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Structure of a Magnetohydrodynamic Shock Wave in a Plasma of Infinite Conductivity

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The structure of a magnetohydrodynamic perpendicular shock wave (magnetic field perpendicular to the direction of propagation) in a plasma of infinite conductivity has been analyzed. For the nonmagnetic case, the width of the shock front is found to be larger than in a non-ionized gas. Further, the width of the shock (in terms of mean free path within the shock) as a function of the shock strength as determined from the Stokes-Navier equations (which applies to weak shocks) is found to join smoothly at $M\simeq1.3$ with the same function as derived from the Mott-Smith analysis (which applies to strong shocks). The magnetic field tends to make the shock front narrower for shocks of moderate strength. It does not have appreciable effect for strong shocks.

Application of the above results to solar radio noise and current theories of the origin of cosmic rays is indicated.

1. INTRODUCTION

RECENTLY de Hoffmann and Teller[1] have extended the Rankine-Hugoniot conditions of classical hydrodynamics to shock waves in an infinitely conducting fluid with superposed magnetic field. The mathematical discontinuity in the physical variables given by the Rankine-Hugoniot conditions at a shock front is, however, not physically possible, and it is well known that considerations of dissipation of energy by viscosity and heat conductivity enable the physical quantities to vary continuously and result in a finite width of the shock front. It is the object of this paper to apply similar considerations to find the structure of a magnetohydrodynamic shock wave in an infinitely conducting plasma (a macroscopically neutral, ionized gas). For simplicity, a plane, longitudinal shock is considered propagating in a direction perpendicular to the applied magnetic field. The shock-structure without the field is also considered as a standard of comparison. The application of the results obtained to solar radio noise and cosmic rays is briefly indicated.

2. BASIC EQUATIONS

We shall assume the shock propagating at nonrelativistic velocities along the x-axis, and make the flow time-independent by referring to a coordinate system moving with the shock front. We shall use the suffixes 0 and 1 to denote the physical variables (velocity u, pressure p, density ρ, and temperature T) in front and back of the shock, respectively. Then we have the following equations describing the flow.

From the conservation of mass, we have

$$\rho u=\rho_0 u_0=m, \text{ say.} \tag{1}$$

The equations of motion and energy are derived in Appendix I. It is shown there that the pressure tensor in our case reduces to a similar form as without a magnetic field in the two extreme cases, *viz.*, $\omega\tau\ll1$ or $\gg1$, where ω is the gyro and τ the collisional frequency. These refer to the ions which, on account of their heavier mass, determine the shock.

The above restrictions on $\omega\tau$ for the validity of our analysis are not so stringent as they might appear to be. They hold in many physical cases of interest, e g., in the cases considered in Sec. 7 below. With the values of the parameters assumed therein, we have in the H II region, $\omega\tau\simeq10^3$ and in the corona at a height of 10^5 km above the photosphere, $\omega\tau\simeq10^7$. We shall assume in what follows that $\omega\tau\ll1$ or $\gg1$. It is true that in the latter case the magnetic field will affect the viscosity μ and the conductivity k by a function[2] of $\omega\tau$. As, in our analysis, we shall be using only the ratio μ/k, we shall ignore this effect.

For our hydromagnetic case, the following extension of Stokes-Navier equation holds:

$$m\frac{du}{dx}=-\frac{d}{dx}\left(p+\frac{H^2}{8\pi}\right)+\frac{4}{3}\frac{d}{dx}\left(\mu\frac{du}{dx}\right), \tag{2}$$

where μ is the coefficient of viscosity (determined primarily by the ions) and H is the transverse magnetic field.

Integrating (2), we have

$$mu-mC=-p-\frac{H^2}{8\pi}+\frac{4}{3}\mu\frac{du}{dx}, \tag{3}$$

where C is an integration constant. We assume uniform conditions both in front of $(x=+\infty)$ and behind $(x=-\infty)$ the shock. At shock entrance, (3) reduces to

$$p_0+\frac{H_0^2}{8\pi}+mu_0=mC. \tag{4}$$

The energy equation can be written as

$$\frac{4}{3}\mu\left(\frac{du}{dx}\right)^2+\frac{d}{dx}\left(k\frac{dT}{dx}\right)-mC_v\frac{dT}{dx}-p\frac{du}{dx}=0, \tag{5}$$

where k is the thermal conductivity and C_v the specific heat at constant volume.

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[1] F. de Hoffmann and E. Teller, Phys. Rev. 80, 692 (1950).

[2] S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University Press, Cambridge, 1953), second edition, p. 337.

No relative motion is possible of an infinitely conducting material with respect to the magnetic lines of force. Hence we have[3]

$$Hu = H_0 u_0. \tag{6}$$

Finally, we may write the equation of state for a perfect gas:

$$p = R\rho T, \tag{7}$$

where $R = \kappa/m_i$, κ being Boltzmann's constant and m_i the (mean) particle mass.

Eliminating p between (3) and (5), we get

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) - mC_v\frac{dT}{dx} - mC\frac{du}{dx} + m\frac{d}{dx}\left(\frac{u^2}{2}\right) + \frac{H^2}{8\pi}\frac{du}{dx} = 0. \tag{8}$$

We can express H as a function of u by (6), integrate (8), and obtain

$$k\frac{dT}{dx} - mE - mCu + \frac{mu^2}{2} - \frac{H_0^2 u_0^2}{8\pi u} = C_1, \tag{9}$$

where C_1 is another constant of integration.

Applying (9) to the front and back of a shock, we obtain the following relation:

$$\begin{aligned} C_1 &= -mE_0 - mCu_0 + \tfrac{1}{2}mu_0^2 - H_0^2 u_0/8\pi \\ &= -mE_1 - mCu_1 + \tfrac{1}{2}mu_1^2 - H_0^2 u_0^2/8\pi u_1. \end{aligned} \tag{10}$$

From (9) and (3), we have

$$\frac{dT}{du} = \frac{4\mu}{3k}\,\frac{C_2 u + Eu + Cu^2 - \frac{1}{2}u^3 + H_0^2 u_0^2/8\pi m}{u^2 - Cu + RT + H_0^2 u_0/8\pi\rho_0 u}, \tag{11}$$

where

$$C_2 = C_1/m = -E_0 - Cu_0 + \tfrac{1}{2}u_0^2 - H_0^2/8\pi\rho_0. \tag{12}$$

Introduce the following dimensionless variables:

$$\Pi = T/T_0, \quad U = u/u_0, \quad \text{and} \quad h_0 = H_0^2/8\pi p_0. \tag{13}$$

Let γ be the ratio of the specific heats at constant pressure and volume (C_p/C_v), which in our case of the plasma we shall take as 5/3. Then $(\gamma-1)h_0$ in (13) is the ratio of the magnetic to the internal energy per unit mass.

Equation (11) yields, on integration,

$$\Pi = \frac{4}{3\gamma}\int_{U_1}^{U} \mu\frac{C_p}{k}\,\frac{U[\pi - \gamma\{1+\frac{1}{2}(\gamma-1)M_0^2\} + (1+\gamma M_0^2)(\gamma-1)U - \frac{1}{2}\gamma(\gamma-1)M_0^2U^2] + h_0(\gamma-1)(U-1)^2}{\Pi/\gamma M_0^2 - U(1-U+1/\gamma M_0^2) - h_0(\gamma M_0^2)^{-1}(U-1/U)}\,dU + \Pi_1. \tag{15}$$

The Prandtl number,[4] $\mu C_p/k$, in (15) is $\gamma/f = \frac{2}{3}$, where $f \simeq 5/2$ for inverse square law of interaction.[5] We shall take it equal to $\frac{3}{4}$, to compare with earlier work.[4] Our conclusions would not be materially affected by this change in the Pradtl number.[6] M_0 in (15) is the Mach number (ratio of stream to sound velocity) in front of the shock, without the magnetic field, which is related to M_{0H}, the corresponding Mach number in the presence of the field, by the following equation[7]:

$$M_0 = M_{0H}(1 + 2h_0/\gamma)^{\frac{1}{2}}. \tag{16}$$

Now,

$$E_1/u_0^2 = C_v T_1/u_0^2 = \Pi_1/\gamma(\gamma-1)M_0^2. \tag{17}$$

By (1), (13), and (17), (10) reduces to

$$\begin{aligned} (\Pi_1 - 1)/\gamma(\gamma-1)M_0^2 + C_0U_1 - \tfrac{1}{2}U_1^2 + h_0/\gamma M_0^2 U_1 \\ = C_0 - \tfrac{1}{2} + h_0/\gamma M_0^2, \end{aligned} \tag{18}$$

where, by (4), (13), and (1)

$$C_0 = C/u_0 = 1 + (1+h_0)/\gamma M_0^2. \tag{19}$$

Also, from (4), (1), (6), (7), and (13),

$$\Pi_1 = \gamma M_0^2 U_1(C_0 - U_1) - h_0/U_1. \tag{20}$$

Equations (18)–(20) determine Π_1 and U_1 (temperature and velocity jump through the shock), as also $\rho_1/\rho_0 = 1/U_1$ and $p_1/p_0 = \Pi_1/U_1$ in terms of the parameters M_0 [hence M_{0H} by (16)] and h_0 in front of the shock. Eliminating Π_1 between (18) and (20), we have the following quadratic equation in U_1:

$$\begin{aligned} \tfrac{1}{2}(\gamma+1)U_1^2 - [\tfrac{1}{2}(\gamma-1) + (1+h_0)/M_0^2]U_1 \\ + h_0(\gamma-2)/\gamma M_0^2 = 0. \end{aligned} \tag{21}$$

In (21), we have omitted the factor $U_1 - 1$, which leads to uniform conditions throughout.

It can be shown that Eq. (21) has a positive and a negative root. The positive root passes over continuously into the nonmagnetic case and is the one we shall consider. The negative root is inadmissible, as it leads to a negative density through the equation of continuity.

We shall further consider only values of $U_1 < 1$, ($M_{0H} > 1$), i.e., compression shocks. Rarefaction shocks too, for which $U_1 > 1$, ($M_{0H} < 1$), are mathematically possible solutions of Eq. (21), but, as shown in Appendix II, they are thermodynamically unstable, on account of increase in entropy.[8]

It is found that for weak shocks the magnetic field, so to say, softens the shock front, that is, decreases the compression across the front. We see, however, from Eqs. (20) and (21), that for M_{0H} (and hence M_0

[3] See reference 1, p. 695, Eq. (20).
[4] M. Morduchow and P. A. Libby, J. Aeronaut. Sci. **16**, 674 (1949).
[5] See reference 2, p. 235.
[6] P. A. Libby, J. Aeronaut. Sci. **18**, 286 (1951).
[7] R. Lüst, Z. Naturforsch. **8a**, 282 (1953).

[8] See H. W. Liepmann and A. E. Puckett, *Introduction to Aerodynamics of a Compressible Fluid* (John Wiley and Sons, Inc., New York, 1947), p. 41.

also)$\to\infty$, $\Pi_1\to\infty$ and $U_1\to(\gamma-1)/(\gamma+1)$, the same as for the nonmagnetic case. Thus, we would expect the magnetic field to lose its importance as the shock strength increases.[9]

Differentiating (15), we can express $d\Pi/dU$ as a function of Π and U. The variation of Π with U can then be traced by the method of isoclines. We note that at the front and back of a shock ($U=1$ and U_1), the integrand in (15) is indeterminate and its value is found by l'Hospital's rule. The singularities at these points lead directly to the Rankine-Hugoniot conditions in the magnetic, as in the nonmagnetic,[10] case.

3. WIDTH OF THE SHOCK FRONT

We shall consider a temperature-dependent viscosity governed by the following relation:

$$\mu=\mu_0(T/T_0)^n, \qquad (22)$$

where $n=2.5$ in our case.[11] Then, from (3), (1), (6), (7), (13), and (19) we have

$$x=\frac{4\mu_0}{3m}\int_{U_c}^{U}\frac{\Pi^n U dU}{\Pi/\gamma M_0{}^2+U^2-C_0U+h_0/\gamma M_0{}^2U}. \qquad (23)$$

In (23), U_c=shock velocity at the inflection point (where $d^2U/dx^2=0$), and we have chosen the origin of x at this point.

We can write the differential form of (23) as

$$\frac{dU}{dx}=\frac{3m}{4\mu_0}\frac{\Pi/\gamma M_0{}^2+U^2-C_0U+h_0/\gamma M_0{}^2U}{\Pi^n U}. \qquad (24)$$

A plot of dU/dx against U from (24) will give at its maximum the value $U=U_c$.

Equation (23) can be numerically integrated to give U, and hence the other physical variables as well, as a function of x. We shall define the shock width as[12]

$$t=(1-U_1)(dU/dx)_{\max}{}^{-1}. \qquad (25)$$

4. MEAN FREE PATH

We shall express the shock width t given by (25) both in terms of λ_0, the mean free path in front of the shock, and λ, the mean free path within the shock. For the latter we shall take the arithmetic mean of the mean free paths (λ_0 and λ_1) in front of and at the back of the shock ($x=\pm\infty$), respectively.

The definition of the mean free path in our case (inverse square law of interaction) is somewhat arbitrary. We shall define it through the expression for the viscosity in gas-kinetic theory[13]:

$$\mu=\delta\rho\bar{c}\lambda, \qquad (26)$$

TABLE I. Width of shock in terms of mean free path. H_0 is defined by Eq. (13) and is a measure of the ratio of the magnetic to the internal energy per unit mass; M_{0H} is the Mach number in front of the shock; t_λ and t_0 are the shock thicknesses in terms of mean free path within and in front of the shock, respectively.

h_0	M_{0H}	t_0	t_λ
0.00	1.100	30.4	29.7
	1.500	9.5	8.3
	2.000	8.5	5.9
	2.500	9.7	5.0
	3.000	11.2	4.1
	4.000	14.7	2.6
0.10	1.039	65.1	64.4
	1.417	9.8	8.8
	1.890	7.9	5.7
	2.362	8.9	4.8
	2.835	10.6	4.1
	3.780	14.2	2.7
2.00	1.085	13.7	13.4
	1.356	4.7	4.2
	1.627	4.7	3.3
	2.169	7.9	2.8
4.00	1.038	16.5	16.4
	1.246	4.4	4.0
	1.661	4.3	2.5

where $\bar{c}$= the mean molecular velocity$=(8\kappa T/\pi m_i)^{\frac{1}{2}}$ and δ is a numerical constant$\simeq\frac{1}{2}$ for smooth, rigid elastic spheres. We shall take $\delta=\frac{1}{2}$. A different δ will only change the relative scale of shock thickness.

Let t_λ and t_0 be the shock thickness in terms of mean free path within and in front of the shock, respectively. Then we have the following relation:

$$2t_0/t_\lambda=1+\lambda_1/\lambda_0. \qquad (27)$$

Now, from (1), (13), (22), and (26):

$$\lambda_1/\lambda_0=\Pi_1{}^2U_1, \qquad (28)$$

where we have put $n=2.5$. Hence

$$t_0/t_\lambda=\tfrac{1}{2}(1+\Pi_1{}^2U_1). \qquad (29)$$

Set

$$x=\lambda_0\xi. \qquad (30)$$

Then, from (7), (24), and (26),

$$\frac{dU}{d\xi}=\frac{3M_0}{8\delta}\left(\frac{\pi\gamma}{2}\right)^{\frac{1}{2}}\frac{\Pi/\gamma M_0{}^2+U^2-C_0U+h_0/\gamma M_0{}^2U}{\Pi^nU}. \qquad (31)$$

From (25), (30), and (31), we have

$$t_0=\frac{8\sqrt{2}}{3(\pi\gamma)^{\frac{1}{2}}}\frac{\delta(1-U_1)}{M_0f_{\max}(\Pi,U)}, \qquad (32)$$

where $f_{\max}(\Pi,U)$ is the maximum value of the third factor on the right-hand side of (31).

The quantity t_λ is obtained from (29) and (32).

5. NONMAGNETIC CASE

We shall first consider the case of the ionized plasma without the magnetic field, i.e., when $h_0=0$. Table I

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[9] H. L. Helfer, Astrophys. J. **117**, 180 (1953). R. Lüst, manuscript read at Eleventh General Assembly, URSI, Hague, 1954 (unpublished).
[10] R. von Mises, J. Aeronaut. Sci. **17**, 552 (1950).
[11] See reference 2, p. 218, Sec. 12.1 (ii).
[12] See reference 4, p. 680.
[13] See reference 2, p. 218, formula (6).

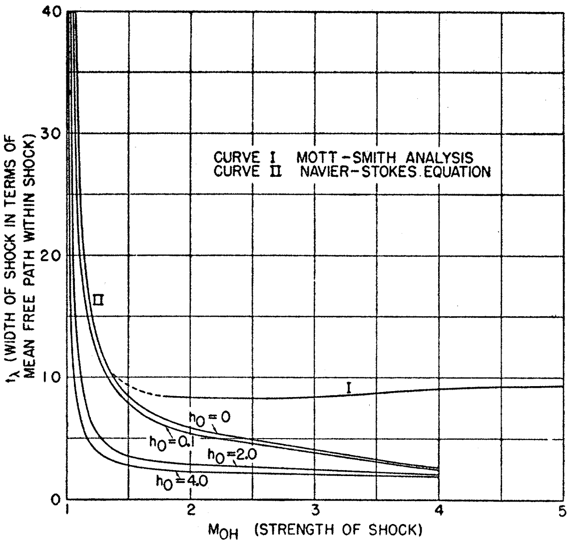

FIG. 1. Width of shock for an ionized gas in terms of mean free path within shock.

gives the values of t_0 and t_λ for values of M_0 from 1 to 4. We see that for $M_0 \simeq 4$, the shock width is two or three times the mean free path for an ionized gas. The shock width turns out to be greater than in the corresponding case for the nonionized gas, on account of the larger value of the exponent n in the relation (22) for the temperature dependence of the viscosity.[14]

The Stokes-Navier equations, however, do not hold for strong shocks.[15] To continue our table to high Mach numbers we use the Mott-Smith interpolation formula.[16]

We assume, as before, that the shock is determined by interionic collisions. We shall use the Mott-Smith notation as far as possible. Using the Chapman-Cowling expression for the viscosity,[17] *viz.*,

$$\mu = \tfrac{5}{8}[A_2(2)]^{-1}(\kappa m_i T/\pi)^{\frac{1}{2}}(2\kappa T/Z^2e^2)^2, \tag{33}$$

we get the following expression for λ from (26):

$$\lambda = 5\kappa^2T^2/2\sqrt{2}nZ^4e^4A_2(2), \tag{34}$$

where n is the particle (ion) density [not to be confused with the n in (22)] and Ze is the charge on the ion assumed to be only of one kind. Thus, the average mean free path within the shock is

$$\bar{\lambda} = \frac{\lambda_\alpha+\lambda_\beta}{2} = \frac{5m_i^2}{4\sqrt{2}Z^4e^4n_0A_2(2)}\left[c_\alpha^4 + \frac{M^2+a-2}{(a-1)M^2}c_\beta^4\right], \tag{35}$$

where

$$a = 2\gamma/(\gamma-1). \tag{36}$$

Again, from Chapman and Cowling,[17]

$$\phi_{11}^{(2)}(g) = 4Z^4e^4m_i^{-2}g^{-3}A_2(2). \tag{37}$$

Thus, we have

$$H(G_0,u_{\alpha\beta}) = -4Z^4e^4\kappa^{-2}A_2(2)(u_{\alpha\beta})^{-1}(1-r^2)(T_\alpha+T_\beta)^{-2}, \quad r<1;$$
$$= 0, \quad r>1. \tag{38}$$

Hence,

$$I_{\alpha\beta}+I_{\beta\alpha} = -8Z^4e^4A_2(2)n_\alpha n_\beta \times[2\pi/\kappa m_i^3(T_\alpha+T_\beta)]^{\frac{1}{2}}A(u_{\alpha\beta}), \tag{39}$$

where

$$A(u_{\alpha\beta}) = \tfrac{1}{4}u_{\alpha\beta}^{-3}[3u_{\alpha\beta}\exp(-u_{\alpha\beta}^2) + (2u_{\alpha\beta}^2-3)\,\mathrm{erf}u_{\alpha\beta}], \tag{40}$$

and

$$\mathrm{erf}u_{\alpha\beta} = \int_0^{u_{\alpha\beta}}\exp(-u^2)du. \tag{41}$$

Mott-Smith's differential equation (29) reduces in our case to

$$u_\alpha(u_\alpha^2+3c_\alpha^2)\frac{dn_\alpha}{dx} + u_\beta(u_\beta^2+3c_\beta^2)\frac{dn_\beta}{dx} + 8Z^4e^4A_2(2)\left[\frac{2\pi}{m_i^3\kappa(T_\alpha+T_\beta)}\right]^{\frac{1}{2}}n_\alpha n_\beta A(u_{\alpha\beta}) = 0. \tag{42}$$

From (35) and (42), we have

$$d\nu_\alpha/dx + B\nu_\alpha(1-\nu_\alpha)/\bar{\lambda} = 0, \tag{43}$$

where

$$B = 10\frac{[\pi(a-2)/a]^{\frac{1}{2}}}{(a-1)(a-3)} \times\frac{[aM^4+2a(a-2)M^2-a+2]^{-\frac{1}{2}}}{M^2(M^2-1)(M^2+a-2)(aM^2+a-2)}[(a-1)^5M^6 + (M^2+a-2)^3(aM^2-1)^2]A(u_{\alpha\beta}). \tag{44}$$

TABLE II. Width of shock in terms of mean free path within shock (Mott-Smith analysis). M_0 = Mach number in front of the shock (without the magnetic field). $u_{\alpha\beta}$, $A(u_{\alpha\beta})$, and B are defined by Eqs. (45), (40), and (44) respectively.

M_0	$u_{\alpha\beta}$	$A(u_{\alpha\beta})$	B	$4/B$
1.5	0.3612	0.0009	0.4418	9.053
2.0	0.5853	0.0080	0.4781	8.366
2.5	0.7377	0.0233	0.4788	8.354
3.0	0.8452	0.0442	0.4781	8.366
4.0	0.9800	0.0818	0.4419	9.052
5.0	1.0563	0.1113	0.4282	9.341
10.0	1.1781	0.1752	0.4252	9.407
∞	1.2247	0.2711	0.4280	9.346

[14] See reference 4, p. 680, Fig. 5. The physical reason for the increase in shock width with increasing n is that the increased viscosity extends the region of influence of the shock wave. See L. Meyerhoff, J. Aeronaut. Sci. **17**, 785 (1950).

[15] C. S. Wang Chang, University of Michigan, Department of Engineering, Report UMH-3-F APL/JHU CM-503 (unpublished).

[16] H. M. Mott-Smith, Phys. Rev. **82**, 885 (1951).

[17] See reference 2, pp. 171 and 172.

We note that $u_{\alpha\beta}$ is a function of M given by

$$u_{\alpha\beta}=(M^2-1)[a(a-2)/2]^{\frac{1}{2}}[M^2(a-1)^2+(M^2+a-2)(aM^2-1)]^{-\frac{1}{2}}. \quad (45)$$

The shock-wave thickness as given by $4/B$ is tabulated in Table II. The complete plot of the shock thickness as a function of M is given in Fig. 1, the two curves being smoothly joined at $M\simeq1.3$. Note that the shock thickness tends to the following finite value as $M\rightarrow\infty$:

$$\lim_{M\to\infty}\left(\frac{4}{B}\right)=\tfrac{2}{5}(a-1)(a-3)[\pi(a-2)]^{-\frac{1}{2}}A^{-1}[(a-2)^{\frac{1}{2}}/\sqrt{2}], \quad (46)$$

where the function A is defined by (40). The Stokes-Navier equation, on the other hand, gives an infinite limit in our case[18] ($n>\frac{1}{2}$). The Mott-Smith analysis is to be preferred, as we have said, for strong shocks.

6. HYDROMAGNETIC SHOCK

In Table III and Fig. 2, we have shown the variation of U within the shock front for $M_{0H}=2$. The density and magnetic field variations can be derived easily therefrom. Table I and Fig. 1 give the shock width in terms of hydromagnetic shock strength M_{0H} for three values of $h_0=0.1$, 2, and 4. For weak shocks, the magnetic field narrows the shock width, but for strong shocks the width tends to the nonmagnetic case. This is in conformity with the result obtained in Sec. 2.

It is true that for strong shocks our analysis based on the Stokes-Navier equation would not apply.[15] As a superposed magnetic field does not affect a steady state Maxwellian distribution,[19] it would not change the Mott-Smith first-order interpolation formula. Higher orders of approximation would seem to be required in this case.

7. CONCLUDING REMARKS

In the foregoing pages we have analyzed the structure of a shock front in an infinitely conducting plasma

TABLE III. Variation of velocity U within shock as a function of distance ξ from inflection point. ($M_{0H}=2.0$.) M_{0H} is the Mach number in front of the shock. h_0 is defined in Eq. (13).

$h_0=0$		$h_0=0.1$		$h_0=2$		$h_0=4$	
ξ	U	ξ	U	ξ	U	ξ	U
−3.09	0.97	−2.66	0.96				
−2.00	0.93	−1.81	0.92	−1.94	0.99	−1.01	0.98
−1.26	0.89	−1.16	0.88	−1.00	0.95	−0.59	0.95
−0.61	0.85	−0.57	0.84	−0.47	0.91	−0.29	0.92
0	0.81	0	0.80	0	0.87	0	0.89
1.23	0.73	1.18	0.72	1.22	0.77	1.02	0.79
2.61	0.65	2.55	0.64	2.68	0.67	2.31	0.69
4.33	0.57	4.39	0.56	4.68	0.57	4.18	0.59
7.18	0.49	7.83	0.48	12.53	0.47	8.81	0.49

[18] See reference 4, p. 681.

[19] H. K. Sen, Phys. Rev. 88, 820 (1952), Appendix I.

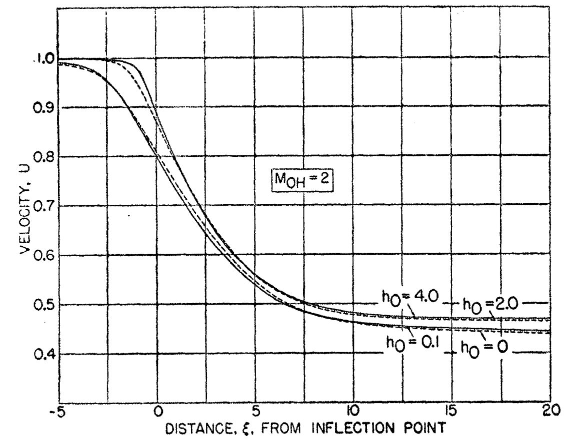

FIG. 2. Variation of velocity within the shock.

when the magnetic field is perpendicular to the direction of propagation. For a parallel field, it is well known that the field has no effect. It would be interesting, though considerably more complicated, to find the shock structure and trace the refraction of the lines of force inside the front when the magnetic field is inclined to the direction of propagation.

It is believed that the physical theory as developed above will have applications in solar, ionospheric, and cosmic problems. We shall herein indicate briefly two such.

First, the gradient of the magnetic field in a shock front may modify the conditions of escape of gyro-frequency radiation in the solar atmosphere. Denisse[20] first noted that a sufficiently rapid *time* variation of the sunspot magnetic field may cut down the reabsorption and facilitate the escape of gyromagnetic radiation, thus accounting for the noise storms in meter wavelengths. A *space* variation of the field inside a shock front may well achieve the same purpose. We shall see if this is so.

In the following we shall use Denisse's notation and results.

Under solar conditions, the electronic collision frequency[21]

$$\nu'=15nT^{-\frac{3}{2}}. \quad (47)$$

For inter-ionic collisions we may take the mean free path to be of the same order as that for electron-ion collisions and given by

$$\lambda=\bar{c}/\nu'=(8\kappa T/\pi m_e)^{\frac{1}{2}}(\nu')^{-1}\simeq4\times10^4T^2/n, \quad (48)$$

from (47).

The mean free path for gyromagnetic radiation of electrons is given by

$$z_0=c(\Delta\omega)^2/\nu'\Omega^2, \quad (49)$$

[20] J. F. Denisse, Ann. astrophys. 10, 1 (1947).

[21] T. G. Cowling in *The Sun*, edited by G. P. Kuiper (University of Chicago Press, Chicago, 1952), p. 538.

where the electronic plasma angular frequency is

$$\Omega=(4\pi ne^2/m_e)^{\frac{1}{2}}, \tag{50}$$

and $(\Delta\omega)/2\pi$ is the Doppler width due to the thermal velocity of the electrons.

From (47), (49), and (50), we have

$$z_0=m_e c T^{\frac{3}{2}}(\Delta\omega)^2/60\pi n^2 e^2. \tag{51}$$

For a shock of moderate strength, we may take the compression to be $\simeq 2$. Hence,

$$\Delta\nu/\nu=(H_1-H_0)/H_0\simeq 1. \tag{52}$$

The change $\Delta\nu$ in (52) takes place in a distance of the order of the shock width, which we may take to be about $5\lambda=2\times10^5 T^2/n$. Hence in a distance z_0 the change in the frequency will be

$$\Delta\nu=5n\nu z_0/10^6 T^2. \tag{53}$$

The Doppler width due to the thermal velocity of the electrons is

$$(\Delta\omega)/2\pi=(\Delta\nu)_d=\nu V/\sqrt{3}c, \tag{54}$$

where

$$V=(3\kappa T/m_e)^{\frac{1}{2}}. \tag{55}$$

From (51) and (53)–(55):

$$\Delta\nu/(\Delta\nu)_d\simeq\nu^2/n\times10^9. \tag{56}$$

At a height $h\simeq10^{10}$ cm above the photosphere, $n\simeq10^8$, $H_0\simeq50$ gauss, $H_1=2H_0\simeq100$ gauss, and $\nu=300$ Mc/sec. Hence, from (56), $\Delta\nu\simeq(\Delta\nu)_d$. Thus the gradient of the magnetic field in the shock front will shift the frequency of the gyromagnetic radiation beyond its Doppler width before it is reabsorbed. The gyromagnetic emission will therefore be considerably more than what would ordinarily be obtained from including the full effects of reabsorption.

We shall next consider the application of our results to current theories of the origin of cosmic rays. The basis of these theories[22] is that the cosmic-ray particles spiral closely around the magnetic lines of force, so that the magnetic moment,

$$\mu=W_\perp/H, \tag{57}$$

remains constant, $W_\perp$ being the energy corresponding to the transverse velocity.

A condition[23] necessary for close spiralling and assumed for the constancy of the magnetic moment is that

$$|\rho dH/dx|\ll|H|, \tag{58}$$

where ρ is the radius of gyration given by

$$\rho=(2mW_\perp)^{\frac{1}{2}}c/eH, \tag{59}$$

and dH/dx is the gradient of the magnetic field.

[22] L. Biermann, Ann. Rev. Nuc. Sci. **2**, 335 (1953).

[23] H. Alfvén, *Cosmical Electrodynamics* (Clarendon Press, Oxford, 1950), pp. 19–23.

Since for shocks of moderate strength, $|dH|\simeq H$ across the shock front, and the shock width is a few mean free paths, condition (58) is equivalent to

$$\rho\ll\lambda, \tag{60}$$

where λ is the mean free path in the front.

In the H I region in interstellar space, the neutral H atoms preponderate over the protons and electrons. The shock is determined by the H atoms, and

$$\lambda_H\simeq(\sigma n_H)^{-1}=10^{15}\text{ cm}, \tag{61}$$

where n_H=concentration of H atoms$=1$ cm^{-3} and σ_H=cross section for H atoms$=10^{-15}$ cm^2.

In the H II regions,[24] $n_i=n_e=10$ cm^{-3}, $\sigma_{i,e}=10^{-11.8}$ cm^2. Hence

$$\lambda_i\simeq\lambda_e=(\sigma_{i,e}n_e)^{-1}\simeq10^{11}\text{ cm}. \tag{62}$$

For[25] $H=10^{-5.5}$ gauss and $W_\perp=10^{10}$ ev,

$$\rho=10^{13}\text{ cm}. \tag{63}$$

From (61)–(63) we see that condition (60) would not be fulfilled in the H II region and would not also hold in H I for proton energies of 10^{14} ev and higher.

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APPENDIX I. THE EQUATIONS OF MOTION AND ENERGY FOR A HYDROMAGNETIC SHOCK

Chapman and Cowling have given expressions for the stress tensor in a magnetic field.[26] We shall use their notation in what follows. They take the magnetic field H in the x direction. Then for unidimensional variation and longitudinal shock propagation in the y-direction:

$$\mathbf{e}^0\equiv\begin{bmatrix}-\frac{1}{3}\partial v_0/\partial y, & 0, & 0\\ 0, & \frac{2}{3}\partial v_0/\partial y, & 0\\ 0, & 0, & -\frac{1}{3}\partial v_0/\partial y\end{bmatrix}. \tag{A-1}$$

In the equation of motion, we have to consider

$$\left(\frac{\partial}{\partial\mathbf{r}}\cdot\mathbf{p}\right)_y=\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}.$$

In our case we need consider only $\partial p_{yy}/\partial y$. Now,

$$p_{yy}=p-2\mu(9+16\omega^2\tau^2)^{-1}[9e^0_{yy}+8(e^0_{yy}+e^0_{zz})\omega^2\tau^2+12e^0_{yz}\omega\tau]$$

$$=p-\frac{36+16\omega^2\tau^2}{27+48\omega^2\tau^2}\mu\frac{\partial v}{\partial y}, \tag{A-2}$$

[24] A. Schlüter, Z. Naturforsch. **5a**, 76 (1950).

[25] See reference 22, p. 352.

[26] See reference 2, pp. 338 and 15. We have used the symbol $\mathbf{e}^0$ for the nondivergent tensor derived from $\mathbf{e}$, as the printer did not have the corresponding symbol used by Chapman and Cowling.

from (A-1). From (A-2), we have

$$p_{yy}=p-\frac{4}{3}\mu\frac{\partial v}{\partial y},\quad \text{when}\quad \omega\tau\ll 1; \tag{A-3}$$

$$=p-\frac{1}{3}\mu\frac{\partial v}{\partial y},\quad \text{when}\quad \omega\tau\gg 1. \tag{A-4}$$

Equation (A-3) is the same as without the magnetic field. Equation (A-4) differs from the nonmagnetic case only in the scale factor 4 for the viscosity. This would involve a change in the Prandtl number $(\mu C_p/k)$ which should not materially affect our conclusions.[6]

Chapman and Cowling have given the formulas for the pressure tensor of a simple gas. As the shock is determined mainly by the ions, we shall use formulas (A-3) and (A-4), with ω, τ, and μ referring to the ions.

To derive the equation of motion we note that the magnetic field introduces a body force $\mathbf{j}\times\mathbf{H}$ per unit volume.[24] With Maxwell's equation (neglecting displacement current in the highly conducting plasma), $\nabla\times\mathbf{H}=4\pi\mathbf{j}/c$, the body force can be written $(c/4\pi)[(\nabla\times\mathbf{H})\times\mathbf{H}]$. Hence we have the following equation of motion for the particular form of the pressure tensor discussed above:

$$\rho\frac{Du}{Dt}=\frac{c}{4\pi}[(\nabla\times\mathbf{H})\times\mathbf{H}]_x-\frac{\partial p}{\partial x}+\frac{4}{3}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right). \tag{A-5}$$

Equation (A-5) can be shown to be equivalent to Eq. (2) of Sec. 2.

For the energy equation, we note that the magnetic field does no work and that the energy dissipated in Joule heat (j^2/σ) vanishes, on account of our assumption of infinite conductivity. Hence the equation of energy remains the same as in the nonmagnetic case.[27]

With neglect of viscosity and conductivity our equations reduce to those considered by Lüst.[28]

[27] Dr. J. Feinstein has drawn the author's attention to the fact that a relativistic treatment is needed when the magnetic field is strong. However, for this to be so, the magnetic field must be very strong, i.e., magnetic energy≫rest energy [F. de Hoffmann and E. Teller, reference 1, p. 697, Eq. (60)]. The present treatment would therefore apply to most physical problems of interest.

[28] See reference 7, p. 278, Eqs. (1) and (3a).

APPENDIX II. THERMODYNAMICAL STABILITY OF A HYDROMAGNETIC SHOCK

We first note that on account of the reasons stated in Appendix I, the magnetic field does not affect the equation of energy. We may therefore use the thermodynamic equation for the change of entropy, *viz.*,[8]

$$S_2-S_1=C_v\ln(Y\eta^{-\gamma}), \tag{B-1}$$

where

$$Y=p_2/p_1,\quad \eta=\rho_2/\rho_1. \tag{B-2}$$

We shall consider the case of a weak shock for which

$$Y=1+y,\quad \eta=1+\epsilon, \tag{B-3}$$

where y and $\epsilon\ll 1$. Then we have[29]

$$h_0(\gamma-1)\epsilon^3+[2\gamma+(\gamma-1)y]\epsilon-2y=0. \tag{B-4}$$

Eliminating y between (B-1) and (B-4), we have

$$(S_2-S_1)/C_v=\ln\{1+\tfrac{1}{2}(\gamma-1)[h_0+\tfrac{1}{6}\gamma(\gamma+1)]\epsilon^3\cdots\};$$

i.e.,

$$(S_2-S_1)/R=\tfrac{1}{2}[h_0+\tfrac{1}{6}\gamma(\gamma+1)]\epsilon^3\cdots. \tag{B-5}$$

For a compression wave $\epsilon>0$, and $S_2>S_1$, from (B-5). $S_2<S_1$ for a rarefaction shock, which is, therefore, thermodynamically unstable.

Our use of Eq. (B-1) seems at first sight to be at variance with de Hoffmann and Teller's inclusion of magnetic terms both in the expressions for pressure and energy,[30] *viz.*,

$$p^*=p+H^2/8\pi, \tag{B-6}$$

$$E^*=E+H^2/8\pi\rho. \tag{B-7}$$

The two procedures are, however, equivalent; this can be seen as follows. For a unit mass of gas, we have the following relation from Eqs. (1) and (6) of Sec. 2:

$$Hv=\text{constant}. \tag{B-8}$$

Now, according to de Hoffmann and Teller:

$$dQ=dE^*+p^*dv=dE+d(H^2v/8\pi)+pdv+(H^2/8\pi)dv$$
$$=dE+pdv, \tag{B-9}$$

from (B-8).

The relation (B-9) shows that there is, in fact, no need to include the magnetic terms in the thermodynamic equations, as was stated in the beginning.

[29] H. L. Helfer, Astrophys. J. 117, 180 (1953), Eq. (19).

[30] See reference 1, pp. 698–700.