be relatively unprecisely determined, and again it will be difficult to determine the statistical distribution of the results. Furthermore, when too many runs are used, the F-ratio test as given below to determine the significance of the $P_4(\cos\theta)$ term will have poor sensitivity because of the large variation of ϵ^2 from run to run.

The data obtained as indicated in the preceding section were divided into a series of 20 runs, and each run was analyzed following the method indicated by Rose.[8] The correlation function obtained, after correction for the finite angular resolution of the detectors, is

$$W(\theta)=1-(0.0685\pm0.0028)P_2(\cos\theta)-(0.0045\pm0.0042)P_4(\cos\theta).$$

The data were next fitted by an expansion out to $P_2(\cos\theta)$. This fitting yielded the correlation function given by $W(\theta)=1-(0.0699\pm0.0025)P_2(\cos\theta)$. Next the following ratio was evaluated for each of the 20 runs:

$$\frac{(\sum_i v_i^2)_{m-l=8}-(\sum_i v_i^2)_{m-l=7}}{(\epsilon^2)_{m-l=7}},$$

where the v^2's are the squares of the residuals and ϵ^2 is the mean square residual as given in Eqs. (27) and (28) of reference 8; m is the number of angles at which data are taken, and l is the number of coefficients in the Legendre polynomial expansion. The values of these ratios were then compared to a table of the F distribution.[9] From this statistical comparison it was determined that the $P_4(\cos\theta)$ term did not contribute significantly to the correlation function.

Thus the correlation function best representing the present data is given by $W(\theta)=1-(0.0699\pm0.0025)\times P_2(\cos\theta)$. Since the theoretical correlation function for the sequence $3(D)2(Q)0$ is given by $W(\theta)=1-(0.07143)P_2(\cos\theta)$, it is concluded that within the accuracy of the present experiment both of the transitions in the cascade are pure multipoles. This result is in agreement with the measured conversion coefficients[3] of the two gamma rays in the cascade.

IV. ACKNOWLEDGMENTS

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[8] M. E. Rose, Phys. Rev. **91**, 610 (1953).

[9] For example, see Paul G. Hoel, *Introduction to Mathematical Statistics* (John Wiley and Sons, Inc., New York, 1947), p. 250.

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Theory of the 0^+ States of O^{16}†

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The usual shell-model interpretation of the 6.06-Mev 0^+ level of O^{16} as two-nucleon excitation is replaced by one-nucleon excitation. The large excitation energy for an individual nucleon is greatly reduced for special linear combinations of the excited shell-model configurations, because of a resonance effect arising from the degeneracy and from the large off-diagonal matrix elements. The state of collective compressional-dilational oscillation of the nucleus is shown to be a particularly favorable such linear combination, and the results of an explicit calculation indicate that the resonance effect is strong enough to account for the 6.06-Mev level. It is also proposed to interpret the 6.91-Mev 2^+ level as a superposition of one-nucleon excited configurations corresponding to spheroidal deformation.

THE shell model, which satisfactorily accounts for many of the observed regularities in the light nuclei, has generally been regarded as inadequate to explain the low-lying excited states of the O^{16} nucleus.[1] The first excited state, which is 0^+ and falls at 6.06 Mev,[2] has been particularly difficult to understand. Because of its even parity it cannot be interpreted as simply an excitation of one nucleon from the filled $1p$ shell to the vacant $2s$–$1d$ shell lying next above. Excitation of two $1p$ nucleons to the $2s$–$1d$ shell does, however, yield even parity states. Therefore, it has generally been attempted to represent the even-parity O^{16} excited states by such configurations. Aside from the difficulties in the level sequence, these interpretations lead to too high an excitation energy to be tenable for the low-lying states. Each nucleon requires 10–15 Mev for promotion into the next shell, leading to a total of 20–30 Mev for two-nucleon excitation. The coupling energy among the two excited nucleons and the two residual holes is unlikely to be sufficiently negative to reduce the net excitation energy to less than 10 Mev. The 6.06-Mev 0^+ level, especially, seems much too low for such an approach to be successful.

We regard the difficulty in accounting for the 0^+ states in O^{16} as not intrinsic in the shell model, but rather as due to an improper choice of shell-model

† Research supported by the Office of Naval Research and by the National Science Foundation.

[1] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).

[2] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).

configuration. The configurations arising from the excitation of one nucleon into the next shell of the same parity have basically the same excitation energy as the configurations of two-nucleon excitation, discussed above. Excited states arise, for example, by replacing one of the $1p$ radial wave functions in the ground-state Slater determinant by a $2p$ radial wave function, but leaving the angular factors, as well as those for spin and isotopic spin, the same. Similarly, one of the $1s$ radial wave functions can be replaced by a $2s$ radial wave function. These possibilities yield a total of 16 different orthogonal configurations. The diagonal element of the Hamiltonian, evaluated for any one of these excited configurations, is, in the oscillator model, roughly $2\hbar\omega$, or of the order of 30 Mev. Therefore, these configurations cannot individually represent a low-lying excited state. The energy of a linear combination of the sixteen configurations is, however, greatly lowered by a resonance among the configurations. This effect results from the high degree of degeneracy present and from the large off-diagonal matrix elements of the Hamiltonian, evaluated with these configurations as a basis. The interaction of the configurations is illustrated in Fig. 1. The dashed lines labeled (a) indicate the interaction of an excited $2p$ nucleon with another nucleon in the $1p$ core. The $2p$ nucleon can fall back into the hole in the $1p$ core, and at the same time pass on its excitation to the $1p$ nucleon, causing the latter to be promoted to the $2p$ level. Similarly, (b) illustrates the exchange of excitation from the p to the s shell. There are also exchange terms in the off-diagonal elements which can be illustrated in a similar fashion.

Rather than attempt to diagonalize the 16×16 energy matrix which the foregoing approach yields, one can be guided by the success which Griffin[3] has had in predicting for the O^{16} nucleus a relatively low-lying state of collective compressional-dilational vibration, the so-called "breathing mode". Following Hill and Wheeler,[4] and representing the nucleon coordinates by x, he writes the states of collective oscillation as

$$\Psi_i(x)=\int\Phi_0(x,\alpha)f_i(\alpha)d\alpha, \tag{1}$$

where

$$\Phi_0(x,\alpha)=\Phi_0(e^{-\alpha}x,0)\cdot\exp(-24\alpha). \tag{2}$$

$\Phi_0(x,0)$ is the Slater determinant of one-nucleon wave functions $u_i(x)$. The $u_i(x)$ are, in principle, solutions of the Hartree-Fock equations. $\Phi_0(x,0)$ represents the O^{16} ground-state configuration. Various values of the scale parameter, α, represent various configurations of dilation and compression. The state of collective oscillation is thus taken to be a linear superposition over these configurations. The coefficients, $f_i(\alpha)$, and the associated energy eigenvalue can be found from application of the Rayleigh-Ritz variational principle, but we

[3] J. J. Griffin, Ph.D. thesis, Princeton University, 1955 (unpublished).
[4] D. L. Hill and J. A. Wheeler, Phys. Rev. 89, 1102 (1953).

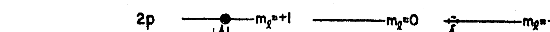

FIG. 1. Interaction of various one-nucleon excited states.

want to point out here an alternative procedure. Equation (1) can be written in terms of shell-model configurations by expanding $\Phi_0(x,\alpha)$ in powers of α. Denoting $\partial^n\Phi_0(x,\alpha)/\partial\alpha^n|_{\alpha=0}$ by $\Phi_0^{(n)}$, one obtains

$$\Psi_i=\sum_{n=0}^{\infty}\left(\frac{1}{n!}\int f_i(\alpha)\alpha^n d\alpha\right)\Phi_0^{(n)}. \tag{3}$$

If one introduces the expressions $u_i(x,\alpha)=u_i(e^{-\alpha}x)\exp(-3\alpha/2)$ and $u_i^{(n)}=\partial^n u_i(x,\alpha)/\partial\alpha^n|_{\alpha=0}$, it follows that $\Phi_0^{(n)}$ is a sum of many Slater determinants, each of which is the result of replacing one or more of the u_i by $u_i^{(\nu)}$, where $\nu\leq n$. Since the effect of the differentiation on the u_i is to introduce more radial nodes into the one-nucleon radial wave functions, it is clear that some or all of the terms in $\Phi_0^{(n)}$ represent excited configurations in the shell model. The wave functions for the ground state and first excited state are particularly simple, and are given approximately by $\Psi_0=\Phi_0$, and by

$$\Psi_1=c_1\Phi_0^{(1)}, \tag{4}$$

where c_1 is a normalization constant.

Thus, the wave function for the first excited level of collective oscillation is simply one particular linear superposition of excited shell model configurations. Approximating the u_i by oscillator wave functions, we find that differentiation with respect to the scale parameter affects the one-particle wave functions as follows:

$$u_{1s}^{(1)}=-(3/2)^{\frac{1}{2}}u_{2s}, \tag{5}$$

$$u_{1p}^{(1)}=-(5/2)^{\frac{1}{2}}u_{2p}. \tag{6}$$

The breathing-mode first excited state wave function has therefore the expansion

$$\Psi_1=\frac{1}{2\sqrt{6}}\sum_{i=1}^{4}\Phi_i+\frac{1}{6}\left(\frac{5}{2}\right)^{\frac{1}{2}}\sum_{i=5}^{16}\Phi_i, \tag{7}$$

where the normalization constant has been fixed as $-1/6$, and the Φ_i are an orthonormal set representing s and p excitation for $i=1,\cdots,4$ and $i=5,\cdots,16$, respectively. Thus, in the breathing mode, the probability of finding an excited p nucleon is five times the probability of finding an excited s nucleon.

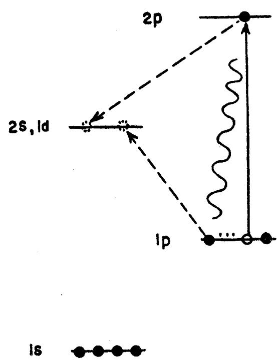

FIG. 2. Mixing of the configurations of one-nucleon and two-nucleon excitation.

The resonance effect discussed above can be exhibited explicitly by direct computation of the expectation value of the energy. In the approximate expression above for the diagonal elements (which, if desired, can of course be improved upon by calculating from first principles), $\hbar\omega$ depends on the radius assumed for the O^{16} nucleus. We have assumed the radii of the O^{16}, O^{15}, and N^{15} nuclei to be all equal, and have employed oscillator wave functions to evaluate the Coulomb energy difference between the O^{15} and N^{15} ground states. Equating this to the experimental Coulomb energy difference gives an O^{16} root-mean-square radius of 2.51×10^{-13} cm, and a value for the diagonal matrix element of $2\hbar\omega=29.7$ Mev. Only the central forces contribute to the breathing mode resonance energy. We chose the Gaussian shape for the interaction potential and determined the range, the strength, and the exchange coefficients from the following five pieces of low-energy experimental data: p-p effective range and scattering length, vanishing singlet odd-state phase shift, α-particle binding energy, and stability of the O^{16} nucleus at the root-mean-square radius given in the foregoing. The two-nucleon interaction determined from these conditions happens to have vanishing spin exchange coefficient and is given by

$$V=-51.9\ \text{Mev}(0.317+0.500P+0.183PQ)\times\exp[-r^2/(1.732\times10^{-13}\text{cm})^2],$$

where P and Q are the space and spin exchange operators, respectively. The contribution to the resonance energy which this force makes, acting within the s shell, is -2.7 Mev. We have found, further, that the resonance between the s and p shells yields -5.6 Mev, while the resonance within the p shell, since there are more nucleons there to interact, gives the major contribution of -12.3 Mev. The total resonance energy is -20.6 Mev, yielding a net excitation energy of $(\Psi_1,(H-E_0)\Psi_1)=9.1$ Mev.

Thus we see that the shell model is capable of describing the states of collective oscillation, and predicts an excited state at an energy much lower than it is generally thought capable of doing. The above result of 9.1 Mev is clearly an overestimate for the first excited 0^+ energy, since it can be lowered by relaxing the strict specification of the wave function given by Eq. (7) to permit an arbitrary ratio between the coefficients of the two separate sums. The s and p shells are then no longer required to "breathe" together, but can dilate by different amounts.[5] Such a generalized wave function can also describe the mode of vibration in which the two shells oscillate out of phase with one another. The first excited level for this type of oscillation will lie considerably higher than that for the in-phase oscillation, however.

An additional lowering of the 0^+ excited states results from the fact that they cannot be pure breathing modes, due to the mixing into the wave function of states of excitation of two $1p$ nucleons into the $2s$–$1d$ shell. Figure 2 illustrates how these states mix with the p-excitation part of the breathing-mode wave function, as a result of the interaction between the excited $2p$ nucleon and one of the nucleons in the $1p$ core. For the s-excitation a similar sort of mixing can occur. The effect of this mixing is to lower the energy still further, and it seems likely that the 6.06-Mev level in O^{16} can be quantitatively accounted for. The next 0^+ state at 11.25 Mev is probably to be interpreted as the breathing mode in its second excited state, with admixtures of other shell-model configurations. The second excited breathing-mode wave function is given by

$$\Psi_2=c_2[\Phi_0^{(2)}+(\Phi_0^{(1)},\Phi_0^{(1)})\Phi_0].$$

The pair-emission lifetime of the 0^+ state provides another experimental check of the correctness of our description.[6,7] The matrix element of the monopole operator $\sum_p r_p^2$, calculated with the breathing mode wave function Ψ_1 is about twice the experimental value. The deviation of the true wave function from Ψ_1, discussed above, will serve to reduce the matrix element, because the admixed states of two-nucleon excitation will not contribute except by reducing the amplitude of Ψ_1.

The other O^{16} even-parity states of $J=2$ and $J=4$ can be interpreted as primarily spheroidal collective oscillations and treated in the shell model in a manner similar to that above. If α is taken to be the parameter describing a spheroidal deformation, with $\alpha=0$ corresponding to the undeformed ground state of O^{16}, then differentiation of the ground-state wave function yields a superposition of states of one-nucleon excitation, in which either a $1s$ nucleon has been excited to the $1d$ shell or a $1p$ nucleon to the $1f$ shell.

[5] B. Jancovici, Compt. rend. **240**, 1608 (1955), has investigated the static effect on the O^{16} ground-state energy of independently dilating the s and p shells. He finds that stability occurs for an s shell contracted about 15% relative to the p shell. *Note added in proof.*—Jancovici's expression for the exchange integral K is in error and should be multiplied by the factor λ_1/λ_0. This causes the two shells to be much more nearly equal in size.

[6] L. I. Schiff, Phys. Rev. **98**, 1281 (1955), discusses the data relevant to this transition, and the predictions of various theoretical models.

[7] P. J. Redmond (private communication) has drawn attention to the fact that the lifetime of an excited s proton is in good agreement with the measured value. See also P. J. Redmond, Phys. Rev. **101**, 751 (1956).