feel that it is not at all certain that one cannot construct theories with interaction based on the anomalous choice of matrices to represent space inversion, Wigner time inversion, and charge conjugation.

ACKNOWLEDGMENTS

Some of the ideas expressed in this paper had their origin while the author was a Fulbright and John Simon Guggenheim Memorial Fellow at the Institute for Theoretical Physics in Copenhagen during 1953 and 1954. It is a pleasure for the author to acknowledge his indebtedness for this opportunity and to express his thanks to Professor Niels Bohr for the hospitality extended to him at the Institute. He is also happy to acknowledge the benefits of discussions there with Professor C. Møller, Dr. R. Haag, and Mr. P. Kristensen. The major part of this work was supported by the U. S. Atomic Energy Commission and by a grant from the National Science Foundation. The work was completed and part of the manuscript prepared at Brookhaven National Laboratory, whose hospitality during the summer of 1955 is gratefully acknowledged.