Potential Scattering of High-Energy Electrons in Second Born Approximation*

R. R. LEWIS, JR.†
University of Notre Dame, Notre Dame, Indiana
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Formulas are derived for the scattering cross section of a high-energy electron in a static potential, correct to second order in the potential, in terms of certain integrals. The potential is that due to a screened, spherical charge distribution. The integrals are evaluated explicitly for Yukawa, exponential and Gaussian charge distributions, and are expanded in power series in k and in $1/k$ for an arbitrary charge distribution. Numerical results are presented.

I. INTRODUCTION

THE elastic scattering of high-energy electrons in nuclei is at present described as scattering from a stationary potential, which is a problem amenable to numerical computation.[1] This has rendered the perturbation treatments of scattering in a potential rather obsolete, especially since they are known to be badly in error at large angles for heavy elements.[2] Unfortunately, however, we have not seen the last of perturbation calculations of scattering; the replacement of a many-body system by a stationary potential is itself an approximation, and any attempt to improve this approximation leads to problems which are *not* amenable to numerical computation, forcing us back to perturbation methods. Inelastic scattering has also only been computed using perturbation theory, or phenomenological potentials. It is felt that the scattering from a stationary potential provides a good testing ground for approximation procedures, since some exact results are known for comparison.

In the course of estimating effects due to second-order virtual transitions of the nucleus during electron scattering,[3] some results have been obtained for the second Born approximation to scattering from a stationary potential. These results consist of closed formulas for the scattering from special models of the charge distribution (Yukawa, Gaussian, exponential)[4] and expansion formulas for the high- and low-energy limit of the scattering from an arbitrary regular charge distribution. Numerical results will also be given.

II. GENERAL FORMULAS

We will derive here the formulas for the scattering cross section of unpolarized electrons incident on a stationary spin-independent potential, keeping only terms of first and second order in the potential. The potential will be that of a shielded, spherical charge distribution $\rho(r)$:

$$V(r)=-Ze^2\int d\mathbf{r}'\frac{\rho(r')e^{-\lambda|r-r'|}}{|r-r'|}. \tag{1}$$

The shielding will prevent divergence difficulties due to the long range of the Coulomb field; we will let $\lambda\rightarrow 0$ in any result which remains finite in that limit.

Using the Green's function for the Dirac Hamiltonian,

$$G(r,r')=-\frac{1}{4\pi}\left[H\left(\frac{1}{i}\nabla_r\right)+\epsilon\right]\frac{e^{ik|r-r'|}}{|r-r'|}, \tag{2}$$

we can write the explicit solution of the Dirac equation, correct to second order, as

$$\Psi(r)=\phi(r)+\int d\mathbf{r}'G(r,r')V(r')\phi(r')+\int\int d\mathbf{r}'d\mathbf{r}''G(r,r')V(r')\times G(r',r'')V(r'')\phi(r'')+\cdots. \tag{3}$$

Here $H(\mathbf{p})=+\boldsymbol{\alpha}\cdot\mathbf{p}+\beta$, and ϵ, k are the energy and momentum, respectively. We use relativistic units $\hbar=m=c=1$ throughout. The incident wave is represented by $\phi(r)=ue^{i\mathbf{k}_i\cdot\mathbf{r}}$. The scattering amplitude obtained from the asymptotic form of (3) is

$$f=-\frac{1}{4\pi}[H(k_f)+\epsilon]Tu, \tag{4}$$

where T is

$$T=\int d\mathbf{r}e^{-i\mathbf{k}_f\cdot\mathbf{r}}V(r)e^{+i\mathbf{k}_i\cdot\mathbf{r}}+\int\int d\mathbf{r}d\mathbf{r}'e^{-i\mathbf{k}_f\cdot\mathbf{r}}\times V(r)G(r,r')V(r')e^{+i\mathbf{k}_i\cdot\mathbf{r}'}+\cdots. \tag{5}$$

Using the familiar rules for summing over spins, we can write the cross section for the scattering of an unpolarized electron into an unspecified final spin state, as

$$d\sigma=\frac{1}{8\pi^2}\frac{1}{4}\,\mathrm{Tr}\{[H(k_f)+\epsilon]T[H(k_i)+\epsilon]T^\dagger\}d\Omega. \tag{6}$$

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1 G. Parzen, Phys. Rev. **80**, 355 (1950); L. Acheson, Phys. Rev. **82**, 488 (1951); D. R. Yennie *et al.*, Phys. Rev. **95**, 500 (1954).

2 D. R. Yennie *et al.*, Phys. Rev. **92**, 1325 (1953).

3 R. R. Lewis, Jr., following paper [Phys. Rev. **102**, 544 (1956)].

4 Some of these results have been mentioned previously in the literature: Vachaspati, Phys. Rev. **93**, 502 (1954); R. N. Wilson, Phys. Rev. **93**, 949(A) (1954).

The first-order term in T is[5]

$$T^{(1)}=(k_f|V|k_i)=-4\pi Ze^2F(K)/(K^2+\lambda^2), \quad (7)$$

where $F(K)=\int d\mathbf{r}e^{i\mathbf{k}\cdot\mathbf{r}}\rho(r)$ is called the "form factor" of the charge distribution. Note that we have chosen $\rho(r)$ normalized to unity.

Keeping terms through second order leads to a simple form for the matrix dependence of T: it is a linear combination of the matrices $\boldsymbol{\alpha}$, β, and 1. It is appropriate to define the coefficients of these matrices in the following way:

$$T=-\frac{4\pi Ze^2}{K^2+\lambda^2}\left[F(K)+\frac{A\boldsymbol{\alpha}\cdot\mathbf{P}}{k}+\frac{2B(\epsilon+\beta)}{k}+\frac{C\boldsymbol{\alpha}\cdot\mathbf{K}}{k}+\frac{D\boldsymbol{\alpha}\cdot(\mathbf{P}\times K)}{k^2}\right]. \quad (8)$$

$A\cdots D$ will be complicated integrals, proportional to Ze^2. We can now perform the trace in (6); discarding the terms proportional to A^2, AB, $\cdots$, D^2 as of higher order, we find for the cross section

$$d\sigma=\left(\frac{Ze^2}{K^2+\lambda^2}\right)^2(P^2+4)F^2\left\{1+\frac{4\epsilon}{k}\frac{P^2}{P^2+4}\frac{A+B}{F}+\frac{4\epsilon}{k}\frac{4}{P^2+4}\frac{2B}{F}\right\}d\Omega, \quad (9)$$

where by A, B we understand the "real parts only" of the corresponding integrals. These integrals will be slowly varying functions of the angle, and so, for very high energies, the first term will always dominate the second, except for angles very near $\theta=180°$, where $P^2=0$. Thus, as long as $P^2\gg 4$, we can write simply

$$d\sigma\cong\left(\frac{Ze^2}{K^2+\lambda^2}\right)^2(P^2+4)F^2\left(1+4\frac{(A+B)}{F}\right)d\Omega \cong d\sigma_1\{1+R\}, \quad (10)$$

where $d\sigma_1$ is the cross section in first order:

$$d\sigma_1=\left(\frac{Ze^2}{K^2+\lambda^2}\right)^2(P^2+4)F^2.$$

III. REDUCTION OF THE INTEGRALS

Introducing the momentum representation of $G(r,r')$

$$G(r,r')=\left(\frac{1}{2\pi}\right)^3\lim_{\delta\to 0}\int d\mathbf{k}'\frac{H(k')+\epsilon}{k^2+i\delta-k'^2}e^{i\mathbf{k}'\cdot(\mathbf{r}-\mathbf{r}')}, \quad (11)$$

and comparing terms in (5) and (8), we find for A, B,

$$A=\frac{Ze^2kK^2}{2\pi^2P^2}\int d\mathbf{k}'\frac{\mathbf{P}\cdot\mathbf{k}'}{k'^2-k^2-i\delta}\times\frac{F(\mathbf{k}'-\mathbf{k}_f)F(\mathbf{k}'-\mathbf{k}_i)}{[|\mathbf{k}'-\mathbf{k}_f|^2+\lambda^2][|\mathbf{k}'-\mathbf{k}_i|^2+\lambda^2]}, \quad (12a)$$

$$B=\frac{Ze^2kK^2}{4\pi^2}\int d\mathbf{k}'\frac{1}{k'^2-k^2-i\delta}\times\frac{F(\mathbf{k}'-\mathbf{k}_f)F(\mathbf{k}'-\mathbf{k}_i)}{[|\mathbf{k}'-\mathbf{k}_f|^2+\lambda^2][|\mathbf{k}'-\mathbf{k}_i|^2+\lambda^2]}. \quad (12b)$$

These are the integrals which we must discuss for various charge distributions. They have been performed explicitly for only a few special models: the Gaussian potential,[6] Coulomb potential,[7] and the Yukawa potential.[8]

The most systematic method of performing such integrals is to reduce them to integrals over denominator factors alone; in the present case, this can be accomplished by using the following representation of $F(K)$:

$$F(K)=\frac{1}{i\pi}\int_C dsF(s)s[K^2-s^2]^{-1}, \quad (13)$$

where the contour C is the real axis, avoiding the singularities at $s=\pm K$ by passing above them. This will reduce the integrals (12) to the desired form

$$A+B=-\frac{Ze^2K^2}{4P^2\pi^4}\int_C dssF(s)\int_C dttF(t)I(s,t), \quad (14)$$

where

$$I(s,t)=k\int d\mathbf{k}'\frac{P^2+2\mathbf{k}'\cdot\mathbf{P}}{k'^2-k^2-i\delta}[(K_1^2+\lambda^2)(K_1^2-s^2)\times(K_2^2+\lambda^2)(K_2^2-t^2)]^{-1}. \quad (15)$$

Here we use the notation $\mathbf{K}_1=\mathbf{k}_i-\mathbf{k}'$, $\mathbf{K}_2=\mathbf{k}'-\mathbf{k}_f$.

If we introduce the further symbols

$$a=K_1^2+\lambda^2,\quad b=K_2^2+\lambda^2,\quad c=k'^2-k^2-i\delta,\quad d=k_1^2-s^2,\quad e=K_2^2-t^2, \quad (16)$$

then $2\mathbf{k}'\cdot\mathbf{P}=4k^2+2c-a-b$, and $I(s,t)$ reduces to

$$I(s,t)=k(P^2+4k^2)\int d\mathbf{k}'[abcde]^{-1}+2k\int d\mathbf{k}'[abde]^{-1}-k\int d\mathbf{k}'[bcde]^{-1}-k\int d\mathbf{k}'[acde]^{-1}.$$

Use of partial fractions expansions further reduces this

[5] We will use the notation $\mathbf{K}=\mathbf{k}_i-\mathbf{k}_f$, $\mathbf{P}=\mathbf{k}_i+\mathbf{k}_f$ throughout.

[6] T. Y. Wu, Phys. Rev. **73**, 934 (1948).
[7] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[8] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

to the final form[9]

$$I(s,t)=\frac{k}{s^2t^2}\Big\{(P^2+4k^2-s^2-t^2)\int d\mathbf{k}'[cde]^{-1}$$
$$-(P^2+4k^2-s^2)\int d\mathbf{k}'[bcd]^{-1}$$
$$-(P^2+4k^2-t^2)\int d\mathbf{k}'[ace]^{-1}+2\int d\mathbf{k}'[de]^{-1}$$
$$-2\int d\mathbf{k}'[ae]^{-1}-2\int d\mathbf{k}'[bd]^{-1}$$
$$+2\int d\mathbf{k}'[ab]^{-1}\Big\}, \quad (17)$$

which are all integrals with denominator factors. Defining

$$M_2(\mu,\nu)=k\int d\mathbf{k}'[(K_1{}^2+\mu^2)(K_2{}^2+\nu^2)]^{-1}, \quad (18a)$$

$$M_3(\mu,\nu)=k^3\int d\mathbf{k}'[(k'^2-k^2-i\delta)(K_1{}^2+\mu^2)(K_2{}^2+\nu^2)]^{-1}, \quad (18b)$$

we can write

$$I(s,t)=\frac{1}{s^2t^2}\Big\{M_3(-is,\,-it)\Big(\frac{P^2+4k^2-s^2-t^2}{k^2}\Big)$$
$$-M_3(0,\,-is)\Big(\frac{P^2+4k^2-s^2}{k^2}\Big)$$
$$-M_3(0,\,-it)\Big(\frac{P^2+4k^2-t^2}{k^2}\Big)+2M_2(-is,\,-it)$$
$$-2M_2(0,\,-is)-2M_2(0,\,-it)+2M_2(0,0)\Big\}. \quad (19)$$

These integrals are performed in Appendix I.

IV. SPECIAL MODELS

(a) Pure Coulomb Potential

As a first application, and a check on our formulas, let us consider the pure Coulomb potential. This is obtained by setting $\rho(r)=\delta(r)$ and $F(K)=1$. Then

$$A=\frac{Ze^2kK^2}{4\pi^2P^2}\Big\{4k^2\int d\mathbf{k}'[abc]^{-1}+2\int d\mathbf{k}'[ab]^{-1}$$
$$-\int d\mathbf{k}'[ac]^{-1}-\int d\mathbf{k}'[bc]^{-1}\Big\},$$

$$B=\frac{Ze^2kK^2}{4\pi^2}\int d\mathbf{k}'[abc]^{-1}\to 0.$$

[9] We have dropped a term proportional to $\int d\mathbf{k}'[abc]^{-1}$ which vanishes as $\lambda\to 0$. This is just the term which gives the nonrelativistic limit of the pure Coulomb potential, which is well known to vanish in second order.

As $\lambda\to 0$, the first terms in each vanish and the other integrals are easily shown to be

$$\int d\mathbf{k}'[ab]=\pi^3/K,\quad \int d\mathbf{k}'[ac]^{-1}=\int d\mathbf{k}'[bc]^{-1}=\pi^3/2k,$$

so that (9) becomes

$$d\sigma=[Ze^2/(K^2+\lambda^2)]^2\{(P^2+4)$$
$$+4Ze^2\epsilon k\sin(\theta/2)[1-\sin(\theta/2)]\}d\Omega, \quad (20)$$

which is a well-known result.[7,10] Note that this gives for R,

$$R=4A=\pi Ze^2\sin(\theta/2)/[1+\sin(\theta/2)] \quad (21)$$

which is a monotonic positive function of the angle, never exceeding $Ze^2\pi/2$, and independent of the energy.

(b) Yukawa Charge Distribution

The charge distribution leading to the least complicated integrals is the Yukawa charge distribution $\rho(r)=\rho_0e^{-ar}/r$. Now $F(K)=a^2/(K^2+a^2)$ and the integral in (14) can be performed by contour integration, giving

$$A+B=(Ze^2K^2a^4/4\pi^2P^2)I(ia,ia)$$
$$=\Big(\frac{Ze^2K^2}{4\pi^2P^2}\Big)\Big\{M_3(a,a)\Big(\frac{P^2+4k^2+2a^2}{k^2}\Big)$$
$$-2M_3(0,a)\Big(\frac{P^2+4k^2+a^2}{k^2}\Big)+2M_2(a,a)$$
$$-4M_2(0,a)+2M_2(0,0)\Big\}. \quad (22)$$

Using the results of the appendix, we find, with some rearrangement

$$R=\frac{4(A+B)}{F}=\Big(\frac{2Ze^2K^2(K^2+a^2)}{a^2P^2}\Big)$$
$$\times\Big\{\frac{k}{K}\frac{P^2+4k^2+2a^2}{[k^2K^2+4k^2a^2+a^4]^{\frac12}}$$
$$\times\arctan\frac{aK}{2[k^2K^2+4k^2a^2+a^4]^{\frac12}}$$
$$+\frac{2k}{K}\arctan\frac{2a^3}{K(K^2+3a^2)}$$
$$-\frac{P^2+4k^2+a^2}{K^2+a^2}\arctan\frac{a}{2k}\Big\}. \quad (23)$$

(c) Exponential Charge Distribution

The exponential charge distribution $\rho(r)=\rho_0e^{-br}$ leads to similar results. Now $F(K)=[b^2/(K^2+b^2)]^2$, and the

[10] H. Feshbach, Phys. Rev. 88, 295 (1952).

integral (14) can again be performed by contour integration, giving

$$A+B=\left(\frac{Ze^2K^2b^6}{16\pi^2P^2}\right)\frac{\partial^2}{\partial\mu\partial\nu}I(i\mu,i\nu)\bigg|_{\mu=\nu=b}$$

$$=\left(\frac{Ze^2K^2b^6}{16\pi^2P^2}\right)\frac{\partial^2}{\partial\mu\partial\nu}\frac{1}{\mu^2\nu^2}$$

$$\times\{M_3(\mu,\nu)(P^2+4k^2+\mu^2+\nu^2/k^2)$$

$$-2M_3(0,\mu)(P^2+4k^2+\mu^2/k^2)+2M_2(\mu,\nu)$$

$$-4M_2(0,\mu)+2M_2(0,0)\}_{\mu=\nu=b}. \quad (24)$$

This is actually a very cumbersome function after the differentiation is carried out, but is not too difficult to differentiate numerically.

(d) Gaussian Charge Distribution

This is the first of the models treated which has a form factor which is not algebraic; in this case the integrals cannot be done explicitly, but only reduced to a single integral. The reduction proceeds as follows:

If $\rho(r)=\rho_0\exp(-\beta^2r^2)$, then $F(K)=\exp(-K^2/4\beta^2)$ and

$$F(K_1)F(K_2)=\exp[-(K_1^2+K_2^2)/4\beta^2]$$

$$=\exp(-K^2/8\beta^2)\exp[-(k'-\tfrac{1}{2}P)^2/2\beta^2].$$

Thus we get from (12) and (13),

$$A+B=\left(\frac{Ze^2kK^2}{2i\pi^3P^2}\right)\exp(-K^2/8\beta^2)\int_C dssJ(s)$$

$$\times\exp(-s^2/2\beta^2), \quad (25)$$

where

$$J(s)=\tfrac{1}{2}\int d\mathbf{k}'(P^2+2\mathbf{k}'\cdot\mathbf{P})/abcf$$

with

$$f=|k'-\tfrac{1}{2}P|^2-s^2.$$

It is important to note that $J(s)$ can be reduced to three-denominator integrals alone. This follows from the fact that the numerator can be expressed as a linear combination of the four denominator factors:

$$P^2+2\mathbf{k}'\cdot\mathbf{P}=2c+(a+b)\left(\frac{4P^2+K^2-4s^2}{K^2+4s^2}\right)$$

$$-4f\left(\frac{2P^2+K^2}{K^2+4s^2}\right),$$

and so

$$J(s)=\int d\mathbf{k}'[abf]^{-1}+\left(\frac{4P^2+K^2-4s^2}{K^2+4s^2}\right)\int d\mathbf{k}'[acf]^{-1}. \quad (26)$$

These integrals are evaluated in Appendix II. The final result for $(A+B)$ is

$$A+B=\left(\frac{Ze^2kK}{\pi P^2}\right)e^{-\alpha}\int_0^\infty\frac{xdx}{x^2+1}\exp(-\alpha x^2)$$

$$\times\left[\log\left(\frac{x+1}{x-1}\right)^2-\sin\frac{\theta}{2}\log\left|\frac{x^2+2qx+1}{x^2-2qx+1}\right|\right]$$

$$+\left(\frac{2Ze^2}{\pi}\right)e^{-\alpha}\,\mathrm{P.V.}\int_0^\infty\frac{xdx}{x^2+1}\frac{1}{x^2-1}$$

$$\times\exp(-\alpha x^2)\log\left|\frac{x^2+2qx+1}{x^2-2qx+1}\right|-\left(\frac{\pi Ze^2}{2}\right)e^{-2\alpha}, \quad (27)$$

where $\alpha=K^2/8\beta^2$, and $q=\csc(\theta/2)$.

The discussion of formulas (23), (24), and (27) is given in Sec. VI.

V. EXPANSION OF THE INTEGRAL FOR AN ARBITRARY MODEL

The integral in Eq. (14) provides a fairly explicit form for the integral arising from an arbitrary charge distribution. For example, we can derive from it the expansion of the integral in powers of k or in inverse powers of k, for an arbitrary "regular" charge distribution. The derivation of these expansion formulas is tedious but straightforward, requiring the expansion of the function $I(s,t)$ and the subsequent termwise integration of (14). We will only present here the leading terms.[11]

At low energies, if we keep only terms through those quadratic in k, we find

$$R\cong[2\pi Ze^2kK(1-K/2k)/P]\{1-K(K+4k)\langle r^2\rangle_{\mathrm{Av}}/6\}, \quad (28)$$

where $\langle r^2\rangle_{\mathrm{Av}}=\int d\mathbf{r}r^2\rho(r)$. The first term is the pure Coulomb result, and, to this order, there are deviations depending on the mean square radius of the charge distribution. This is explicit verification, in second Born approximation, of Feshbach's[12] theorem on the model independence of the low-energy electron scattering.

At high energies, if we restrict ourselves to charge distributions having a derivative at the origin,[13] we find for the leading term,

$$R\simeq-8Ze^2(1/k)\langle 1/r\rangle_{\mathrm{Av}}. \quad (29)$$

Note that this is the result of expanding in powers of $1/k$, and $1/k\sin\theta/2$; i.e., we must hold the angle fixed and let the energy increase. The limit (29) is undoubtedly reached more slowly at small angles than at large.

This result is somewhat surprising, in view of the

[11] For further details see R. R. Lewis, Jr., Ph.D. thesis, University of Michigan, 1954 (unpublished).

[12] H. Feshbach, Phys. Rev. **84**, 1206 (1951).

[13] Except for a multiplicative factor, the same result applies to the Yukawa charge distribution.

fact that the phase shifts do not approach the first Born approximation phase shifts in the high-energy limit.[14] Of course, we have only proved that the second Born approximation is small compared to the first in the high-energy limit, yet one would guess intuitively that the same result was correct to all orders of the Born approximation. This is *not* inconsistent with Parzen's result for the phase shifts, due to the nonuniform approach to this high-energy limit at different angles. This nonuniformity means that the usual relations between phase shifts and scattering amplitudes do not hold for the leading terms in the high-energy limit. Thus, the correct scattering amplitude could approach, at any angle, the first Born approximation scattering amplitude, even though the phase shifts do not approach the first Born approximation phase shifts.

We should point out that the imaginary part of the second Born approximation scattering amplitude, which we have always dropped, is in fact independent of the energy in the high-energy limit, for any finite shielding constant λ. Thus, the high-energy limit of the scattering amplitude in second order, differs from the first Born approximation only by a phase factor, which diverges as $\lambda\rightarrow 0$.

TABLE I. The ratio R of the second-order term to the first-order term $(d\sigma_2/d\sigma_1)$, divided by Ze^2, for the Yukawa charge distribution.

Energy	$\theta=30°$	$\theta=60°$	$\theta=90°$	$\theta=120°$	$\theta=150°$
$k/a=0.1$	0.636	1.01	1.24	1.37	1.39
$k/a=0.5$	0.450	0.411	0.168	−0.116	−0.323
$k/a=1.0$	0.197	−0.250	−0.805	−1.24	−1.45
$k/a=1.5$	−0.005	−0.626	−1.15	−1.46	−1.62
$k/a=10$	−0.326	−0.380	−0.390	−0.394	−0.396

VI. RESULTS

Numerical calculations of formulas (23), (24), and (27) have been carried out. The values of R/Ze^2 for the Yukawa charge distribution for various energies and angles are given in Table I. In general, they are positive for small momentum transfers, and negative for large momentum transfers, and never exceed about 1.6. Thus for a light nucleus like copper ($Z=29$), the corrections would be at most $\sim$35%, while for a heavy nucleus like gold ($Z=79$) the corrections would be at most 90%. Since the large-angle corrections are negative, this represents an almost complete cancellation for heavy nuclei. For long wavelengths, R approaches the pure Coulomb result (21), and for short wavelengths it approaches a value decreasing like $1/E$ and independent of the angle.

The values of R/Ze^2 for the exponential charge distribution have been computed at a single energy, such that $k/b=1.10$; these results are given in Table II. They are also positive at sufficiently small angles, and negative at large angles, which is in agreement with the sign of the deviations of the first Born approximation from the correct result.[15] The root occurs at about 25°, while the correct value of the root is near 65° for $Z=79$. The deviations are now much larger, leading to ridiculous results for heavy nuclei; since R is large and negative, the "correction" term becomes larger than the first Born approximation, giving a *negative* result for the cross section. This indicates the seriousness of the errors involved in discarding higher order terms, which complete the square and prevent negative values of the cross section. Thus, one can only apply the exponential model results to fairly light nuclei, where they still represent sizeable corrections ($\sim$75% for copper at 150°). For high energies, the ratio R decreases like $1/E$.

TABLE II. The ratio R of the second-order term to the first-order term $(d\sigma_2/d\sigma_1)$, divided by Ze^2, for the exponential charge distribution.

Energy	$\theta=30°$	60°	90°	120°	150°
$k/a=1.10$	−1.85	−1.50	−2.62	−3.93	−3.57

Similar results have been obtained for the Gaussian charge distribution; here the energy was chosen so that $k/\beta=3.1$. The root now occurs at about 30°, and the ratio R/e^2Z reaches even larger values at large angles, due mostly to the very rapid decrease of the first Born approximation. These results are given in Table III. In this case, not even the sign of the second-order terms seems correct; exact calculations show that for heavy nuclei, the first Born approximation always gives too small a cross section, and the second Born approximation decreases it further at large angles. For the Gaussian charge distribution, R increases exponentially with the energy, at high energies. This increase in R probably results for any charge distribution whose form factor falls off faster than any power of the energy at high energies (i.e., exponentially, etc.).

VII. CONCLUSIONS

Several conclusions can be reached from these results:

(a) The second Born approximation provides a good criterion for the validity of the first Born approximation. Thus whenever the first Born approximation is seriously in error, as at large angles for heavy nuclei, or at high energies for "special" models like uniform and Gaussian, the second Born approximation provides very *large* corrections (not however in the correct direction

TABLE III. The ratio R of the second-order term to the first-order term $(d\sigma_2/d\sigma_1)$, divided by Ze^2, for the Gaussian charge distribution.

Energy	$\theta=30°$	60°	90°	120°	150°
$k/a=3.10$	0.005	−3.10	−10.9	−35.8	−73.4

[14] G. Parzen, Phys. Rev. 80, 261 (1950).

[15] D. R. Yennie *et al.*, Phys. Rev. 95, 506 (1954), Fig. 2.

always). When the corrections are small, they are of the correct sign, and probably provide a considerable improvement in the Born approximation. It would be of interest to check this with exact computations for medium weight nuclei.

(b) The Born expansion must converge very slowly for heavy nuclei, large angles, and $kr\approx 1$, if at all. The behavior at higher energies will depend very much on the character of the charge distribution: in particular, on the behavior of the charge distribution at very small radii. The second Born approximation indicates that, for charge distributions with a derivative at the origin, such as the exponential model, the first Born approximation becomes valid again at very high energies.

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APPENDIX I. GENERAL THREE DENOMINATOR INTEGRAL

The integrals required in this work are all special cases of the three denominator integral, or simpler two denominator integrals.

$$I(\lambda;\mathbf{q}_1,\mu_1;\mathbf{q}_2,\mu_2)=\int d\mathbf{q}[(q^2+\lambda^2)(|\mathbf{q}-\mathbf{q}_1|^2+\mu_1^2)(|\mathbf{q}-\mathbf{q}_2|^2+\mu_2^2)]^{-1}. \quad (1)$$

The following properties of this function follow from examination of the integrand:

(a) The function is even in λ, μ_1, and μ_2.

(b) The function is symmetric under the simultaneous interchange of $\mathbf{q}_1$ and μ_1, with $\mathbf{q}_2$ and μ_2.

(c) The function is analytic in λ, μ_1, and μ_2, except for points on the imaginary axes, where the value of the integral depends on whether the axis is approached through positive or negative values of the real part. Using property (1), we can restrict ourselves to evaluating the function for values of the parameters with *positive* real parts only; the function can be evaluated at other points by symmetry.

Using the integral representation

$$\frac{1}{ab}=\int_0^1 dx[ax+b(1-x)]^{-2}, \quad (2)$$

we can write (1) in the form

$$I=\int_0^1 dx\int d\mathbf{q}(q^2+\lambda^2)^{-1}[|\mathbf{q}-\mathbf{Q}|^2+\Delta^2]^{-2},$$

where

$$\mathbf{Q}=x\mathbf{q}_1+(1-x)\mathbf{q}_2,$$
$$\Delta^2=(\mathbf{q}_1-\mathbf{q}_2)^2x(1-x)+x\mu_1^2+(1-x)\mu_2^2,$$

Thus

$$I=-\frac{1}{2}\int_0^1 dx\frac{1}{\Delta}\frac{\partial}{\partial\Delta}\int d\mathbf{q}[(q^2+\lambda^2)(|\mathbf{q}-\mathbf{Q}|^2+\Delta^2)]^{-1}$$
$$=-\frac{1}{2}\int_0^1 dx\frac{1}{\Delta}\frac{\partial}{\partial\Delta}\left\{\frac{\pi^2 i}{Q}\log\left(\frac{\lambda+\Delta-iQ}{\lambda+\Delta+iQ}\right)\right\}$$
$$=\pi^2\int_0^1 dx\Delta^{-1}[\lambda^2+x(q^2+\mu_1^2)+(1-x)(q_2^2+\mu_2^2)+2\lambda\Delta]^{-1}, \quad (3)$$

which is a rational function of x and Δ. Note that the symmetry (*b*) is obvious, by simply substituting $u=1-x$ in the integral.

The last integral can be performed by reducing the integrand to rational form, using linear fractional transformations. First put $z=(\mu_1/\mu_2)[x/(1-x)]$: then

$$\int_0^1 dx=\mu_1\mu_2\int_0^\infty dz(\mu_2z+\mu_1)^{-2},$$
$$\Delta=\mu_1\mu_2[S(z)]^{\frac{1}{2}}/(\mu_2z+\mu_1),$$
$$S(z)=z^2+[(\mathbf{q}_1-\mathbf{q}_2)^2+\mu_1^2+\mu_2^2]z/\mu_1\mu_2+1,$$

and so

$$I=\pi^2\int_0^\infty dzS^{-\frac{1}{2}}[\mu_2z(\lambda^2+q_1^2+\mu_1^2)+\mu_1(\lambda^2+q_2^2+\mu_2^2)+2\lambda\mu_1\mu_2\sqrt{S}]^{-1}, \quad (4)$$

which has the advantage that the roots of $S(z)$ are negative definite for real positive λ, μ_1, μ_2. This integral can now be put in rational form by introducing a transformation based on the roots of $S(z)$. Let

$$S(z)=(z+z_1)(z+z_2),$$

where

$$z_{1,2}=[(\mathbf{q}_1-\mathbf{q}_2)^2+\mu_1^2+\mu_2^2\mp\{[(\mathbf{q}_1-\mathbf{q}_2)^2+\mu_1^2+\mu_2^2]^2-4\mu_1^2\mu_2^2]\}^{\frac{1}{2}}/2\mu_1\mu_2.$$

Then, introducing

$$u=z_1^{\frac{1}{2}}[(z+z_2)/(z+z_1)]^{\frac{1}{2}},$$

we find

$$I=\pi^2\int_{\sqrt{z_1}}^{\sqrt{z_2}} du[au^2+2bu+c]^{-1}, \quad (5)$$

with

$$a=\mu_1z_2^{\frac{1}{2}}(\lambda^2+q_2^2+\mu_2^2)-\mu_2z_1^{\frac{1}{2}}(\lambda^2+q_1^2+\mu_1^2),$$
$$b=\lambda\mu_1\mu_2(z_2-z_1), \quad (6)$$
$$c=\mu_2z_2^{\frac{1}{2}}(\lambda^2+q_1^2+\mu_1^2)-\mu_1z_1^{\frac{1}{2}}(\lambda^2+q_2^2+\mu_2^2).$$

This has the disadvantage that a, b, and c are not positive-definite for real positive λ, μ_1, and μ_2, and so we perform one further linear fractional transformation:

$$v=z_2^{\frac{1}{2}}\left(\frac{u-z_1^{\frac{1}{2}}}{z_2^{\frac{1}{2}}-u}\right),$$

which gives

$$I=2\pi^2\int_0^\infty dv[\alpha v^2+2\beta v+\gamma]^{-1}, \tag{7}$$

where

$$\begin{aligned}\alpha&=\mu_2(z_1^{\frac{1}{2}}+z_2^{\frac{1}{2}})[q_1^2+(\lambda+\mu_1)^2],\\ \beta&=\mu_1\mu_2\lambda(2+z_1+z_2)+\mu_2(\lambda^2+q_1^2+\mu_1^2)\\ &\quad+\mu_1(\lambda^2+q_2^2+\mu_2^2),\\ \gamma&=\mu_1(z_1^{\frac{1}{2}}+z_2^{\frac{1}{2}})[q_2^2+(\lambda+\mu_2)^2].\end{aligned} \tag{8}$$

Now the integral is a simple function of the roots of the quadratic: introduce v_1, and v_2 by

$$v_{1,2}=[\beta\pm(\beta^2-\alpha\gamma)^{\frac{1}{2}}]/\alpha,$$

and then, by a simple integration,

$$I(\lambda;\mathbf{q}_1,\mu_1;\mathbf{q}_2,\mu_2)=\pi^2(\beta^2-\alpha\gamma)^{-\frac{1}{2}}\log\left[\frac{\beta+(\beta^2-\alpha\gamma)^{\frac{1}{2}}}{\beta-(\beta^2-\alpha\gamma)^{\frac{1}{2}}}\right], \tag{9}$$

with

$$\begin{aligned}\alpha\gamma&=[(\mathbf{q}_1-\mathbf{q}_2)^2+(\mu_1+\mu_2)^2][q_1^2+(\mu_1+\lambda)^2]\\ &\quad\times[q_2^2+(\lambda+\mu_2)^2],\\ \beta&=\lambda[(\mathbf{q}_1-\mathbf{q}_2)^2+(\mu_1+\mu_2)^2]+\mu_2(\lambda^2+q_1^2+\mu_1^2)\\ &\quad+\mu_1(\lambda^2+q_2^2+\mu_2^2).\end{aligned} \tag{10}$$

This function is clearly single valued, even when we cross a branch cut of $(\beta^2-\alpha\gamma)^{\frac{1}{2}}$; i.e., we can choose *either* square root. Thus the function is obviously analytic even for complex values of λ, μ_1, and μ_2. The only remaining problem is the specification of the branch of the logarithm; examination shows that one must take the arguments of numerator and denominator from $-\pi$ to $+\pi$.

The integral $M_3(\mu,\nu)$ defined in (19b) is a special case:

$$M_3(\mu,\nu)=k^3I(-ik;\mathbf{k}_i\mu;\mathbf{k}_f\nu). \tag{11}$$

The integral $M_2(\mu,\nu)$ can be done directly by introducing polar coordinates:

$$M_2(\mu,\nu)=\frac{\pi^2ik}{K}\log\left(\frac{\mu+\nu-iK}{\mu+\nu+iK}\right) \tag{12}$$

or by using

$$M_2(\mu,\nu)=k\lim_{\lambda\to\infty}\lambda^2I(\lambda;\mathbf{k}_i,\mu;\mathbf{k}_f,\nu).$$

Let us list the various special cases used in the text:

$$\text{Re } M_3(\mu,\nu)=\pi^2k^3[k^2(K^2+\mu^2+\nu^2)^2-P^2\mu^2\nu^2]^{-\frac{1}{2}}\left\{\arctan\left[\frac{k[K^2+(\mu+\nu)^2]+[k^2(K^2+\mu^2+\nu^2)^2-P^2\mu^2\nu^2]^{\frac{1}{2}}}{\mu\nu(\mu+\nu)}\right]-\arctan\left[\frac{k(K^2+(\mu+\nu)^2]-[k^2(K^2+\mu^2+\nu^2)^2-P^2\mu^2\nu^2]^{\frac{1}{2}}}{\mu\nu(\mu+\nu)}\right]\right\},$$

$$\text{Re } M_3(\mu,\mu)=2\pi^2k^3\{[k^2K^2+4k^2\mu^2+\mu^4]^{-\frac{1}{2}}/K\}\times\arctan\{\tfrac{1}{2}K\mu[k^2K^2+4k^2\mu^2+\mu^4]^{-\frac{1}{2}}\},$$

$$\text{Re } M_3(0,\mu)=[\pi^2k^2/(K^2+\mu^2)]\arctan(\mu/2k),$$

$$\text{Re } M_2(\mu,\nu)=(2\pi^2k/K)\arctan[K/(\mu+\nu)],$$

$$\text{Re } M_2(0,\mu)=(2\pi^2k/K)\arctan(K/\mu),$$

$$\text{Re } M_2(0,0)=\pi^3k/K.$$

APPENDIX II. GAUSSIAN CHARGE DISTRIBUTION INTEGRALS

The two integrals which appear for the Gaussian charge distribution are special cases of the integral in the previous Appendix. Substitution into the formulas derived there show that, as $\lambda\to0$, the integrals approach

$$\int d\mathbf{k}'[abf]^{-1}=\frac{\pi^2}{iK(s^2+K^2/4)}\log\left[\frac{i(s-K/2)^2}{-i(s+K/2)^2}\right],$$

$$\int d\mathbf{k}'[acf]^{-1}=\frac{\pi^2i}{k(s^2-K^2/4)}\log\left[\frac{\lambda[P^2/4-(s+k)^2]}{2ik(s^2-K^2/4)}\right].$$

In each case, the argument of the numerator and denominator factors is to be chosen between $-\pi$ and π.

The final answer can be expressed in terms of real integrals by substituting the above functions into (25) and (26) and deforming the contour onto the real axis. Considerable simplification then results due to the fact that the odd portions of the above integrals give no contribution, and only the real parts need be kept. The only care that need be taken is in properly treating the contribution near the poles at $s=\pm K/2$, in the second integral.