Scattering of Electrons from Nuclear Charge Fluctuations*

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An attempt is made to discuss the influence of non-potential-scattering effects in the elastic scattering of electrons from nuclei. It is shown that in the second order of perturbation theory, if we assume the major contributions to come from low-lying nuclear states, the scattering is completely determined by specifying the average charge density of the ground state, and the density of the fluctuations about this charge distribution. Numerical computations are presented for one simplified model of these distributions.

I. INTRODUCTION

THE elastic scattering of electrons from nuclei is generally treated[1] as the scattering from a static potential, while it actually involves the scattering from a many-body system. Although the experiments performed to date are consistent with analyses in terms of static potential,[2] there are in principle corrections to this analysis, which can *not* be ascribed to a static potential. It is of some interest to estimate these corrections, particularly since electron scattering from light nuclei represents a sufficiently simplified limit of the general many-body problem to admit a comparison of the two points of view. This limit, which is called the static,[3] or adiabatic, limit, pertains to the scattering of a light, fast particle from a system of heavy, slowly moving bodies. There is little energy transfer from the particle to the system, and so the scattering is mainly elastic. We can take advantage of this simplicity by expanding the scattering amplitude in powers of the velocity ratio, and, at the same time, in powers of the interaction potential; that is, we will compute the plural scattering of a particle, in the static limit. It will be shown that this limit is then sufficiently tractable to allow detailed calculation of the deviations from potential scattering.

Specifically, we will calculate terms of first and second order in the potential, neglecting terms of the order of the velocity ratio. The static approximation, as we shall see, involves the neglect of terms proportional both to the *unperturbed* velocity of the proton, and to its *recoil* velocity during the collision, which restricts the electron energy to a range from, say, 1 Mev to 300 Mev. At higher energies than this, there will be considerable excitation of the nucleus, and the static approximation will break down. The accuracy of the expansion in powers of the potential is discussed elsewhere,[4] and limits the application of these calculations to light or medium weight nuclei, say $Z<40$.

The only previous calculation of such effects for the nuclear scattering of electrons was carried out under different conditions of validity.[5] For atomic scattering of electrons, these approximations have long been in use.[6] A discussion of both of these predecessors of this work will be given in Sec. VI.

The principal result of the present treatment is a proof that, within the above restrictions, the deviations from potential scattering can be related to a single further property of the ground state; namely, the pair correlation function. Thus, so long as only low-lying states of the nucleus are excited, and we can neglect third- and higher order terms, the scattering can be represented as potential scattering from the charge distribution, and fluctuation scattering from the pair correlations. By making rather crude assumptions regarding these two properties of the ground state, we can calculate the scattering in complete detail. A further result, already well known,[7] is that the total inelastic cross section also depends on these same two functions, suggesting that future experiments should attempt to measure *both* the elastic and total inelastic cross sections, in order to determine simultaneously both of these features of nuclear ground states.

II. DERIVATION OF THE SCATTERING AMPLITUDE

Two alternative, equivalent derivations of the relevant formulas are available. The first starts from the "stationary states" formulation of scattering theory, and the second from the "time-dependent" formulation. We shall present the first here, as somewhat more familiar.

We want to consider the scattering of a Dirac particle from a system of many bodies, with which it interacts via a central, spin independent potential (the Coulomb potential). The time-independent equations of motion for the wave function describing the entire system are thus

$$\{\mathcal{H}_D(r)+\mathcal{H}_0(\xi_1\cdots\xi_N)+\sum_{i=1}^{N}V(r-\xi_i)-E\}\Psi(r,\xi_1\cdots\xi_N)=0. \quad (1)$$

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If we expand the wave function Ψ in terms of the (complete) set of nuclear states $\varphi_n(\xi_1\cdots\xi_N)$, with energies E_n,

$$\Psi(r,\xi_1\cdots\xi_N)=\sum_n \psi_n(r)\varphi_n(\xi_1\cdots\xi_N),$$

we find the following equations for the various energy electrons present:

$$\{\epsilon_n-\mathcal{H}_D(r)\}\psi_n(r)=\sum_m \mathcal{V}_{nm}(r)\psi_m(r). \tag{2}$$

In the above equations $\mathcal{H}_D$ is the Dirac Hamiltonian, $\mathcal{H}_0$ is the entire nuclear Hamiltonian with neutron coordinates suppressed, $V(r)$ is the Coulomb potential. Also, $E_n+\epsilon_n=E$ and $\mathcal{V}_{nm}(r)=(\varphi_n|\sum_i V(r-\xi_i)|\varphi_m)$.

We can rewrite (2) as integral equations by introducing the Green's functions satisfying

$$\{\epsilon_n-\mathcal{H}_D(r)\}G(\epsilon_n,\mathbf{r},\mathbf{r}')=\delta(r-r') \tag{3}$$

and appropriate boundary conditions. Using the boundary condition that there be an incident wave $\phi(r)$ only for the ground state of the nucleus ($n=0$), and outgoing scattered waves in every state, we find

$$\psi_n(r)=\delta_{n,0}\phi(r)+\sum_m\int \mathbf{r}'G(\epsilon_n,\mathbf{r},\mathbf{r}')\mathcal{V}_{nm}(r')\psi_m(r'). \tag{4}$$

The scattering amplitudes, correct to second order in V can be obtained by iterating (4) in the familiar way. Using

$$G(\epsilon_n,\mathbf{r},\mathbf{r}')\underset{r\to\infty}{\simeq}-\frac{1}{4\pi}\{\mathcal{H}_D(k_f)+\epsilon_n\}\frac{e^{ik_nr}}{r}e^{-i\mathbf{k}_f\cdot\mathbf{r}}, \tag{5}$$

we find for the elastic scattering amplitude

$$f_0\cong-\frac{1}{4\pi}\{\mathcal{H}_D(k_f)+\epsilon_0\}\Big\{\int d\mathbf{r}e^{-i\mathbf{k}_f\cdot\mathbf{r}}\mathcal{V}_{00}(r)\phi(r)$$
$$+\sum_m\int\int d\mathbf{r}d\mathbf{r}'e^{-i\mathbf{k}_f\cdot\mathbf{r}}\mathcal{V}_{0m}(r)$$
$$\times G(\epsilon_m,r,r')\mathcal{V}_{m0}(r')\phi(r')+\cdots\Big\}. \tag{6}$$

To simplify the notation, let us write

$$\mathcal{V}_{nm}(r)=(\phi_n|\textstyle\sum_i V(r-\xi_i)|\phi_m)$$
$$=\int d\xi V(r-\xi)(\phi_n|\textstyle\sum_i\delta(r-\xi_i)|\phi_m),$$

and then call

$$\rho_{nm}^{(1)}(r)=(\phi_n|\textstyle\sum_i\delta(r-\xi_i)|\phi_m),$$
$$\rho_{nm}^{(2)}(r,r')=(\phi_n|\sum_{i\neq j}\delta(r-\xi_i)\delta(r'-\xi_j)|\phi_m). \tag{7}$$

These are the (transition) densities of single particles, and of pairs of particles. In terms of these quantities, and in the momentum representation of the integrals, (6) becomes

$$f_0\cong-\frac{1}{4\pi}\{\mathcal{H}_D(k_f)+\epsilon_0\}\Big\{(k_f|V|k_i)(k_f|\rho_{00}^{(1)}|k_i)$$
$$+\Big(\frac{1}{2\pi}\Big)^3\int d\mathbf{k}'(k_f|V|k')(k'|V|k_i)$$
$$\times\sum_m\frac{(k_f|\rho_{0m}^{(1)}|k')(k'|\rho_{m0}^{(1)}|k_i)}{\epsilon_m+i\delta-\mathcal{H}_D(k')}+\cdots\Big\}u_i. \tag{8}$$

We have set $\phi(r)=e^{i\mathbf{k}_i\cdot\mathbf{r}}u_i$.

III. THE STATIC APPROXIMATION

Had we assumed the nucleus to remain in its ground state during the scattering, only the $m=0$ term would be present in (8); this single term is the only second-order term included in a calculation with a static potential. The other terms, in which the nucleus is (virtually) excited during scattering, are our primary concern. Their contribution to the scattering has been called the dispersion contribution,[5] in analogy with calculations of the light scattering. Since the sum includes all excited states of the nucleus, the scattering depends on more detailed features of the nucleus. Rather than relate the scattering to details of the excited-state wave functions, we can, by very general relationships of quantum theory, carry out the sums and relate the scattering to matrix elements involving only the ground-state wave functions, and details of the nuclear Hamiltonian. It is the purpose of this section to show that, in the static approximation, these sums can be easily carried out, and lead to particularly simple results.

This simplification results whenever the major contributions to the sum in (8) come for energies $\epsilon_m\approx\epsilon_0$; i.e., whenever the excitation of highly excited states is negligible. In this case we can expand $G(\epsilon_m)$ about $G(\epsilon_0)$, and carry out the sums over m term-wise. The expansion is in powers of the ratio of the virtual excitation energy to the initial electron energy, and will converge rapidly if the electron is not too energetic. It breaks down at high energies due to the excitation of continuum states, such as (e,ep) reactions, which carry off a large fraction of the electron energy. When the electron wavelength becomes much shorter than the nuclear radius, these reactions will predominate over elastic scattering. At 300 Mev, an electron can give up about 40% of its energy in a collision with a free proton, and the static approximation will no longer be valid.

If we replace ϵ_m by ϵ_0 in (8), the sum can be carried out as follows:

$$\sum_m\rho_{0m}^{(1)}(r)\rho_{m0}^{(1)}(r')=\sum_m(\phi_0|\textstyle\sum_i\delta(r-\xi_i)|\phi_m)$$
$$\times(\phi_m|\textstyle\sum_j\delta(r'-\xi_j)|\phi_0)$$
$$=(\phi_0|\textstyle\sum_{ij}\delta(r-\xi_i)\delta(r'-\xi_j)|\phi_0)$$
$$=\delta(r-r')\rho_{00}^{(1)}(r)+\rho_{00}^{(2)}(r,r'). \tag{9}$$

Thus, in the static limit the scattering depends on the nuclear state only through these two features of the ground-state wave function; i.e., its single-particle distribution function (charge distribution) and its pair distribution function.

The succeeding terms in the expansion about this static limit will involve the same sum with higher powers of the energy transfer $\epsilon_0-\epsilon_m$. For example, the next term will be, after taking its momentum matrix elements,

$$\sum_m (k_f|\rho_{0m}^{(1)}(r)|k')(\epsilon_0-\epsilon_m)(k'|\rho_{m0}^{(1)}(r')|k_i)$$
$$=\sum_m(\phi_0|\sum_i \exp(i\mathbf{K}_2\cdot\xi_i)|\phi_m)(E_m-E_0)$$
$$\times(\phi_m|\sum_j \exp(+i\mathbf{K}_1\xi_j)|\phi_0), \quad (10)$$

where $\mathbf{K}_1=\mathbf{k}_i-\mathbf{k}'$ and $\mathbf{K}_2=\mathbf{k}'-\mathbf{k}_f$ are the momentum transfers in the two collisions. This sum can be performed, using

$$E_m|\phi_m)=\mathcal{H}_0|\phi_m).$$

If we treat the protons nonrelativistically, and neglect exchange forces, it follows that the sum (10) is

$$\frac{1}{4M}\sum_{i\neq j}(\phi_0|\exp(iK_1\cdot\xi_i+iK_2\cdot\xi_j)$$
$$\times[2\mathbf{p}_j\cdot\mathbf{K}_1-2\mathbf{p}_i\cdot\mathbf{K}_2+K_1^2-K_2^2]|\phi_0)$$
$$+\frac{1}{4M}\sum_i(\phi_0|\exp(i\mathbf{K}\cdot\xi_i)$$
$$\times[2\mathbf{p}_i\cdot(\mathbf{K}_1-\mathbf{K}_2)+2K_1^2-K^2]|\phi_0), \quad (11)$$

where $\mathbf{K}=\mathbf{K}_1+\mathbf{K}_2$ is the total momentum transfer, M the proton mass, and $\mathbf{p}_i$ the momentum operator for the ith proton.

Examination of (11) shows that the terms, when divided by ϵ_0, are all proportional to a proton velocity divided by c, with terms involving the unperturbed velocity p/M and the recoil velocity K/M both appearing. Actually, the unperturbed terms would drop out to this order, due to their approximately isotropic orientation with respect to the electron direction; that is, matrix elements of $\mathbf{p}_i$ should be zero, or very small for most nuclei. Thus, in neglecting (11), we are dropping terms of order K/Mc, and of order $(p/Mc)^2$.

IV. INTRODUCTION OF THE CORRELATION FUNCTION

It is convenient for several reasons to rewrite Eq. (9) by adding and subtracting the $m=0$ term to the expression. This enables us to separately discuss the potential scattering terms, and also clarifies the physical interpretation of the additional "dispersion contribution." Writing $\rho_{00}^{(2)}(r,r')$ as

$$\rho_{00}^{(2)}(r,r')=\rho_{00}^{(1)}(r)\rho_{00}^{(1)}(r')$$
$$-[\rho_{00}^{(1)}(r)\rho_{00}^{(1)}(r')-\rho_{00}^{(2)}(r,r')], \quad (12)$$

corresponds to the expansion of the pair distribution function in terms of its "cluster functions," a procedure commonly employed in classical statistical mechanics. The first term in (12) represents the random (asymptotic) portion of the pair distribution function, and the second term the fluctuations, due to correlations, about this random value. Presumably for $\mathbf{r}\gg\mathbf{r}'$, the first term will predominate[8]; that is, the second term has a shorter "range" than the first.

As a final further simplification, again made in analogy with statistical mechanics, we will assume that the fluctuation density has the form

$$\rho_{00}^{(1)}(r)\rho_{00}^{(1)}(r')-\rho_{00}^{(2)}(r,r')$$
$$\cong\tfrac{1}{2}[\rho_{00}^{(1)}(r)+\rho_{00}^{(1)}(r')]g(|\mathbf{r}-\mathbf{r}'|). \quad (13)$$

We will refer to $g(r)$ as the correlation function. Notice that (13) implies that $\int d\mathbf{r}g(r)=1$. It should be pointed out that (13) is appropriate for distributions of classical bodies (i.e., obeying classical statistics) in an homogeneous, isotropic medium, and may be far from the truth for an actual nucleus. Our point of view is frankly phenomenological at this point, since we have little knowledge on which to base more accurate assumptions.

Combining these assumptions and definitions, Eq. (8) becomes, in the static approximation

$$f_0\cong-\frac{1}{4\pi}\{\mathcal{H}_D(k_f)+\epsilon_0\}\Big\{(k_f|V|k_i)(k_f|\rho_{00}^{(1)}|k_i)$$
$$+\left(\frac{1}{2\pi}\right)^3\int\mathbf{dk}'\frac{(k_f|V|k')(k'|V|k_i)}{\epsilon_0+i\delta-\mathcal{H}_D(k')}$$
$$\times(k_f|\rho_{00}^{(1)}|k')(k'|\rho_{00}^{(1)}|k_i)+\left(\frac{1}{2\pi}\right)^3(k_f|\rho_{00}^{(1)}|k_i)$$
$$\times\int\mathbf{dk}'\frac{(k_f|V|k')(k'|V|k_i)}{\epsilon_0+i\delta-\mathcal{H}_D(k')}$$
$$+\left(\frac{1}{2\pi}\right)^3\int d\mathbf{k}'\frac{(k_f|V|k')(k'|V|k_i)}{\epsilon_0+i\delta-\mathcal{H}_D(k')}\mathcal{G}(k')\Big\}u_i, \quad (14)$$

where

$$\mathcal{G}(k')=-\int\int d\mathbf{r}_1d\mathbf{r}_2e^{i\mathbf{K}_1\cdot\mathbf{r}_1+i\mathbf{K}_2\cdot\mathbf{r}_2}$$
$$\times\{\rho_{00}^{(1)}(r_1')\rho_{00}^{(1)}(r_2)-\rho_{00}^{(2)}(r_1,r_2)\}$$
$$=-\tfrac{1}{2}(k_f|\rho_{00}^{(1)}|k_i)\Big\{\int d\mathbf{r}e^{i\mathbf{K}_1\cdot\mathbf{r}}g(r)$$
$$+\int d\mathbf{r}e^{i\mathbf{K}_2\cdot\mathbf{r}}g(r)\Big\}.$$

[8] These statements are true only if we neglect terms of order $1/N$. Notice for example, that a properly defined density of fluctuations would give a total fluctuation of zero, when averaged over the entire nucleus, while actually $(1/N^2)\int\int d\mathbf{r}_1d\mathbf{r}_2\{\rho'(r_1)\rho'(r_2)-\rho^2(r_1r_2)\}=1/N$. Thus our correlation function and fluctuation density are defined slightly differently than is customary in statistical mechanics, differing by terms of order $1/N$. It is deemed appropriate to define the functions as in (12), however, since they appear that way in the total inelastic cross section.

These terms are readily interpreted physically; they all describe single or double scattering events, depending on whether one or two proton coordinates appeared. Thus the first and third terms are the first- and second-order contributions to single scattering. The second and fourth terms are the double scattering, which we have, in (12), divided into double scattering from the average density, and from the fluctuations in density, respectively.

V. CROSS-SECTION FORMULAS

The final formulas for the differential cross section for the scattering of unpolarized electrons can now be deduced from (14) in the usual manner. We will simply collect the results, using a terminology very similar to that used in the preceding paper[4] for ease of comparison.

Writing the cross section in the form,

$$d\sigma(\theta)=d\sigma_1(\theta)[1+R_1+R_2+R_3], \tag{15}$$

with

$$d\sigma_1(\theta)=\left(\frac{Ze^2}{2k^2\sin^2(\theta/2)}\right)^2[1+k^2\cos^2(\theta/2)]F^2(K)d\Omega, \tag{16}$$

we find, for the three second-order correction terms,[9]

$$R_1=\frac{Ze^2kK^2}{\pi^2P^2F(K)}\int d\mathbf{k}' \times\frac{(P^2+2\mathbf{k}'\cdot\mathbf{P})F(K_1)F(K_2)}{(k'^2-k^2-i\delta)(K_1^2+\lambda^2)(K_2^2+\lambda^2)}, \tag{17a}$$

$$R_2=\frac{e^2kK^2}{\pi^2P^2}\int d\mathbf{k}'\frac{(P^2+2\mathbf{k}'\cdot\mathbf{P})}{(k'^2-k^2-i\delta)(K_1^2+\lambda^2)(K_2^2+\lambda^2)}, \tag{17b}$$

$$R_3=-\frac{e^2kK^2}{\pi^2P^2}\int d\mathbf{k}'\frac{(P^2+2\mathbf{k}'\cdot\mathbf{P})}{(k'^2-k^2-i\delta)(K_1^2+\lambda^2)(K_2^2+\lambda^2)} \times\tfrac{1}{2}[G(K_1)+G(K_2)]. \tag{17c}$$

A simple derivation gives for the total inelastic differential cross section, correct to order $(Ze^2)^2$,

$$d\sigma_{\text{inelastic}}(\theta)=\frac{1}{Z}d\sigma_1(\theta)\left[1-\frac{1}{Z}\int\int d\mathbf{r}_1d\mathbf{r}_2e^{i\mathbf{K}\cdot(\mathbf{r}_1-\mathbf{r}_2)} \times\{\rho_{00}^{(1)}(r_1)\rho_{00}^{(1)}(r_2)-\rho_{00}^{(2)}(r_1r_2)\}\right], \tag{18}$$

and using (13), this becomes

$$d\sigma_{\text{inelastic}}=\frac{1}{Z}d\sigma_1(\theta)[1-G(K)]. \tag{19}$$

[9] The notation is as follows: $\mathbf{k}_i$=initial wave number; $\mathbf{k}_f$=final wave number; $\mathbf{K}_1=\mathbf{k}_i-\mathbf{k}'$; $\mathbf{K}_2=\mathbf{k}'-\mathbf{k}_f$; $\mathbf{K}=\mathbf{K}_1+\mathbf{K}_2$; $\mathbf{P}=\mathbf{k}_i+\mathbf{k}_f$; $F(K)=(1/Z)\int d\mathbf{r}e^{i\mathbf{K}\cdot\mathbf{r}}\rho_{00}^{(1)}(r)$, $G(K)=\int d\mathbf{r}e^{i\mathbf{K}\cdot\mathbf{r}}g(r)$. In computing the integrals, we are to let δ, λ approach zero, and keep only the real parts.

We propose to compute these integrals with the choice of exponential models for both the charge density $\rho_{00}^{(1)}(r)$ and the correlation function $g(r)$:

$$\rho_{00}^{(1)}(r)=\rho_0e^{-br}, \qquad g(r)=g_0e^{-ar}. \tag{20}$$

The integration of R_1 for this model has been carried out elsewhere[4]; the evaluation of R_2 and R_3 is readily carried out by the same methods:

$$R_2=\pi e^2\sin(\theta/2)/[1+\sin(\theta/2)],$$

$$R_3=\frac{2e^2K^2}{P^2}\left\{\frac{4k^2K^2+8k^2a^2+a^4}{(K^2+a^2)^2}\arctan\left(\frac{a}{2k}\right)-\frac{2k}{K}\arctan\left(\frac{a}{K}\right)\right\}. \tag{21}$$

The numerical results for these models are presented in the next section.

As an alternative to introducing a special assumption regarding the form of $g(r)$, it may be appropriate to simply use the "model-independent" value of R_3, derived on the basis of "short-range" correlations. Thus, one readily proves that for correlation lengths short compared to the wavelength $1/k$, the value of R_3 approaches

$$R_3=-\pi e^2\sin(\theta/2)/[1+\sin(\theta/2)] +\tfrac{1}{6}\pi e^2K^2\langle r^2\rangle_{\text{Av}}+O((kr)^3), \tag{22}$$

with

$$\langle r^2\rangle_{\text{Av}}=\int_0^\infty r^4g(r)dr\Big/\int_0^\infty r^2g(r)dr.$$

Under this assumption we find

$$d\sigma(\theta)\cong d\sigma_1(\theta)[1+R_1+\pi e^2K^2\langle r^2\rangle_{\text{Av}}/6], \tag{23}$$

and

$$d\sigma_{\text{inelastic}}\cong\frac{1}{Z}d\sigma_1(\theta)K^2\langle r^2\rangle_{\text{Av}}/6. \tag{24}$$

VI. DISCUSSION

A. Numerical Results: Application to Light Nuclei

We have carried out computations based on the models (20), for several different values of k/a, keeping $k/b=1.10$; that is, for several different values of the ratio of nuclear radius to correlation length. The results are summarized in Table I.

The principal features of the fluctuation scattering contribution are that it is always of order e^2, in contrast to the second-order potential scattering which is of order Ze^2, and that it is most important near $k(\langle r^2\rangle_{\text{Av}})^{\frac{1}{2}}\approx 1$. This follows from the fact that at low energies R_2 and R_3 nearly cancel, and potential scattering predominates,

TABLE I. Values of the correction factors R_1/Ze^2, and R_3/e^2 for exponential models of the charge distribution and correlation function.

	Energy	$\theta=30°$	$\theta=60°$	$\theta=90°$	$\theta=120°$	$\theta=150°$
R_1/Ze^2	$k/b=1.10$	−0.19	−1.5	−2.6	−3.9	−3.6
R_3/e^2	$k/a=0.1$	−0.65	−1.0	−1.3	−1.5	−1.6
	$k/a=0.25$	−0.57	−0.78	−0.80	−0.76	−0.72
	$k/a=0.50$	−0.43	−0.39	−0.26	−0.17	−0.11
	$k/a=1.0$	−0.19	−0.043	0.011	0.025	0.031
	$k/a=1.5$	−0.066	0.012	0.018	0.014	0.011

and at energies such that $k(\langle r^2\rangle_{\text{AV}})^{\frac{1}{2}}\gg 1$, the fluctuation term falls off with the energy. It is clear that there is only one circumstance for which the fluctuation term can become important; namely, when there is strong cancellation between the first- and second-order potential scattering. This is known to happen however, at large momentum transfers. It is unfortunate that it is just such a cancellation that a second-order calculation cannot adequately describe. There is perhaps a region of atomic weights, say from $Z=20$ to 40, for which the second-order calculation achieves fair accuracy, and for which there is considerable interference between single and double potential scattering. The fact that R_2 and R_3 usually have opposite signs, and are both of order $e^2=1/137$ forces us to conclude, however, that only for unusually strong cancellation, reducing the potential scattering by at least a factor of ten, will the "dispersion" terms become appreciable.

B. High-Energy Limit

Schiff[5] has presented a discussion of the static approximation to the first- and second-order scattering, valid in the limit of wavelengths shorter than the nuclear radius. We have repeated this calculation, with different results. For example, let us compare his results for the second order scattering from a *static potential*, with our results. As stated in the discussion following his Eq. (31), Schiff's method gives, for the ratio of second-order to first-order contributions in a static potential,[10]

$$R_1\simeq -48Ze^2/(kR)^3\sin^4(\theta/2), \tag{25}$$

while more detailed calculations,[11] for most potentials, lead to a considerably larger result:

$$R_1\simeq -8Ze^2/kR. \tag{26}$$

We have set

$$1/R=\int_0^\infty r\rho(r)dr\Big/\int_0^\infty r^2\rho(r)dr.$$

Examination of Schiff's technique indicates that it is equivalent to assuming that, in momentum space, the major contributions come in such a way that the total momentum transfer is shared equally by the two collisions; i.e., the double scattering proceeds mainly through two scatterings of $\theta/2$ degrees. The detailed calculations lead to a different conclusion, that the momentum is transferred predominantly in a single large-angle scattering, followed (or preceded) by a small-angle scattering. One easily sees that the former assumption leads to a much *smaller* estimate of the scattering than the latter, as is shown by comparing (25) and (26).

One can attempt to repeat Schiff's discussion on the basis of this revised estimate (26). If we assume for the moment that (26) and the static approximation are simultaneously valid, then we would be led to conclude that the "dispersion contribution" was small, ($\sim$1%) since there is no appreciable cancellation between first- and second-order potential scattering. This conclusion would presumably apply even for *heavy* nuclei, if we assume, as is probably true, that higher order terms vanish as rapidly as second-order terms in this limit ($kR\gg 1$). It is probably not appropriate to assume that the wavelength is also short compared to the correlation length, and so the value of R_3 need not be small (i.e., one might still assume $R_3\approx e^2=1/137$). There is, however, considerable doubt that (26) and the static approximation have any common range of validity. For example, (26) is certainly *not* yet valid for gold at 125 Mev ($kR\sim 4$); for a gold nucleus, $kR\sim 10$ at $\sim$300 Mev, and the static approximation is of doubtful validity.

Thus it should be emphasized that the region of most interest, i.e., heavy elements, large angles, energies around 100–200 Mev, is completely inaccessible for either of these methods of estimating the fluctuation scattering. There seems to be no *a priori* reason for it to be negligible in this region; on the contrary, the large cancellation between the various orders of potential scattering, evidenced by the complete failure of the Born approximation, make it likely that the fluctuation scattering is considerably enhanced, and may contribute an amount greater than the experimental errors.

C. Dispersion Contribution to Atomic Scattering

The dispersion contribution to atomic scattering of electrons, or "polarization correction" as it is there called, has been estimated by the same techniques,[6] keeping second-order terms and making the static approximation. The result, however, is quite different: the dispersion contribution now makes up the major portion of the small-angle scattering, diverging logarithmically at 0 degree, while the potential scattering remains finite. The origin of this difference is readily discovered: the interaction energy now consists of a static nuclear potential, plus the interaction with the atomic electrons, giving a potential which falls off rapidly at large distances for a neutral atom (exponentially for a spherical electronic charge distribution). The potential scattering at small angles is therefore

[10] We have corrected a trivial error in the over-all sign of the second-order contributions.

[11] R. R. Lewis, Jr., preceding paper [Phys. Rev. **102**, 537 (1956)], Eq. (29).

finite, characteristic of a short range potential. The dispersion contribution however comes from the interaction with the *atomic electrons alone*, since the atom cannot be excited via the static nuclear potential. Thus the dispersion contribution is characteristic of scattering in a *long-range potential* (as in the nuclear case), and contributes very strongly to the small-angle scattering.

The present analyses in atomic scattering, however, always neglect the fluctuation scattering portion of the dispersion contribution; it would be of some interest to search for visible effects of these terms.

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The calculation of the wave function of a particle subjected to the field of two isolated potential wells is arranged in such a way as to bring into evidence the penetration factors for passage through regions of negative kinetic energy between the wells. The Schrödinger equation is replaced by an equivalent integral equation and the eigenfunction system of the latter is used for the expansion of the wave function. The dependence of the kernel on energy and the transformation from the reference system of the integral equation to that of the wave equation are considered. Extensions to three-dimensional problems and to many nuclear particles are discussed and some characteristic differences from the one-dimensional case are pointed out. The presence of more than one tunnelling factor in the general case and the participation of tongues or tentacles in configuration space which correspond to the temporary formation of nuclear aggregates such as deuterons or alpha particles are considered.

I. INTRODUCTION

IN calculations of stripping reactions taking place in collisions of heavy nuclear aggregates it is useful to be able to calculate the process by means of adiabatic wave functions such as occur in the work of Wigner and Pelzer.[1] In the molecular problem considered by them these are the wave functions of the electron system for fixed positions of the nuclei. Modifications of this method, employing a semiclassical discussion in which the motion of the heavy particles is treated classically while the motion of the transferred particles is calculated quantum mechanically, are also possible.[2] In both procedures it is essential to be able to calculate the wave function of a nucleon for fixed positions of the nuclear aggregates to the field of which it is subjected. For well separated nuclear aggregates, the fields of which do not overlap, it is especially desirable to have convenient approximations for such wave functions. If the problem is treated by usual perturbation theory, the expansion involves the wave functions of the continuum of the eigenfunction system of the nucleon in the field of one of the nuclei and the rather small probability of transfer which may result on account of the presence of a region of negative kinetic energy which must be traversed by the nucleon becomes apparent as the result of an interference between the effects of different regions of the continuum. It is desirable to have a treatment which is free of this complication.

A possible treatment of this type consists of expansions of the Wigner[3] type in which boundary conditions for internal wave functions are used. An infinite number of such eigenfunction systems can be introduced by changing the nuclear radius or the boundary condition. Such a procedure has many advantages but its flexibility regarding the introduction of the nuclear radius makes the barrier tunnel effects entering in first approximation flexible also. A complete calculation of all effects which is free of the initially assumed radius is possible but lengthy.

In the present note a treatment is described in which the tunnel effect factors are present throughout in a simple manner in the case of one-dimensional potential

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[1] H. Pelzer and E. Wigner, Z. physik. Chem. **15**, 445 (1932).

[2] This extension is readily made. Some considerations regarding the role of accelerations in causing gravitational effects and their relationship to the terms occurring in Pelzer and Wigner's paper will be discussed in another publication. The reference to the work of Wigner and Pelzer and of the use of wave functions for fixed nuclei is not meant in the sense of considering the expansion of the wave function in terms of such eigenfunctions as being the best from the viewpoint of convergence. It has nevertheless some definite advantages inasmuch as it provides a clear and definite starting point. In cases of slow convergence caused by resonance, linear combinations capable of improving convergence can nevertheless be introduced. This matter will be discussed in connection with an application as part of a paper which is being prepared in collaboration with Dr. M. E. Ebel.

[3] E. P. Wigner, Phys. Rev. **70**, 606 (1946); E. P. Wigner and L. Eisenbud, Phys. Rev. **72**, 29 (1947); R. G. Thomas, *The Theory of Nuclear Exchange Collisions*, Los Alamos informal report (unpublished).