Analytic Approach for the Pion-Proton Scattering Phase Shifts*

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A simple method of solving for the phase shifts of the pion-proton scattering is presented. The rapid solution afforded can be utilized, as the Ashkin diagrams have been employed, to give starting values to an electronic computer or alternatively to analyze with more ease the variation of the phase shifts as a function of the input data in terms of the coefficients of the angular distributions. A new plot of a function of the total cross section *versus* the pion energy is introduced. The nearly straight line resulting should help to evaluate the experimental data.

INTRODUCTION

THE first analyses of pion-proton scattering were performed by Fermi *et al.*[1] with an electronic computer. A thorough statistical investigation is necessary to extract the greatest accuracy and maximum consistency from the experimental information, but the essence of the physics is thereby obscured. Extensive calculations have shown that the phase shifts[2] δ_{11} and δ_{13} are small and erratic. A good assumption is then to take these phase shifts equal to zero.[3,4] We shall see that the remaining four phase shifts can be easily evaluated analytically. Increased insight into the nature of the solutions results as a consequence. Of course our conclusions do not differ essentially from those reached by others using fast digital calculations or Ashkin diagrams,[3] but we offer our method in the hope that its simplicity will help us understand the behavior of the pion-proton scattering.

POSITIVE PION-PROTON SCATTERING

We first develop our formulas for the case of the $\pi^+ - p$ scattering. Here our method is basically the transformation of the graphical or geometrical procedure of Ashkin to an algebraic guise.

Given the experimental data in terms of the coefficients[2,5] A_+, B_+, C_+ of the angular distribution[6]

$$\lambda\!\!\!^{-}{}^{-2}d\sigma/d\Omega = A_+ + B_+\cos\theta + C_+\cos^2\theta,$$

we get for the S phase shift δ_3:

$$\sin 2\delta_3 = -\mu(D-\Sigma^2)^{\frac{1}{2}} + (\Sigma-2)\left(\frac{1}{L}-\mu^2\right)^{\frac{1}{2}}, \qquad (1)$$

where $D=4(A_+ + B_+ + C_+)$, $\Sigma = 2(A_+ + C_+/3)$, $L=D-4\Sigma+4$, and $\mu=(\Sigma-2-2B_+)/L$. Again we may prefer to use the equivalent formulas:

$$\cos(\beta - 2\delta_3) = (\Sigma-2-2B_+)/L^{\frac{1}{2}}, \qquad (2a)$$

$$\cos\beta = (\Sigma-2)/L^{\frac{1}{2}}. \qquad (2b)$$

The multiplicity of the allowable solutions comes from the ambiguity of the signs of the square roots in Eq. (1) or alternatively the choice of the branches of the cosine functions in Eqs. (2a) and (2b). We illustrate our procedure with the data[5] at 120 Mev, where

$$A_+ = 0.200, \quad B_+ = -0.360, \quad \text{and} \quad C_+ = 1.040.$$

We get

$$D=3.520, \quad \Sigma=1.093, \quad L=3.147,$$
$$(D-\Sigma^2)^{\frac{1}{2}}=1.525, \qquad \mu=-0.0593,$$
$$(1/L-\mu^2)^{\frac{1}{2}}=0.5606, \qquad \Sigma-2=-0.9067,$$
$$\sin 2\delta_3=-0.4178, \qquad \delta_3=-12.35°.$$

If we use Eqs. (2), we have $\cos(\beta-2\delta_3)=-0.1052$ and $\cos\beta=-0.5111$; $\beta-2\delta_3=-96.04°$, $\beta=-120.74°$, and $\delta_3=-12.35°$ as before.

To calculate δ_{33}, we can use

$$\cos(2\delta_{33}-\theta) = \frac{|b+3|^2+9+4C_+-|b|^2}{6|b+3|}, \qquad (3)$$

where

$$b+3 = (-\Sigma+4-\cos 2\delta_3) + i[(D-\Sigma^2)^{\frac{1}{2}} - \sin 2\delta_3]$$
$$\equiv |b+3|(\cos\theta + i\sin\theta) \equiv X+iY.$$

For the 120-Mev data, we have

$$X=1.9981, \qquad Y=1.9425, \qquad |b+3|^2=7.7657,$$
$$|b+3|=2.7867, \quad \cos\theta=0.7170, \qquad \theta=44.19°,$$
$$|b|^2=(X-3)^2+Y^2=4.777, \quad \cos(2\delta_{33}-\theta)=0.9658,$$

$2\delta_{33}-\theta=\pm 15.03°$, $\delta_{33}=29.61°$ (Fermi), and $\delta_{33}=14.58°$ (Yang).

Thus our method gives both the Fermi and Yang solutions at the same time. It is interesting to note that the Fermi solution $(\delta_3,\delta_{33},\delta_{31})$ and the Yang solution $(\alpha_3,\alpha_{33},\alpha_{31})$ are related to each other by the following equations:

$$\alpha_3=\delta_3, \qquad (4a)$$

$$\alpha_{31}-\alpha_{33}=\delta_{33}-\delta_{31}, \qquad (4b)$$

$$\tan(\alpha_{33}-\delta_{31})=\tfrac{1}{3}\tan(\delta_{33}-\delta_{31}). \qquad (4c)$$

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[6] W. Rarita, Phys. Rev. **99**, 630(A) (1955).

The special case when $\delta_{31}=0$ was derived by de Hoffmann *et al.*[5]

To obtain δ_{31}, we use

$$\cos 2\delta_{31}=X-2\cos 2\delta_{33}. \quad (5)$$

At 120 Mev we have for the Fermi solution:

$$\cos 2\delta_{31}=0.9746, \quad \delta_{31}=6.47°.$$

Our values check those quoted by de Hoffmann *et al.*[5]

POSITIVE AND NEGATIVE PION-PROTON SCATTERING

We first observe that in general

$$\sin^2\delta_1+2\sin^2\delta_{13}+\sin^2\delta_{11}=(3\sigma_- - \sigma_+)/8\pi\lambda\!\!\!^{-2}. \quad (6)$$

Thus $\sin^2\delta_1 \leq (3\sigma_- - \sigma_+)/\sigma_0$, where $\sigma_0 \equiv 8\pi\lambda\!\!\!^{-2}$. If we now make the explicit assumption that $\delta_{11}=\delta_{13}=0$, we get

$$\sin^2\delta_1=(3\sigma_- - \sigma_+)/\sigma_0. \quad (7)$$

In principle, we can compute $|\delta_1|$ from just the measurements of the total cross sections alone. The error in this method is, however, large because there occurs the difference of two large numbers with sizable errors.

TABLE I. Positive angular distribution coefficients. (a) Original coefficients of Anderson *et al.* (reference 7). (b) A more precise set of coefficients computed from their final phase shifts. (c) Coefficients obtained from our least-squares fit.

	A_+	B_+	C_+
(a)	0.960±0.101	0.411±0.172	3.395±0.345
(b)	0.917±0.073	0.345±0.145	3.340±0.181
(c)	0.960±0.101	0.273±0.141	3.365±0.305

In order to make a self-consistent calculation,[4] we have to examine the relations that the coefficients A_+, B_+, C_+, A_0, B_0, C_0 and A_-, B_-, C_- must obey under our condition of $\delta_{11}=\delta_{13}=0$. By making use of the equations of reference 5, we can show that

$$3(B_- + B_0)=B_+, \quad (8a)$$

$$3(C_- + C_0)=C_+, \quad (8b)$$

$$9C_0=18C_-=2C_+. \quad (8c)$$

If our data satisfy Eqs. (8) within the experimental error, we can be assured that δ_{11} and δ_{13} are small and can therefore be set equal to zero.

We now explore two additional methods of computing δ_1. They are based on Eqs. (9):

$$2A_- - A_0=\tfrac{1}{6}(|s|^2+sa^*+s^*a), \quad (9a)$$

$$2B_- - B_0=\tfrac{1}{6}(sb^*+s^*b), \quad (9b)$$

where

$$s=\exp(2i\delta_1)-1, \quad a=\exp(2i\delta_3)-1,$$

and

$$b=2\exp(2i\delta_{33})+\exp(2i\delta_{31})-3.$$

TABLE II. 189-Mev phase shifts.

	Anderson *et al.*	Orear	Our values
δ_3	−11.3°±3.2°	−10.3°	−11.1°±1.8°
δ_{33}	98.8°±3.6°	89°	93.1°±9.4°
δ_{31}	−11.6°±5.1°	0	−13.0°±5.5°
δ_1	−2.8°±4.5°	15°	17.1°±8.0°
δ_{13}	−2.1°±3.8°	0	0
δ_{11}	−2.6°±7.5°	0	0

Another equation which may be useful is

$$\sin^2\delta_1=\tfrac{1}{2}[3(A_- + A_0)-A_+]. \quad (10)$$

We will illustrate our procedure with the recent accurate data obtained at Chicago[7] at 189 Mev.

Using a least-squares fit, we first choose the best values of B_+, C_+ which satisfy Eqs. (8). In Table I, we enter our results as (c). For comparison, we also give (a) the original coefficients of Anderson *et al.*,[7] and (b) a more precise set computed from their final phase shifts.

We use Eq. (2) and find $\cos(\beta-2\delta_3)=0.6751$, $\cos\beta=0.9030$, $\beta-2\delta_3=47.54°$ and $\beta=25.44°$, $\delta_3=-11.05°$. At 189 Mev, we note that the signs of both angles $\beta-2\delta_3$ and β have changed from their assignment at 120 Mev. Of course we have to determine as a function of the meson energy E_L when $\cos(\beta-2\delta_3)$ and $\cos\beta$ go through 1 in order to get a continuous (or here an analytical) change in δ_3 *vs* E_L. By tracking or following the cosine function, we find the critical region when $\cos(\beta-2\delta_3)$ and $\cos\beta$ go through 1 is for both of them about 169 Mev. This behavior accounts for the large number of solutions found by de Hoffmann *et al.*[5] at this energy. The multiplicity of solutions arises from the various choices of the sign for β and $\beta-2\delta_3$.

The value of $\delta_3=-11.1°$ is the Fermi solution, i.e., the continuous extension of the solution at low energies. Anderson *et al.*[7] give $\delta_3=-11.3°\pm3.2°$. With our choice of δ_3, we proceed to δ_{33}. We have $\theta=210.92°$ and $2\delta_{33}-\theta=\pm24.77°$,

$$\delta_{33}=93.08° \text{ (Fermi)}; \quad \delta_{33}=117.85° \text{ (Yang)}.$$

The function $(D-\Sigma^2)^{\frac{1}{2}}$ has changed sign, as it becomes zero when $\mathrm{Im}(a+b)=0$ and this happens at about 177 Mev.

The behavior of Eq. (2) as a function of energy is smooth and no new branching of solutions occurs. Of

TABLE III. Comparison of the (a) Brookhaven and (b) Bethe assumptions.

		$R\equiv\frac{1}{n}\left(\sum^n\left\|\frac{\Delta y}{\epsilon}\right\|^2\right)^{\frac{1}{2}}$ St. line	Cubic	$\Delta y/\epsilon$ St. line 181 Mev	189 Mev	Cubic 181 Mev	189 Mev
(a)	Brookhaven	0.738	0.699	7.3	4.1	6.4	3.2
(b)	Bethe	0.478	0.462	−2.2	−3.2	−1.7	−2.8
(c)	Russian σ_+	0.353	0.336				
(d)	Russian σ_-	0.661					

[7] Anderson, Davidson, Glickman, and Kruse, Phys. Rev. **100**, 279 (1955).

TABLE IV. Resonance parameters.

	E_r (Mev)	$(\sigma_+-\sigma_0)_r$ (mb)
(a) Brookhaven	188	7.5
(b) Bethe	194	7.5
(c) Russian σ_+	198	16.1
(d) Russian σ_-	196	32.9

course these conclusions can be obtained equally well, and were so arrived at, with the use of Ashkin diagrams.[3]

The use of Eq. (5) gives $\delta_{31}=-13.01°$.

To complete our phase-shift analysis, we have to determine δ_1. Equation (7) gives $|\delta_1|=10.8°\pm 9.5°$. Equation (10) has a resulting value of $|\delta_1|=13.8° \pm 12.4°$.

We rewrite Eq. (9a) in the form

$$2A_--A_0=\tfrac{1}{3}[\cos 2(\delta_1-\delta_3)-2\cos 2\delta_1-\cos 2\delta_3+2]. \quad (11)$$

Clementel *et al.*[8] have developed a similar point of view.

For $2A_--A_0=-0.044\pm 0.080$, we get $\delta_1=10°\pm 15°$.

Our most reliable determination of δ_1 comes from Eq. (9b), which we rewrite as

$$2B_--B_0=\tfrac{1}{3}[2\cos 2(\delta_{33}-\delta_1)+\cos 2(\delta_{31}-\delta_1) -3\cos 2\delta_1-2\cos 2\delta_{33}-\cos 2\delta_{31}+3]. \quad (12)$$

Then $\delta_1=17.1°$ from $2B_--B_0=0.113$. In Table II we summarize our results and compare them to those of Anderson *et al.*[7] and Orear.[9]

ENERGY DEPENDENCE OF THE CROSS SECTION

A new plot of a function of the total cross section *versus* the energy will be introduced, in which the resulting nearly straight line should aid in evaluating the experimental data. We first derive the relation for σ_+ *vs* ω, the center-of-mass energy of the pion. We avail ourselves of the relation given by Chew and Low[10] that $(k^3/\omega^*)\cot\delta_{33}$ *vs* ω^* is almost a straight line. k is the momentum of the meson and ω^* is ω plus the kinetic energy of the proton. Serber and Lee[11] and Dyson, Castillejo, and Dalitz[12] have shown that in general $(k^3/\omega^*)\cot\delta_{33}-(1/\omega^*)$ is an analytic function of ω^*. As $\sigma_+=\sigma_0(\sin^2\delta_{33}+\frac{1}{2}(\sin^2\delta_3+\sin^2\delta_{31}))$ and $\sin^2\delta_3+\sin^2\delta_{31}$ is small and believed to vary slowly with energy, we can surmise that $[\sigma_+-(\sigma_+-\sigma_0)_r]/\sigma_0=\sin^2\delta_{33}+\epsilon(\omega)$, where $\epsilon(\omega)$ is small and slowly varying. Further, as $k=\lambda\!\!\!^{-\,-1}\sim\sigma_0^{-\frac{1}{2}}$, we can transform the Chew-Low relation into the following one: $y\equiv(1/\omega\sigma_0)[(\sigma_0-\sigma_c)/\sigma_c\sigma_0]^{\frac{3}{2}}$ *vs* ω should be nearly a straight line. Here $\sigma_c=\sigma_+-(\sigma_+-\sigma_0)_r$, where the subscript r refers to the resonance energy ω_r. The advantage of this representation is that we can test our data for smoothness without the intermediary of knowledge of the angular distribution. Incidentally, since y varies chiefly as $(\sigma_0-\sigma_c)^{\frac{3}{2}}$ we can use this equation as a convenient interpolation formula. Also we can see from this expression that y is more sensitive to errors in ω and σ_+ near resonance than away from resonance.

The recent Brookhaven[13] data have been analyzed according to the above prescription. A least-squares fit, first to a linear and then to a cubic function of ω, was made. Moreover, the 181- and 189-Mev data were considered (a) to be above resonance as preferred by Lindenbaum and Yuan[13] and (b) to be below resonance as advocated by Bethe *et al.*[5] We enter our results in Table III. In Table III, $\Delta y\equiv$deviation of y from the least-squares fit. $\epsilon\equiv$experimental error in y due to $\Delta\sigma_+$. We ignored the error due to $\Delta\omega$. A slightly greater weight is thus given to the data around resonance. The quantity R is taken as a measure of the least-squares fit.

TABLE V. δ_1 from the Russian data.[a]

E_L (Mev)	η	δ_1	$\delta_1{}^l$	$\delta_1{}^u$
140	1.354	0°	0	11.9°
184	1.376	4.3°	0	13.4°
197	1.650	17.9°	11.7°	22.7°
216	1.742	13.0°	0°	19.5°
226	1.789	21.3°	12.6°	26.0°

[a] See reference 14.

A similar analysis was performed on the Russian data[14] for energies from 140 to 229 Mev. We summarize our results in Table IV. $E_r\equiv$laboratory resonance energy. The π^+-p Russian data[14] from 140 to 335 Mev (we excluded the data at 363 and 393 Mev) were studied only for the straight-line case. $\sigma_c=3\sigma_- -(3\sigma_--\sigma_0)_r$, or we have $\sigma_+\to 3\sigma_-$ in our formulas. For comparison, we recall that Bethe *et al.*[5] give 195 Mev for E_r.

Assuming that $y=(\sigma_0-\sigma_c)^{\frac{3}{2}}$ is linear in ω, and interpolating, we obtained Table V from Eq. (7). Here η is the pion momentum in units of $m_\pi c$. If we assume that $\delta_1\sim\eta$, then $\delta_1/\eta=6.7°$. Orear[9] gives 9.2° for this quantity. In Table V, $\delta_1{}^l$ is the lower limit and $\delta_1{}^u$ the upper limit of δ_1.

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