finite, characteristic of a short range potential. The dispersion contribution however comes from the interaction with the *atomic electrons alone*, since the atom cannot be excited via the static nuclear potential. Thus the dispersion contribution is characteristic of scattering in a *long-range potential* (as in the nuclear case), and contributes very strongly to the small-angle scattering.

The present analyses in atomic scattering, however, always neglect the fluctuation scattering portion of the dispersion contribution; it would be of some interest to search for visible effects of these terms.

ACKNOWLEDGMENTS

The author wishes to acknowledge the guidance and stimulation received from Professor G. E. Uhlenbeck during the course of this work.

PHYSICAL REVIEW VOLUME 102, NUMBER 2 APRIL 15, 1956

Transfer of Nuclear Particles*

G. BREIT
Yale University, New Haven, Connecticut
(Received December 16, 1955)

The calculation of the wave function of a particle subjected to the field of two isolated potential wells is arranged in such a way as to bring into evidence the penetration factors for passage through regions of negative kinetic energy between the wells. The Schrödinger equation is replaced by an equivalent integral equation and the eigenfunction system of the latter is used for the expansion of the wave function. The dependence of the kernel on energy and the transformation from the reference system of the integral equation to that of the wave equation are considered. Extensions to three-dimensional problems and to many nuclear particles are discussed and some characteristic differences from the one-dimensional case are pointed out. The presence of more than one tunnelling factor in the general case and the participation of tongues or tentacles in configuration space which correspond to the temporary formation of nuclear aggregates such as deuterons or alpha particles are considered.

I. INTRODUCTION

IN calculations of stripping reactions taking place in collisions of heavy nuclear aggregates it is useful to be able to calculate the process by means of adiabatic wave functions such as occur in the work of Wigner and Pelzer.[1] In the molecular problem considered by them these are the wave functions of the electron system for fixed positions of the nuclei. Modifications of this method, employing a semiclassical discussion in which the motion of the heavy particles is treated classically while the motion of the transferred particles is calculated quantum mechanically, are also possible.[2] In both procedures it is essential to be able to calculate the wave function of a nucleon for fixed positions of the nuclear aggregates to the field of which it is subjected. For well separated nuclear aggregates, the fields of which do not overlap, it is especially desirable to have convenient approximations for such wave functions. If the problem is treated by usual perturbation theory, the expansion involves the wave functions of the continuum of the eigenfunction system of the nucleon in the field of one of the nuclei and the rather small probability of transfer which may result on account of the presence of a region of negative kinetic energy which must be traversed by the nucleon becomes apparent as the result of an interference between the effects of different regions of the continuum. It is desirable to have a treatment which is free of this complication.

A possible treatment of this type consists of expansions of the Wigner[3] type in which boundary conditions for internal wave functions are used. An infinite number of such eigenfunction systems can be introduced by changing the nuclear radius or the boundary condition. Such a procedure has many advantages but its flexibility regarding the introduction of the nuclear radius makes the barrier tunnel effects entering in first approximation flexible also. A complete calculation of all effects which is free of the initially assumed radius is possible but lengthy.

In the present note a treatment is described in which the tunnel effect factors are present throughout in a simple manner in the case of one-dimensional potential

* This research was supported by the U. S. Atomic Energy Commission and by the Office of Ordnance Research, U. S. Army.

[1] H. Pelzer and E. Wigner, Z. physik. Chem. **15**, 445 (1932).

[2] This extension is readily made. Some considerations regarding the role of accelerations in causing gravitational effects and their relationship to the terms occurring in Pelzer and Wigner's paper will be discussed in another publication. The reference to the work of Wigner and Pelzer and of the use of wave functions for fixed nuclei is not meant in the sense of considering the expansion of the wave function in terms of such eigenfunctions as being the best from the viewpoint of convergence. It has nevertheless some definite advantages inasmuch as it provides a clear and definite starting point. In cases of slow convergence caused by resonance, linear combinations capable of improving convergence can nevertheless be introduced. This matter will be discussed in connection with an application as part of a paper which is being prepared in collaboration with Dr. M. E. Ebel.

[3] E. P. Wigner, Phys. Rev. **70**, 606 (1946); E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947); R. G. Thomas, *The Theory of Nuclear Exchange Collisions*, Los Alamos informal report (unpublished).

fields and in which qualitatively the same situation applies in the many-body problem. This treatment has some features in common with the Wigner[3] and the Kapur—Peierls[4] approach. The similarity consists in an eigenfunction system which is not that corresponding to the Schrödinger energy equation. The procedure will be first described for transfer between two one-dimensional potential wells.

II. TWO ONE-DIMENSIONAL POTENTIAL WELLS

The particle coordinate is called ξ and the energy enters through the parameter κ^2. A positive κ^2 corresponds to a negative energy. In the absence of the potential wells, the wave equation is

$$\mathfrak{L}u=0 \tag{1}$$

with

$$\mathfrak{L}=(d^2/d\xi^2)-\kappa^2-\mathcal{U}(\xi). \tag{1.1}$$

The function $\mathcal{U}(\xi)$ represents the effect of a general field such as a Coulomb field which may be present in addition to the potential wells. In an application ξ may represent the distance of the particle from a fixed point in which case the boundaries are at $\xi=0$ and $\xi=\infty$; or else it may stand for the distance along a line in space corresponding to boundaries at $\xi=\pm\infty$. In either case, one can introduce two functions v and w satisfying

$$\mathfrak{L}v=0, \quad \mathfrak{L}w=0,$$

with v satisfying the boundary conditions on the right boundary and w those on the left. In the well-known manner, one forms

$$K(\xi,\xi')=v(\xi_>)\,w(\xi_<), \tag{1.2}$$

with

$$(\xi_<,\xi_>)\equiv(\xi,\xi'), \quad (\xi<\xi'), \tag{1.3}$$

$$(\xi_>,\xi_<)\equiv(\xi,\xi'), \quad (\xi>\xi'); \tag{1.4}$$

and employing the normalization

$$wdv/d\xi-vdw/d\xi=1,$$

there results

$$\mathfrak{L}\int K(\xi,\xi')\rho(\xi')d\xi'=\rho(\xi), \tag{1.5}$$

where ρ is an arbitrary density function.

The two potential wells modify the wave equation so that it becomes

$$(\mathfrak{L}-V_1-V_2)u=0, \tag{2}$$

the potential well V_1 being taken to be to the left of V_2. Conversion to an integral equation is accomplished by the substitution

$$u(\xi)=\int K(\xi,\xi')\rho(\xi')d\xi'. \tag{3}$$

Substitution of Eq. (3) in Eq. (2) gives, taking account of Eq. (1.5),

$$\rho(\xi)=[V_1(\xi)+V_2(\xi)]u(\xi). \tag{3.1}$$

The density in the integral representation of u, for finite u thus has a value only in the regions occupied by V_1, V_2. On multiplying Eq. (3) by $V_1(\xi)+V_2(\xi)$, it follows from (3.1) that

$$\rho(\xi)-[V_1(\xi)+V_2(\xi)]\int K(\xi,\xi')\rho(\xi')d\xi'=0. \tag{3.2}$$

This integral equation will be used in place of the original Eq. (2). The method used does not lead exactly to Eq. (3.2) in the first step. Instead, the modification

$$\rho(\xi)-\lambda[V_1(\xi)+V_2(\xi)]\int K(\xi,\xi')\rho(\xi')d\xi'=0, \tag{3.3}$$

with λ nearly 1, will be satisfied and a correction for $\lambda-1$ will be made later through a modification of the energy parameter κ^2. The reason for $\lambda\neq1$ is that the proximity of V_2 modifies the energy of a stationary state in V_1.

If the two potential wells were at an infinite distance from each other, the wave function in either could also be treated by means of an integral eauation such as Eq. (3.3). The energy parameter κ^2 will be chosen to correspond to the value of the energy which gives a solution of the corresponding differential equation for one of the potential wells, say V_1. The potential well V_2 will be supposed to be at an infinite distance in this adjustment of κ^2. Two sets of eigenfunctions can be introduced by means of

$$\begin{aligned} u_{1n}(\xi)-\lambda_{1n}\int K(\xi,\xi')V_1(\xi')u_{1n}(\xi')d\xi'=0,\\ u_{2m}(\xi)-\lambda_{2m}\int K(\xi,\xi')V_2(\xi')u_{2m}(\xi')d\xi'=0. \end{aligned} \tag{4}$$

To these there correspond two density functions

$$\rho_{1n}=V_1u_{1n}, \quad \rho_{2m}=V_2u_{2m}, \tag{4.1}$$

and the corresponding equations for these are

$$\begin{aligned} \rho_{1n}(\xi)-\lambda_{1n}V_1(\xi)\int K(\xi,\xi')\rho_{1n}(\xi')d\xi'=0,\\ \rho_{2m}(\xi)-\lambda_{2m}V_2(\xi)\int K(\xi,\xi')\rho_{2m}(\xi')d\xi'=0, \end{aligned} \tag{4.2}$$

which are similar to (3.2). The ρ_{1n} has a value only in V_1, ρ_{2m} only in V_2. One of the λ_{1n} is 1. It follows from the first Eq. (4.2) and its complex conjugate that the normalization can be such that

$$\int \rho_{1n}^*\rho_{1m}d\xi/V_1=\int u_{1n}^*V_1u_{1m}d\xi=-\delta_{nm}, \tag{4.3}$$

[4] R. L. Kapur and R. Peierls, Proc. Roy. Soc. (London) **A166**, 277 (1938).

provided all the λ_{1n} are distinct. The sign in $-\delta_{nm}$ is taken such as to make Eq. (4.3) possible for an attractive potential well. Equations (4), (4.2) are such that the differential equations satisfied by u_{1n}, u_{1m} are

$$(\mathfrak{L}-\lambda_{1n}V_1)u_{1n}=0, \quad (\mathfrak{L}-\lambda_{2m}V_2)u_{2m}=0. \tag{4.4}$$

The set of eigenfunctions corresponds therefore to potential well depths adjusted to make κ^2 correspond to a possible energy, and in this respect the procedure differs from that of Wigner[3] and of Wigner and Eisenbud.[3] The eigenfunctions u_{1n}, u_{1m} satisfy the same differential equation independently of n and m in the space between the potential wells and each of them has a barrier penetration type of decrease as one moves away from the parent well. The expansions used in the present note are not intended to furnish a representation in terms of energy in the form of resonance terms. The energy of the transformed particle enters through K, and if one employs the bilinear expansion the energy dependence can be seen in a manner similar to that of dispersion theories of nuclear reactions.

In order to solve Eq. (3.3), the density function ρ will be represented as a sum of two parts confined respectively to regions occupied by V_1 and V_2. This representation is consistent with Eq. (3.1). These densities ρ_1, ρ_2 may be expanded respectively in terms of the ρ_{1n} and ρ_{2m} so that

$$\rho_1=\sum_n b_{1n}\rho_{1n}, \quad \rho_2=\sum_m b_{2m}\rho_{2m}. \tag{5}$$

In many-dimensional generalizations the expansions will at least partly involve integrals rather than sums, and even in the one-dimensional case the expansion may involve the Gibbs' phenomenon. These circumstances do not interfere, however, with the possibility of using them. Writing the equation separately for the regions occupied by V_1 and V_2, one has from Eq. (3.3)

$$\begin{aligned}\rho_1(\xi_1)/[\lambda V_1(\xi_1)]&=\int K(\xi_1,\xi_1')\rho_1(\xi_1')d\xi_1'\\ &\quad+\int K(\xi_1,\xi_2')\rho_2(\xi_2')d\xi_2',\\ \rho_2(\xi_2)/[\lambda V_2(\xi_2)]&=\int K(\xi_2,\xi_1')\rho_1(\xi_1')d\xi_1'\\ &\quad+\int K(\xi_2,\xi_2')\rho_2(\xi_2')d\xi_2'.\end{aligned} \tag{5.1}$$

In most of the discussion the potential wells are supposed to be isolated in such a way that there is no overlap of the regions occupied by V_1 and V_2. The equations hold, however, even if there is an overlap, the total potential being then $V=V_1+V_2$. The separation of ρ into $\rho_1+\rho_2$ is made by defining

$$\rho_1=V_1u, \quad \rho_2=V_2u.$$

Equations (5.1), (5.2) combined with (3.3) are then equivalent to $\rho_1=[V_1/(V_1+V_2)]\rho$, $\rho_2=[V_2/(V_1+V_2)]\rho$, which are consistent with the definitions of ρ_1, ρ_2 and with Eq. (3.1). The applicability of the equations to overlapping potentials is not especially useful in the one-dimensional problem under discussion. It is convenient to use it in some many-dimensional problems.

The applicability of the equations to overlapping potentials can be seen also by noting that in (5.1) the limits of integration are automatically determined by the presence of factors V_1, V_2 in ρ_1, ρ_2. The specification of variables as ξ_1, ξ_2 is therefore only convenient but not necessary and Eqs. (5.4) apply also if the ranges of ξ_1 and ξ_2 overlap. Introducing the expansions of Eq. (5), there result two sets of linear equations on the b_{1n} and b_{2m}. From the first of the two Eqs. (5.1), a particular b_{1n} can be isolated on the left by multiplication with ρ_{1n}^* and integration over ξ_1, making use of Eq. (4.3). On the right side, there occur then quantities as follows:

$$\begin{aligned}&\int\int u_{1n}^*(\xi)V_1(\xi)K(\xi,\xi')V_1(\xi')u_{1\nu}(\xi')d\xi d\xi'\\ &\qquad=\int u_{1n}^*(\xi)V_1(\xi)u_{1\nu}(\xi)d\xi/\lambda_{1\nu}=V_{1n\nu}/\lambda_{1\nu},\end{aligned} \tag{5.2}$$

$$\begin{aligned}&\int\int u_{1n}^*(\xi)V_1(\xi)K(\xi,\xi')V_2(\xi')u_{2m}(\xi')d\xi d\xi'\\ &\qquad=\int u_{1n}^*(\xi)V_1(\xi)u_{2m}(\xi)d\xi/\lambda_{2m}=V_{1nm}/\lambda_{2m},\end{aligned} \tag{5.3}$$

use having been made of Eq. (4.1). In Eq. (5.2) the $V_{1n\nu}$ may be replaced by $-\delta_{n\nu}$ in accordance with Eq. (4.3). Performing the substitutions and going through similar manipulations with the second equation in (5.1), one finds

$$\begin{aligned}b_{1n}/\lambda&=b_{1n}/\lambda_{1n}-\textstyle\sum_m V_{1nm}b_{2m}/\lambda_{2m},\\ b_{2m}/\lambda&=b_{2m}/\lambda_{2m}-\textstyle\sum_n V_{2mn}b_{1n}/\lambda_{1n}.\end{aligned} \tag{5.4}$$

The V_{1nm} are defined as in Eq. (5.3) and the V_{2mn} by an equation obtained by interchanging 1 with 2 and m with n in (5.3). One has

$$\begin{aligned}&(V_{1nm}/\lambda_{2m})^*\\ &=\int\int u_{1n}(\xi)V_1(\xi)K^*(\xi,\xi')V_2(\xi')u_{2m}^*(\xi')d\xi d\xi'\\ &=\int\int u_{2m}^*(\xi)V_2(\xi)K(\xi,\xi')V_1(\xi')u_{1n}(\xi')d\xi d\xi'\\ &=\int u_{2m}^*(\xi)V(\xi)u_{1n}(\xi)d\xi/\lambda_{1n}=V_{2mn}/\lambda_{1n}.\end{aligned} \tag{5.5}$$

Here use is made of the Hermitean character of K and of the first of the two lines in Eq. (4). According to Eq. (5.5), the matrix of coefficients in (5.4) is Her-

mitean. The quantity V_{2mn} vanishes unless the function u_{1n} has an appreciable value in the potential well V_2. For sufficiently large separations of the two potential wells, the system of Eqs. (5.4) can be satisfied by $\lambda=\lambda_{1n}$ for a particular n and only $b_{1n}\neq 0$; similarly $\lambda=\lambda_{2m}$ for a particular m, and for this λ only $b_{2m}\neq 0$. These solutions correspond to uncoupled ρ_{1n}, ρ_{2m} and to eigenfunctions u_{1n}, u_{2m} satisfying Eq. (4). Only the solution with $\lambda_{1n}=1$ satisfies the original Eq. (3.2) for an infinite interwell distance, all other solutions corresponding to modified potentials as is seen from (4.4). It is supposed that the potential wells are not identical and that there are no accidental degeneracies in the systems λ_{1n}, λ_{2m}.

In the case of attractive potential wells the presentation can be conveniently put in a symmetric form through the introduction of quantities

$$\sigma_{in}=(-V_i)^{\frac{1}{2}}u_{in}, \quad (i=1, 2) \tag{5.51}$$

$$G(\xi,\xi')=(-V)^{\frac{1}{2}}K(\xi,\xi')(-V)^{\frac{1}{2}}, \tag{5.52}$$

leading to

$$\sigma_{in}(\xi)+\lambda_{in}\int G(\xi,\xi')\sigma_{in}(\xi,\xi')d\xi'=0, \tag{5.53}$$

$$\int \sigma_{in}{}^{*}\sigma_{i\nu}d\xi=\delta_{n\nu}, \tag{5.54}$$

and to Eq. (5.4) as before.

The proximity of the potential wells couples the b_{1n} to the b_{2m} as in Eq. (5.4). For large interwell distances the b_{2m} are of order V_{2mn} which contain the wave function barrier penetration factor once. The $b_{1\nu}$ for $\nu\neq n$ occur as a result of coupling of the $b_{1\nu}$ to the b_{2m} and expressions for them contain the barrier penetration factor twice. For small barrier penetration factors the largest effects are therefore on the b_{2m}. From (3.1) and (3.3), it follows that

$$u=\lambda\int K(\xi,\xi')\rho(\xi')d\xi', \tag{6}$$

a modification of (3) required by the introduction of λ. Substitution of

$$\rho=\rho_1+\rho_2, \tag{6.1}$$

making use of Eq. (5) gives

$$\mu/\lambda=\textstyle\sum_n u_{1n}/\lambda_{1n}+\sum_m u_{2m}/\lambda_{2m} \tag{6.2}$$

for the space eigenfunction. Equations (5.4) can be satisfied only for certain λ. Among these, only the particular λ which becomes 1 for large interwell distances is of interest. Since this λ is also not exactly 1, it is Eq. (3.3) rather than (3.2) that is satisfied. From (6), (1.5), and (3.1) it follows that

$$(\mathfrak{L}-\lambda V_1-\lambda V_2)u=0. \tag{7}$$

The problem solved so far is thus not quite the intended one, the potential well depths being slightly wrong, the depths of V_1 and V_2 having been changed in the same ratio. It will be seen presently that the change of depth can be compensated by a change in κ^2. This is natural because the κ^2 used so far corresponds to uncoupled potential wells and the coupling produces a change in the energy of a stationary state. The potential wells adjust themselves in the solution obtained so far in such a way as to accommodate the originally chosen κ^2. It will be necessary therefore to change κ^2 in such a way as to make λ return to its original value, $\lambda=1$. A change in κ^2 gives a change in K which makes it possible to satisfy

$$\rho(\xi)-[V_1(\xi)+V_2(\xi)]\int(K+\delta K)(\xi,\xi')\rho(\xi')d\xi'=0. \tag{7.1}$$

In order to see how this happens it is desirable to know how K depends on κ^2. This question is considered in the next section. It should be noted however that for an approximate correction it suffices to apply the ordinary first-order perturbation formula.

III. CHANGE IN ENERGY; ENERGY REFERENCE SYSTEM

The useful property of the kernel is Eq. (1.5), which can be written as

$$\mathfrak{L}K(\xi,\xi')=\delta(\xi-\xi'). \tag{8}$$

Employing the definition of $\mathfrak{L}$ as in Eq. (1.1), one obtains

$$-K(\xi,\xi')+\mathfrak{L}\partial K(\xi,\xi')/\partial(\kappa^2)=0. \tag{8.1}$$

Here and below, K without explicit specification of value of κ^2 is meant to be evaluated for the value κ^2 rather than $\kappa^2+\delta\kappa^2$. One has therefore, making use of (8.1) and of partial integration,

$$\int K(\xi,\xi')K(\xi,\xi'')d\xi=\int\{\mathfrak{L}K(\xi,\xi')\}\partial K(\xi,\xi'')/\partial(\kappa^2)d\xi$$
$$=\partial K(\xi',\xi'')/\partial(\kappa^2). \tag{8.2}$$

In matrix notation, the above relation may be written as

$$\partial K(\xi',\xi'')/\partial(\kappa^2)=(K^2)(\xi',\xi''), \tag{9}$$

the right side of the equation being meant to be the (ξ',ξ'') matrix element of K^2. From (9) it follows, on successive differentiation, that

$$\partial^n K(\xi',\xi'')/\partial(\kappa^2)^n=n!(K^{n+1})(\xi',\xi''). \tag{9.1}$$

Expansion in Taylor series gives, on applying (9.1),

$$K(\xi',\xi'';\kappa^2+\delta\kappa^2)=\sum_0^\infty(\delta\kappa^2)^n(K^{n+1})(\xi',\xi'';\kappa^2), \tag{9.2}$$

so that the kernel for $\kappa^2+\delta\kappa^2$ can be expressed in terms of powers of K, the multiplications being meant in a matrix sense. The expansion can be verified by noting

that

$$\mathfrak{L}K^{n+1}(\xi,\xi')=\mathfrak{L}\int K(\xi,\xi'')(K^n)(\xi'',\xi')d\xi''$$
$$=(K^n)(\xi,\xi') \quad (10)$$

as a consequence of (8). Applying $\mathfrak{L}$ to the right side of Eq. (9.2) and remembering (8) and (10), it is seen that

$$(\mathfrak{L}-\delta\kappa^2)\sum_0^\infty(\delta\kappa^2)^nK^{n+1}(\xi,\xi';\kappa^2)=\delta(\xi-\xi'),$$

which means that the right side of Eq. (9.2) is K for the changed κ^2.

Symbolically the expansion of $K(\xi',\xi'';\kappa^2+\delta\kappa^2)$ can be obtained by writing

$$K=\mathfrak{L}^{-1}, \quad (11)$$

and

$$K(\xi',\xi'',\kappa^2+\delta\kappa^2)=(\mathfrak{L}-\delta\kappa^2)^{-1}$$
$$=\mathfrak{L}^{-1}+\sum_1^\infty(\delta\kappa^2)^n\mathfrak{L}^{-n}=\sum_0^\infty(\delta\kappa^2)^n(K^{n+1})(\xi',\xi'';\kappa^2). \quad (11.1)$$

This result enables one to calculate the change $\delta\kappa^2$ needed to compensate for $\lambda-1$ to any order. The first order calculation is especially simple. In this case, (3.3) and (7.1) give, on subtraction,

$$(\lambda-1)\int K(\xi,\xi')\rho(\xi')d\xi'$$
$$=\left\{\int[\partial K(\xi,\xi')/\partial(\kappa^2)]\rho(\xi')d\xi'\right\}\delta(\kappa^2).$$

Upon multiplying by $\rho(\xi)d\xi$, integrating, and making use of (8.2), this condition becomes

$$(\lambda-1)\int\int\rho(\xi)K(\xi,\xi')\rho(\xi')d\xi d\xi'$$
$$=[\delta(\kappa^2)]\int\int\rho(\xi)K(\xi,\xi'')K(\xi'',\xi')\rho(\xi')d\xi d\xi'd\xi'',$$

and making use of (3.3) and (6) as well as of $\lambda\cong1$, one has

$$[\delta(\kappa^2)]\int u^2d\xi=(\lambda-1)\int u^2Vd\xi, \quad (11.2)$$

which is the Rayleigh-Schrödinger first-order formula. The expansion of K in powers of $\delta\kappa^2$ can be related to the representation

$$K(\xi,\xi';\kappa^2)=\sum_\mu\psi_\mu(\xi)\psi_\mu^*(\xi')/(\kappa_\mu^2-\kappa^2), \quad (11.3)$$

where the ψ_μ, κ_μ^2 are the eigenfunction and eigenvalue system of $\mathfrak{L}$. According to this expansion and disregarding questions of convergence,

$$K(\xi,\xi';\kappa^2+\delta\kappa^2)=\sum_{n=0}^\infty\sum_\mu\psi_\mu(\xi)\psi_\mu^*(\xi')(\delta\kappa^2)^n/(\kappa_\mu^2-\kappa^2)^{n+1}.$$

Comparing this expression with Eq. (11.1), one obtains

$$(K^{n+1})(\xi',\xi'';\kappa^2)=\sum_\mu\psi_\mu(\xi')\psi_\mu^*(\xi'')/(\kappa_\mu^2-\kappa^2)^{n+1}, \quad (11.4)$$

a relation which is readily verified by means of Eq. (11.3) if one makes use of the orthogonality of the ψ_μ. In this derivation of the expansion, there is a difficulty regarding convergence of series whenever $|\kappa_\mu^2-\kappa^2|<\delta(\kappa^2)$. Since the convergence of Eq. (11.1) rather than of parts of the series is of interest, it does not follow that (11.1) will be inapplicable under the same conditions. On the other hand, the convergence of the series for K has not been studied by the writer and may possibly be a difficulty in some applications. By means of the representation (11.3), one can formally relate the λ_{1n}, u_{1n} system to that of the κ_μ^2, ψ_μ. For simplicity the subscript $1n$ will be written as n in this connection. Expanding

$$u_n(\xi)=\sum_\mu a_{n\mu}\psi_\mu(\xi), \quad (11.5)$$

and substituting in Eq. (4), one finds

$$(1/\lambda_n)c_{n\mu}-\sum_\nu v_{\mu\nu}c_{n\nu}=0, \quad (11.6)$$

where

$$c_{n\mu}=(\kappa_\mu^2-\kappa^2)^{\frac{1}{2}}a_{n\mu}, \quad (11.7)$$

and

$$v_{\mu\nu}=(\kappa_\mu^2-\kappa^2)^{-\frac{1}{2}}(\kappa_\nu^2-\kappa^2)^{-\frac{1}{2}}\int\psi_\mu^*V\psi_\nu d\xi. \quad (11.8)$$

The matrix $\|v_{\mu\nu}\|$ is Hermitean, and according to Eq. (11.6) the $c_{n\mu}$ can be made to form a unitary matrix. According to (11.8), a diagonal element of $(\psi_\mu,V\psi)_\nu$ changes sign on transformation to $v_{\mu\nu}$ if $\kappa_\mu^2<\kappa^2$ and retains its sign if $\kappa_\mu^2>\kappa^2$, in agreement with the radical difference in the eigenvalue spectra of the κ_μ^2 and that of the $1/\lambda_m$. It should be noted that if $\|c_{n\mu}\|$ is made unitary, the normalization of Eq. (4.3) is not reproduced for orthonormal ψ_μ. The fact that the transition from the κ_μ^2 to the $1/\lambda_m$ can be represented by a unitary matrix puts the change to the λ_m on the same footing as that of other changes of reference systems in quantum mechanics.

Correction for effect of $\lambda-1$.—On replacing κ^2 by $\kappa^2+\delta\kappa^2$ in $\mathfrak{L}$, an equation like (3.3) is obtained but with $K(\xi,\xi';\kappa^2)$ replaced by $K(\xi,\xi';\kappa^2+\delta\kappa^2)$. The $\delta\kappa^2$ can now be adjusted so as to make $\lambda=1$. The desired equation is then satisfied exactly for an adjusted energy parameter which corresponds to the energy modified by coupling of the potential wells. The correction will be carried out here only to the first order in $\lambda-1$. If one denotes by n the particular n for which $\lambda\cong1$, it follows from Eq. (5.4) that

$$b_{2m}\cong[\lambda_{2m}/(1-\lambda_{2m})]V_{2mn}b_{1n}, \quad (12)$$

and substituting this value into the first line of Eq. (5.4), one has

$$\lambda-1\cong\sum_m|V_{1nm}|^2/[\lambda_{2m}(1-\lambda_{2m})], \quad (12.1)$$

a second-order effect in the matrix elements. The change

in λ which must be produced to change its value back to 1 is

$$\delta\lambda = -(\lambda - 1), \tag{12.2}$$

in agreement with (22.1). When this change is made in Eq. (7), the value of κ^2 changes by

$$\delta\kappa^2 = -(\delta\lambda)\int (V_1+V_2)u^2 d\xi, \tag{13}$$

with the normalization

$$\int u^2 d\xi = 1. \tag{13.1}$$

In applications one needs also the change in the wave function u associated with $\delta\kappa^2$ especially because $\partial u/\partial\xi$ and $\partial^2 u/\partial\xi^2$ are responsible for transitions between potential energy curves calculated for fixed nuclei. The calculation of the required change δu can be made by introducing the logarithmic derivative

$$y = (\partial u/u\partial\xi), \tag{14}$$

which experiences a first-order change

$$\delta y = (\partial y/\partial\lambda)\delta\lambda + [\partial y/\partial(\kappa^2)]\delta\kappa^2$$

as a result of the combined action of changes in λ and in κ^2. One obtains in the well-known manner

$$\partial[u^2\partial y/\partial\lambda]/\partial\xi = (V_1+V_2)u^2,$$
$$\partial[u^2\partial y/\partial(\kappa^2)]/\partial\xi = u^2,$$

and hence

$$\delta y = u^{-2}\int^{\xi} [(V_1+V_2)\delta\lambda + \delta(\kappa^2)]u^2 d\xi. \tag{14.1}$$

If the left boundary is $\xi = -\infty$, an auxiliary function $\hat{u}$ can be introduced, differing from u only in its normalization so that

$$u = N\hat{u}, \tag{14.2}$$

and defined so that

$$\hat{u} \sim e^{\kappa\xi}, \quad (\xi = -\infty).$$

One has then

$$\ln\hat{u} = \kappa\xi + \int_{-\infty}^{\xi} (\partial\hat{u}/\hat{u}\partial\xi - \kappa)d\xi,$$
$$\delta(\ln\hat{u}) = \xi\delta\kappa + \int_{-\infty}^{\xi} (\delta y - \delta\kappa)d\xi. \tag{14.3}$$

In (14.3), δy is available from (14.1). If the left boundary is not $\xi = -\infty$, a function $\hat{u}$ with arbitrary but convenient normalization at the left boundary can be introduced and an equation like (14.2) can be derived. The second term is the same as in (14.3) provided the lower limit $-\infty$ is replaced by the value of ξ at the left boundary. The first term $\xi\delta\kappa$ becomes replaced by $(\xi - \xi_l)\delta\kappa$, where ξ_l is the value of ξ at the left boundary. It is assumed here that the boundary condition at $\xi = \xi_l$ gives a fixed value of y. The correctness of the statement just made is verified by noting that it is obvious for $\xi = \xi_l$ and that differentiation of (14.3) with respect to ξ gives a correct equation. With the known $\delta(\ln\hat{u})$ the other necessary quantities can be obtained as follows. One has

$$\delta\int \hat{u}^2 d\xi = 2\int \hat{u}^2\delta(\ln\hat{u})d\xi,$$

and hence, from Eq. (14.2),

$$\delta N = -N^3\int \hat{u}^2\delta(\ln\hat{u})d\xi.$$

Upon employing these relations, it follows that

$$\delta u = -\left[u \Big/ \int u^2 d\xi\right]\int u^2(\delta\ln\hat{u})d\xi + u\delta\ln\hat{u}, \tag{15}$$

the result being written in a form homogeneous in u so that in the result the normalization of Eq. (13.1) may be dropped. The change just calculated has been obtained on the supposition that δy is caused by combined action of $\delta\lambda$ and $\delta(\kappa^2)$ as in (14.1) and continuing u from the left boundary to the right. If $\delta(\kappa^2)$ is taken from Eq. (13), the boundary condition at the right boundary is satisfied automatically to within the first order in $\delta\lambda$. The correction to u caused by $\delta\lambda$ is of second order in the V_{2mn}, V_{1nm} and may be neglected in the order of accuracy used for Eq. (12).

IV. EXTENSIONS

The procedure used can be extended to somewhat more general situations. The extension to three-dimensional potential wells is immediate regarding the discussion down to Eq. (5.4). The potential wells need not be spherically symmetric. The auxiliary field $\mathfrak{V}$ may be supposed to be present. The construction of K has to be modified, but in many cases K can be constructed and in case of $\mathfrak{V}=0$ the construction is standard. The correction to κ^2 arising from $\lambda - 1$ also can be made in the same way as in the one-dimensional case. The correction to u arising from $\delta\kappa^2$ can probably not be made by a formula as compact as Eq. (15) but the Rayleigh-Schrödinger perturbation method is available. At this point the energy continuum enters, partly defeating the aim of avoiding anything but pure tunnelling terms. On the other hand, the correction to u for $\delta(\kappa^2)$ need not be made in the first-order calculation for u and $\delta(\kappa^2)$ contains the square of the barrier penetration factor.

If the potential well V_1 is spherically symmetric, the eigenfunctions u_{1n} can be expressed as $Y_{Lm}R_L(r)$, where the Y_{Lm} are spherical harmonics of polar angles. The eigenfunctions are thus somewhat similar to the Schrödinger eigenfunction for a fixed potential. The

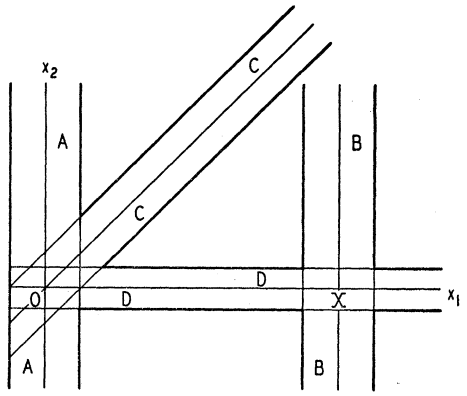

FIG. 1. Illustration of a simple many-body process. Two particles 1 and 2 move along a straight line on which they have coordinates x_1, x_2. Particle 1 is acted on by a potential in strip A when $x_1 \cong 0$, in strip B when $x_1 \cong X$ and also in strip C when $x_1 \cong x_2$. Particle 2 is free outside of strips C and D. Nuclear tentacle formation takes place in C.

$R_L(r)$ contain effects of the centrifugal barriers which depend on L. In the three-dimensional case, there are thus different tunnelling factors for different eigenfunctions.

The extension of the method to many-body problems is possible but lacks uniqueness regarding the choice of the eigenfunction systems. It nevertheless has possibilities which may merit a brief discussion. A system consisting of two particles 1, 2 moving along the same straight line and having coordinates x_1, x_2, respectively, will be considered as an illustration. In Fig. 1 the two vertical strips A, B represent locations of particle 1 with coordinate x_1 in the vicinity of $x_1=0$ for A and of $x_1=X$ for B. Particle 2 is supposed to interact with 1 so that there is a region C in the vicinity of $x_1=x_2$ within which there is a potential. Particle 2 is also supposed to be exposed to an attractive potential in strip D in the vicinity of $x_2=X$. For large X the potential energy region around 0 consisting of strips A, C, D and their common portions can produce binding of 1 and 2 in the vicinity of 0 giving a kind of a compound nucleus. As X decreases the wave function can leak into B so that particle 1 becomes transferred to the vicinity of X. The leakage can take place through channel D, particle 2 remaining permanently bound to 0. If there is enough energy and if the binding of 1 to 2 is large enough, these particles can escape together via C and afterwards attach themselves to X. Such attachment corresponds to there being an appreciable value of the wave function u in the vicinity of the intersection of channels B and C. Finally some leakage can occur without special use of D and C, corresponding to escapes of 1 and 2 followed by attachment of the particles to 0 and X or else to attachment of 1 to X and the escape of 2.

The calculation of these processes involves the knowledge of the eigenfunction system. This system is more complicated than in the one dimensional case as may be seen even by considering an isolated strip A. Since this strip represents the binding potential of 1 to 0 it corresponds to a potential $V_A(x_1)$ and is independent of x_2. The associated wave equation is

$$[\partial^2/\partial x_1^2+\partial^2/\partial x_2^2-\kappa^2-V_A(x_1)]u=0.$$

Representing u as

$$u=(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{+\infty} u_\omega(x_1)\exp(i\omega x_2)d\omega \qquad (16)$$

one has

$$[d^2/dx_1^2-\kappa^2-\omega^2-V_A(x_1)]u_\omega(x_1)=0. \qquad (16.1)$$

The eigenfunction system thus contains one-dimensional functions $u_\omega(x_1)$ with effectively different energy parameters $\kappa^2+\omega^2$. Large ω give large kinetic energies to 2 and decrease the energy available to 1. The decrease of u with x_1 becomes accordingly steeper and there is less large distance leakage. One expects therefore the general situation to correspond to different tunnelling factors for 1 depending on the kinetic energy acquired by 2.

On the other hand, if there is no interaction between 1 and 2, transfer of 1 must occur as in the one-particle problem for 1 alone. In this case there is no channel C and the potential may be represented as

$$V=V_A+V_D+V_B.$$

The part V_A+V_B depends on x_1 only, V_D on x_2 only, the wave equation is separable and u consists therefore of two factors depending on x_1 and x_2, respectively, verifying the expectation regarding reduction to a one-particle problem. From the point of view of the integral equation, this means that a density function $\rho_D(x_2)$ remains unaffected by the coupling of $\rho_A(x_1)$ to $\rho_B(x_1)$. The coupling of A to B does not involve the modifications in the tunnel factors which have been considered in connection with the eigenfunction system of channel A. The introduction of the interaction between particles 1 and 2 brings in the whole set of functions u_ω of Eqs. (16), (16.1) and in the general case the transcription of Eq. (5.4) contains barrier penetration factors with $\kappa^2+\omega^2$ in place of κ^2. There are several possible choices of the eigenfunction system. One possibility is to introduce the systems belonging to V_A, V_B, V_C, V_D separately. The coupled equations connecting A, C, D then include among their solutions the stable state in which particles 1 and 2 are bound to 0. Another possible choice is to treat $V_A+V_C+V_D$ in one step and to have the stable state included in the eigenfunction system. With this choice it is convenient to have κ^2 correspond to the energy of the stable state. The tunnelling factors then include among them the penetrabilities from parts of C to B and can be appreciably greater than those from A to B. This circumstance will be especially pronounced if 1 and 2 attract each other and if the binding of the compound 1+2 to 0 is not excessive. The wave function u decreases then only mildly as one proceeds in C away from 0, and the

penetration to channel B is seriously affected. This situation is encountered for example if the absolute value of the binding energy of a deuteron to the remainder of the nucleus is not too great. The deuteron has then an appreciable probability of being found away from the nucleus and the proton or neutron can then find itself sufficiently close to a bombarding nuclear projectile to leave its original nucleonic partners and to join those in the projectile. Unless B is very close to A the most probable leakage is to a part of B between D and the intersection with C. For the larger distances the shortest path is closer to D indicating a qualitative preference for a transfer of particle 1 alone. An attractive force insufficient to produce binding such as between two protons also gives a concentration in a channel like C but with a more rapid decay.

These considerations indicate that "resonating group structure" in nuclei[5] is likely to have special importance in relation to questions of nuclear radii. The nucleus when considered in many-dimensional space is only partly similar to a compactly filled volume associated with a spherical or spheroidal three-dimensional nuclear model. This picture appears to be more properly supplemented by the addition of a set of tentacles corresponding to regions of configuration space in which there are deuterons, tritons, or alpha particles in states corresponding to negative kinetic energies of relative motion of these particles with respect to the residual nucleus. One may expect therefore that in some nuclear reactions the apparent nuclear radius calculated from reaction yield data on the assumption of spherical nuclear shape will be somewhat larger than the radius of the main region of configuration space occupied by the nucleons. One would expect these phenomena to depend on the particular reaction but in most cases a change in the effective radius is to be expected. The effect of configuration space tentacles exists not only for reactions depending on particle transfer but also for reactions depending on the approach of a projectile to a distance within the range of force of the nucleons in the target. Thus if in Fig. 1 the exponentially decaying wave-function branch in C comes close to the intersection with B, then the direct action of X on the deuteron becomes possible any place within the width of B which corresponds to the range of force.

In transfer reactions it is not too likely that a tentacle caused by a deuteron, triton, or alpha-particle formation will be more effective in causing transfer than the direct transfer of a single particle. The reason is the increase in the mass of the particle which decreases its penetration along a channel such as C. It is therefore more probable in a transfer reaction for a nucleon to escape the nuclear surface directly than to be carried part of the way inside a deuteron or a heavier particle although the latter process exists in addition. Thus the binding of a proton in N^{14} is 8.2 mMU, that of a neutron 11.4 mMU, while H^2, He^3, and He^4 are bound in it by 11.0, 22.2, and 12.5 mMU, respectively. Since the barrier penetration exponents contain products of binding energies and masses, the latter three bindings are equivalent on the single-nucleon basis to 22.0, 66.6, and 49.0 mMU, respectively. In this case it is probable therefore that there will be more direct neutron transfers to another nucleus than through the negative-energy regions of heavier combinations just mentioned. On the other hand, it is conceivable that some of these particles give wave functions of an especially favorable symmetry for neutron transfer. It is therefore premature to exclude the possibility of an effectively larger nuclear radius in some cases in a transfer reaction taking place by tunnelling, and one expects the tunnelling of a single particle to be partly replaced by contact with the tentacles at shorter distances.

The two three-dimensional potential wells, the character of the eigenfunction system of a single strip, such as A, and the formation of tentacles have the common feature of spoiling the purity of the approximately exponentially varying barrier penetration factors expected for single-nucleon transfer on the simplest view. For sufficiently large distances between the projectile and target, however, the most slowly varying factor becomes the most important one and is then likely[6] to be determined by single-nucleon transfer.

[5] J. A. Wheeler, Phys. Rev. **52**, 1083, 1107 (1937).

[6] Breit, Hull, and Gluckstern, Phys. Rev. **87**, 74 (1952).