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Properties of Half-Integral Spin Dirac-Fierz-Pauli Particles*

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A method is developed to eliminate extraneous components from the Dirac-Fierz-Pauli equations for half-integral spin particles. Starting from the Rarita-Schwinger formulation, Hamiltonian forms of the equations for the independent components of the wave functions are obtained. The process is carried out in detail for the field-free spin 3/2 and spin 5/2 particles and the reduced equations are quantized. It is shown that the interaction with the electromagnetic field can be introduced in an infinite variety of ways. A one-parameter class of equations containing the interaction is obtained for each spin. By reducing these equations in the nonrelativistic limit it is shown that the gyromagnetic ratio is in each case independent of the parameter and has the unique value of the reciprocal of the spin. For the cases of spin 3/2 and spin 5/2, the quadrupole moment is also obtained. It has the unique value $+(5/3)(\hbar/mc)^2$ for spin 3/2. For spin 5/2, it depends on the parameter but is close to $+(28/15)(\hbar/mc)^2$ over most of its range.

INTRODUCTION

THE discovery in recent years of many new particles whose spins and moments have not yet been measured has revived the interest in theories of particles with spins greater than unity. Two types of theories have been discussed extensively, those of Dirac, Fierz, and Pauli[1–3] (hereafter referred to as DFP) and those of Bhabha[4] though these by no means exhaust all possibilities. The distinctive features of the DFP half-integral spin equations are that the covariant wave functions describing field-free particles transform according to irreducible representations of the orthochronous Lorentz group (group of four-dimensional rotations and space inversions) and that each equation describes a particle with only one spin state. In that sense, the DFP half-integral spin equations may be regarded as the simplest theories of such particles.

The principal purpose of this paper is to derive some of the properties of half-integral spin DFP particles by reducing the equations so as to eliminate the extraneous components contained in the covariant wave functions. In Sec. 1, the convenient form of the field-free equations given by Rarita and Schwinger[5] is derived and the interaction with the electromagnetic field is introduced. In Sec. 2, a general method for reducing these equations is described and is carried out completely for the field-free spin 3/2 and spin 5/2 equations. The reduced equations are quantized. In Sec. 3, the reduction of the equations containing the interaction with the electromagnetic field is carried out in the nonrelativistic limit to obtain the magnetic moments of particles with arbitrary half-integral spins. For the cases of spin 3/2 and spin 5/2, the nonrelativistic expansion is carried further and the intrinsic quadrupole moments of these particles are obtained.

In what follows, the letters $\alpha, \beta, \gamma, \cdots \dot{\alpha}, \dot{\beta}, \dot{\gamma}, \cdots$ will be used for spinor indices; $a, b, c, \cdots$ for Dirac four-spinor indices; $\mu, \nu, \kappa, \cdots$ for four-vector indices; $i, j, k, \cdots$ for space-vector indices. The units in which $\hbar = c = 1$ will be used. The summation convention is used for all indices.

1. COVARIANT EQUATIONS

The field-free DFP equations for a particle of spin $n+1/2$ are[1,2]

$$p_{\gamma\dot{\alpha}}A^{\dot{\alpha}\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\epsilon_1\epsilon_2\cdots\epsilon_n}=mB^{\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\gamma\epsilon_1\epsilon_2\cdots\epsilon_n},$$
$$p^{\gamma\dot{\alpha}}B^{\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\gamma\epsilon_1\epsilon_2\cdots\epsilon_n}=mA^{\dot{\alpha}\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\epsilon_1\epsilon_2\cdots\epsilon_n}, \tag{1.1}$$

where m is the mass of the particle and the component wave functions A and B are each completely symmetric in their dotted and undotted spinor indices. $p_{\gamma\dot{\alpha}}$ is the momentum operator written as a covariant spinor. Since a spinor with one dotted and one undotted index transforms like a four-vector[6,7] one can, by pairing $\dot{\beta}_i$ and ϵ_i replace n dotted and n undotted spinor indices of both A and B by a n symmetric traceless four-vector indices. For $n=0$, Eqs. (1.1) are equivalent to the Dirac spin 1/2 equation and the two spinors $A^{\dot{\alpha}}$, B_γ transform like a Dirac four-spinor.[6] Therefore, Eqs. (1.1) can be written as a Dirac equation whose wave function has, besides the four-spinor index, n symmetric traceless four-vector indices:

$$\left(\gamma_\mu{}^{ab}\frac{\partial}{\partial x_\mu}+m\delta^{ab}\right)\Psi^b{}_{\nu_1\nu_2\cdots\nu_n}=0. \tag{1.2}$$

The wave function Ψ will be called an nth rank spin tensor. So far only the symmetry of A and B in

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$\dot{\beta}_1, \dot{\beta}_2, \cdots, \dot{\beta}_n$ and in $\epsilon_1, \epsilon_2, \cdots \epsilon_n$ has been used. The additional symmetry conditions required are

$$A^{\dot{\alpha}\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\epsilon_1\epsilon_2\cdots\epsilon_n}-A^{\dot{\beta}_1\dot{\alpha}\dot{\beta}_2\cdots\dot{\beta}_n}_{\epsilon_1\epsilon_2\cdots\epsilon_n}=\mathcal{A}^{\beta_2\cdots\dot{\beta}_n}_{\epsilon_1\epsilon_2\cdots\epsilon_n}=0,$$
$$B^{\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\gamma\epsilon_1\epsilon_2\cdots\epsilon_n}-B^{\dot{\beta}_1\dot{\beta}_2\cdots\beta_n}_{\epsilon_1\gamma\epsilon_2\cdots\epsilon_n}=\mathcal{B}^{\dot{\beta}_1\dot{\beta}_2\cdots\dot{\beta}_n}_{\epsilon_2\cdots\epsilon_n}=0. \quad (1.3)$$

The fact that an antisymmetric dotted or undotted second-rank spinor transforms like a scalar has been used. Writing the functions $\mathcal{A}$, $\mathcal{B}$ as an $(n-1)$th rank spin-tensor one finds that Eqs. (1.3) require that the function $\Psi^b{}_{\nu_1\nu_2\cdots\nu_n}$ shall not contain an $(n-1)$th rank spin tensor which in turn does not contain an $(n-2)$th rank spin tensor and so on. Because of the tracelessness of Ψ it is sufficient to require that an $(n-1)$th rank spin tensor shall vanish. This is satisfied only by the condition

$$\gamma_\mu{}^{ab}\Psi^b{}_{\mu\nu_2\cdots\nu_n}=0. \quad (1.4)$$

Equations (1.2) and (1.4) are the Rarita-Schwinger equations for a particle of spin $n+1/2$. It is only necessary to require that Ψ by symmetric in all its vector indices. The trace condition follows from Eqs. (1.2) and (1.4). Another consequence of these equations is

$$(\partial/\partial x_\mu)\Psi^b{}_{\mu\nu_2\cdots\nu_n}=0. \quad (1.5)$$

Henceforth, the four-spinor indices will be dropped and the notation $\partial_\mu=\partial/\partial x_\mu$ will be used. The adjoints of the Rarita-Schwinger equations can be written

$$\partial_\mu\Psi^\dagger{}_{\nu_1\nu_2\cdots\nu_n}\gamma_\mu-m\Psi^\dagger{}_{\nu_1\nu_2\cdots\nu_n}=0,$$
$$\Psi^\dagger{}_{\mu\nu_2\cdots\nu_n}\gamma_\mu=0, \quad (1.6)$$

where

$$\Psi^\dagger{}_{\nu_1\nu_2\cdots\nu_n}=(-1)^r\Psi^*{}_{\nu_1\nu_2\cdots\nu_n}\gamma_4,$$

and r is the number of times the index 4 occurs in $\nu_1\nu_2\cdots\nu_n$.

Equations (1.2) and (1.4) are derivable from any one of a set of Lagrangians obtained from the densities

$$\mathcal{L}=\Psi^\dagger{}_{\kappa\nu_2\cdots\nu_n}[(\gamma_\mu\partial_\mu+m)\delta_{\kappa\lambda}+A(\gamma_\kappa\partial_\lambda+\gamma_\lambda\partial_\kappa)+(\tfrac{3}{2}A^2+A+\tfrac{1}{2})\gamma_\kappa\gamma_\rho\partial_\rho\gamma_\lambda-(3A^2+3A+1)m\gamma_\kappa\gamma_\lambda]\Psi_{\lambda\nu_2\cdots\nu_n}; \quad (1.7)$$
$$A\neq-1/2,$$

where the parameter A may assume any real value except $-1/2$. Equations (1.2), (1.4), and (1.5) are obtained by varying $\Psi^\dagger$ and operating on the resulting equation with γ_κ and with ∂_κ. It can easily be shown in this way that (1.7) defines the only possible Lagrangians.

Equations Containing the Electromagnetic Interaction

The interaction with the electromagnetic field is now introduced by writing Lagrangian densities with the following properties: (a) They shall be relativistically invariant and gauge invariant. (b) For vanishing fields they shall reduce to (1.7). (c) The equations derived from them shall contain as many subsidiary conditions as are embodied in Eq. (1.4) in order that the wave function shall have as many linearly independent components as in the field-free case and shall hence describe a particle with the same spin states. By a subsidiary condition is here meant any equation which is not an equation of motion for some component of Ψ (does not contain its time derivative) and which thus permits the elimination of that component in terms of others.

Conditions (a) and (b) are clearly satisfied by replacing in (1.7) the gradient operators ∂_μ by the gauge-invariant derivatives,

$$D_\mu=\partial_\mu-ieA_\mu, \quad (1.8)$$

and adding all possible invariant terms which contain the electromagnetic field tensor explicitly. It can easily be shown that all such additional terms lead to terms in the resulting equations which make it impossible to satisfy the requirement (c). The remaining Lagrangian densities (1.7) with the substitution (1.8) satisfy the requirements (a), (b), and (c) for all real values of A except $-1/2$. The resulting possible equations and subsidiary conditions for a particle of spin $n+1/2$ interacting with the electromagnetic field are

$$[(\gamma_\mu D_\mu+m)\delta_{\kappa\lambda}+A\partial_\kappa\gamma_\lambda+\tfrac{1}{2}(A+1)\gamma_\kappa\gamma_\rho D_\rho\gamma_\lambda-(\tfrac{3}{2}A+1)m\gamma_\kappa\gamma_\lambda]\Psi_{\lambda\nu_2\cdots\nu_n}=0, \quad (1.9)$$

$$3(A+\tfrac{1}{2})m^2\gamma_\lambda\Psi_{\lambda\nu_2\cdots\nu_n}=ie[\gamma_\kappa F_{\kappa\lambda}+\tfrac{1}{2}A\gamma_\kappa F_{\kappa\rho}\gamma_\rho\gamma_\lambda]\Psi_{\lambda\nu_2\cdots\nu_n}, \quad (1.10)$$

where $F_{\mu\nu}=-(i/e)[D_\mu,D_\nu]$ is the electromagnetic field tensor. It should be noted that since Ψ no longer satisfies Eq. (1.4) the spinor representation of the wave function is no longer that of Eq. (1.1) but there are lower rank spinors present. These are the auxiliary spinors of Fierz and Pauli.[3]

2. REDUCTION OF THE FIELD-FREE EQUATIONS

The wave function $\Psi_{\nu_1\nu_2\cdots\nu_n}$ of Eqs. (1.2) and (1.4) contains parts which under space rotations transform according to the representations $D_{\frac{1}{2}}\times D_0$, $D_{\frac{1}{2}}\times D_1$, $D_{\frac{1}{2}}\times D_2$, $\cdots$, $D_{\frac{1}{2}}\times D_{n-1}$, $D_{\frac{1}{2}}\times D_n$ of the three-dimensional rotation group.[8] These parts correspond to the traceless components $\Psi_{44\cdots4}$, $\Psi_{44\cdots4i_n}$, $\Psi_{44\cdots4i_{n-1}i_n}$, $\cdots$, $\Psi_{4i_2\cdots i_n}$, $\Psi_{i_1i_2\cdots i_n}$, which will be denoted by their angular momentum quantum numbers l, m_l, $\frac{1}{2}$, $m_{\frac{1}{2}}$ as follows: $\langle 00\frac{1}{2}m_{\frac{1}{2}}|$, $\langle 1m_1\frac{1}{2}m_{\frac{1}{2}}|$, $\langle 2m_2\frac{1}{2}m_{\frac{1}{2}}|$, $\cdots\langle nm_n\frac{1}{2}m_{\frac{1}{2}}|$. The irreducible parts of $\langle lm_l\frac{1}{2}m_{\frac{1}{2}}|$ which transform according to $D_{l+\frac{1}{2}}$ and $D_{l-\frac{1}{2}}$ will be written $\langle l\frac{1}{2}l+\frac{1}{2}m_{l+\frac{1}{2}}|$ and $\langle l\frac{1}{2}l-\frac{1}{2}m_{l-\frac{1}{2}}|$. Also, $\langle 00\frac{1}{2}m_{\frac{1}{2}}|=\langle 0\frac{1}{2}\frac{1}{2}m_{\frac{1}{2}}|$.

Using the representation of the γ matrices:

$$\gamma_i=\rho_2\sigma_i, \quad \gamma_4=\rho_3, \quad (2.1)$$

[8] The fact that each representation occurs twice because of the positive-negative energy degree of freedom of the four-spinor index is unimportant here. This degree of freedom will be acted upon only by the ρ matrices of Eq. (2.1).

where the σ_i acting on $\langle lm_l\frac{1}{2}m_{\frac{1}{2}}|$ are twice the angular momentum 1/2 matrices and ρ_1, ρ_2, ρ_3 is a set of Pauli matrices, Eq. (1.4) can be written

$$\Psi_{4\nu_2\cdots\nu_n}=i\rho_1\sigma_j\Psi_{j\nu_2\cdots\nu_n}. \tag{2.2}$$

This equation has components of the type

$$\Psi_{i_1i_2\cdots i_{l-1}44\cdots 4}=i\rho_1\sigma_j\Psi_{ji_1i_2\cdots i_{l-1}4\cdots 4}. \tag{2.3}$$

Considering only the part which is traceless in i_1, i_2, $\cdots i_{l-1}$, the left-hand side contains the functions $\langle l-1\frac{1}{2}l-\frac{1}{2}m_{l-\frac{1}{2}}|$ and $\langle l-1\frac{1}{2}l-3/2m_{l-\frac{3}{2}}|$. Hence also the right hand side can contain only parts transforming according to $D_{l-\frac{1}{2}}$ and $D_{l-\frac{3}{2}}$. But $\Psi_{ji_1i_2\cdots i_{l-1}4\cdots 4}$ contains only the function $\langle l\frac{1}{2}l-\frac{1}{2}m_{l-\frac{1}{2}}|$ which transforms according to $D_{l-\frac{1}{2}}$. One concludes therefore that $\sigma_j\Psi_{ji_1i_2\cdots i_{l-1}4\cdots 4}$ is proportional to $\langle l\frac{1}{2}l-\frac{1}{2}m_{l-\frac{1}{2}}|$ as can be verified by comparing the matrices σ_j with the Wigner coefficients $\langle l\frac{1}{2}l-\frac{1}{2}m_{l-\frac{1}{2}}|lm_l\frac{1}{2}m_{\frac{1}{2}}\rangle$. Equation (1.4) has then the following consequence (where C_l is a nonvanishing constant):

$$\langle l-1\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}|=i\rho_1C_l\langle l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}|,\quad l=1,2,3,\cdots,n. \tag{2.4}$$

According to Eq. (1.2), each function $\langle lm_l\frac{1}{2}m_{\frac{1}{2}}|$ satisfies the Dirac equation

$$i\partial_t\langle lm_l\tfrac{1}{2}m_{\frac{1}{2}}|=\langle m_{\frac{1}{2}}|H_{\frac{1}{2}}|m_{\frac{1}{2}}'\rangle\langle lm_l\tfrac{1}{2}m_{\frac{1}{2}}'|, \tag{2.5}$$

where

$$H_{\frac{1}{2}}=\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}+\rho_3m.$$

Decomposing (2.5) into its irreducible parts, one obtains

$$\begin{aligned}i\partial_t\langle l\tfrac{1}{2}l+\tfrac{1}{2}m_{l+\frac{1}{2}}|&=\langle l\tfrac{1}{2}l+\tfrac{1}{2}m_{l+\frac{1}{2}}|H_{\frac{1}{2}}|l\tfrac{1}{2}l+\tfrac{1}{2}m_{l+\frac{1}{2}}'\rangle\langle l\tfrac{1}{2}l+\tfrac{1}{2}m_{l+\frac{1}{2}}'|\\&\quad+\langle l\tfrac{1}{2}l+\tfrac{1}{2}m_{l+\frac{1}{2}}|H_{\frac{1}{2}}|l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}\rangle\langle l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}|,\end{aligned} \tag{2.6}$$

$$\begin{aligned}i\partial_t\langle l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}|&=\langle l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}|H_{\frac{1}{2}}|l\tfrac{1}{2}l+\tfrac{1}{2}m_{l+\frac{1}{2}}\rangle\langle l\tfrac{1}{2}l+\tfrac{1}{2}m_{l+\frac{1}{2}}|\\&\quad+\langle l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}|H_{\frac{1}{2}}|l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}'\rangle\langle l\tfrac{1}{2}l-\tfrac{1}{2}m_{l-\frac{1}{2}}'|,\\&\qquad l=1,2,3,\cdots,n,\end{aligned} \tag{2.7}$$

$$i\partial_t\langle 0\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|=\langle 0\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|H_{\frac{1}{2}}|0\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}'\rangle\langle 0\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}'|. \tag{2.8}$$

Equations (2.4), (2.6), (2.7), and (2.8) can now be used to eliminate all functions except $\langle n\frac{1}{2}n+\frac{1}{2}m_{n+\frac{1}{2}}|$. The latter is then found to satisfy an equation of the form

$$\begin{aligned}i\partial_t\langle n\tfrac{1}{2}n+\tfrac{1}{2}m_{n+\frac{1}{2}}|\\=\langle n+\tfrac{1}{2}m_{n+\frac{1}{2}}|H_{n+\frac{1}{2}}|n+\tfrac{1}{2}m_{n+\frac{1}{2}}'\rangle\langle n\tfrac{1}{2}n+\tfrac{1}{2}m_{n+\frac{1}{2}}'|,\end{aligned} \tag{2.9}$$

which is the reduced spin $n+1/2$ equation.

The reduction is most conveniently carried out successively for spin 3/2, 5/2, 7/2, and so on. Having obtained $H_{n-\frac{1}{2}}$, one finds $H_{n+\frac{1}{2}}$ in the following way. Equations (2.4) through (2.8) for $l<n$ are identical to the corresponding spin $n-1/2$ equations. They can therefore be replaced by

$$\begin{aligned}i\partial_t\langle n-1\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}|=\langle n-\tfrac{1}{2}m_{n-\frac{1}{2}}|H_{n-\frac{1}{2}}|n-\tfrac{1}{2}m_{n-\frac{1}{2}}'\rangle\\\times\langle n-1\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}'|.\end{aligned} \tag{2.10}$$

The remaining Eq. (2.4) is now used to express $\langle n-1\frac{1}{2}n-\frac{1}{2}m_{n-\frac{1}{2}}|$ in terms of $\langle n\frac{1}{2}n-\frac{1}{2}m_{n-\frac{1}{2}}|$. Equation (2.10) becomes then

$$\begin{aligned}i\partial_t\langle n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}|\\=\langle n-\tfrac{1}{2}m_{n-\frac{1}{2}}|\rho_1H_{n-\frac{1}{2}}\rho_1|n-\tfrac{1}{2}m_{n-\frac{1}{2}}'\rangle\\\times\langle n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}'|.\end{aligned} \tag{2.11}$$

Subtracting (2.11) from Eq. (2.7) for $l=n$, one obtains

$$\begin{aligned}\{\langle n-\tfrac{1}{2}m_{n-\frac{1}{2}}|\rho_1H_{n-\frac{1}{2}}\rho_1|n-\tfrac{1}{2}m_{n-\frac{1}{2}}'\rangle\\-\langle n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}|H_{\frac{1}{2}}|n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}'\rangle\}\langle n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}'|\\=\langle n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}|H_{\frac{1}{2}}|n\tfrac{1}{2}n+\tfrac{1}{2}m_{n+\frac{1}{2}}\rangle\langle n\tfrac{1}{2}n+\tfrac{1}{2}m_{n+\frac{1}{2}}|.\end{aligned} \tag{2.12}$$

The operator on the left-hand side can be diagonalized by multiplying the equation by a proper operator. One obtains then a partial differential equation with constant coefficients which may be solved for $\langle n\frac{1}{2}n-\frac{1}{2}m_{n-\frac{1}{2}}|$ by the Fourier transform method. Using this solution to eliminate $\langle n\frac{1}{2}n-\frac{1}{2}m_{n-\frac{1}{2}}|$ from Eq. (2.6) for $l=n$, one obtains the reduced spin $n+1/2$ equation. This procedure will now be illustrated for the case of spin 3/2. The results for spin 5/2 will then be stated.

Spin 3/2

For $n=1$, Eq. (2.12) becomes

$$\begin{aligned}\{\langle\tfrac{1}{2}m_{\frac{1}{2}}|\rho_1H_{\frac{1}{2}}\rho_1|\tfrac{1}{2}m_{\frac{1}{2}}'\rangle-\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|H_{\frac{1}{2}}|1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}'\rangle\}\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}'|\\=\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|H_{\frac{1}{2}}|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}\rangle\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|.\end{aligned} \tag{2.13}$$

Introducing the expression (2.5) for $H_{\frac{1}{2}}$, this becomes

$$\begin{aligned}\langle\tfrac{1}{2}m_{\frac{1}{2}}|(4/3)\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}-2\rho_3m|\tfrac{1}{2}m_{\frac{1}{2}}'\rangle\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}'|\\=\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}\rangle\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|.\end{aligned} \tag{2.14}$$

This equation is now multiplied by the operator on the left-hand side, and one obtains

$$\begin{aligned}[(4/9)p^2+m^2]\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|\\=-\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}+\tfrac{1}{2}\rho_3m|1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}'\rangle\\\times\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}'|\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}\rangle\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|.\end{aligned} \tag{2.15}$$

The solution of Eq. (2.15) may be indicated formally by

$$\begin{aligned}\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|=-\frac{1}{(4/9)p^2+m^2}\\\times\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|(\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}+\tfrac{1}{2}\rho_3m)P_{\frac{1}{2}}\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}\rangle\\\times\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|,\end{aligned} \tag{2.16}$$

where the projection operator $P_{\frac{1}{2}}$ is used to indicate summation over the $\langle 1\frac{1}{2}\frac{1}{2}m_{\frac{1}{2}}'|$ states.

Equation (2.6) becomes, for $n=1$,

$$\begin{aligned}i\partial_t\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|=\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|H_{\frac{1}{2}}|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}'\rangle\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}'|\\+\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|H_{\frac{1}{2}}|1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}\rangle\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|.\end{aligned} \tag{2.17}$$

Using Eq. (2.16) to eliminate $\langle 1\frac{1}{2}\frac{1}{2}m_{\frac{1}{2}}|$, one obtains

$$i\partial_t\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}| = \langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}+\rho_3 m - \frac{\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}P_{\frac{1}{2}}(\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}+\frac{1}{2}\rho_3 m)P_{\frac{1}{2}}\rho_1\boldsymbol{\sigma}\cdot\mathbf{p}}{(4/9)p^2+m^2}|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}'\rangle \times\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|. \quad (2.18)$$

The operator on the right-hand side is the spin 3/2 Hamiltonian. The angular momentum operators occurring in it can all be expressed in terms of the angular momentum 3/2 matrices $J_i^{\frac{3}{2}}$. Denoting the function $\langle 1\frac{1}{2}\frac{3}{2}m_{\frac{3}{2}}|$ by $\Psi_{\frac{3}{2}}$, one obtains

$$i\partial_t\Psi_{\frac{3}{2}} = H_{\frac{3}{2}}\Psi_{\frac{3}{2}},$$

$$H_{\frac{3}{2}} = \rho_1\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{p}\left[1+\frac{p^2-(\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{p})^2}{(4/9)p^2+m^2}\right] +\rho_3 m\left[1+\frac{1}{2}\frac{p^2-(\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{p})^2}{(4/9)p^2+m^2}\right], \quad (2.19)$$

where

$$\Sigma_i^{\frac{3}{2}} = \tfrac{2}{3}J_i^{\frac{3}{2}}.$$

This result can be checked by observing that $(H_{\frac{3}{2}})^2 = p^2+m^2$. It is interesting to note that with the substitution $\boldsymbol{\Sigma}^{\frac{3}{2}}\rightarrow\boldsymbol{\sigma}$ one obtains $H_{\frac{3}{2}}\rightarrow H_{\frac{1}{2}}$.

Equation (2.19) can now easily be quantized. From (1.7) the charge density is found to be $e\Psi^\dagger_\mu\gamma_4\Psi_\mu$. Eliminating the extraneous components of Ψ by means of (2.4) and (2.16), one obtains for the total charge

$$C_{\frac{3}{2}} = e\int d^3x\Psi_{\frac{3}{2}}^*\rho_{\frac{3}{2}}\Psi_{\frac{3}{2}},$$
$$\rho_{\frac{3}{2}} = 1-\frac{1}{2}\frac{p^2-(\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{p})^2}{(4/9)p^2+m^2}, \quad (2.20)$$

which is positive definite for $m\neq 0$. The energy is

$$E_{\frac{3}{2}} = \int d^3x\Psi_{\frac{3}{2}}^* H_{\frac{3}{2}}\rho_{\frac{3}{2}}\Psi_{\frac{3}{2}}. \quad (2.21)$$

Requiring $\partial_t\Psi_\alpha = i[E_{\frac{3}{2}},\Psi_\alpha]$, where α indicates the eight degrees of freedom of the field operator $\Psi_{\frac{3}{2}}$, the anticommutation relations satisfied by $\Psi_{\frac{3}{2}}$ are found to be

$$\{\Psi_\alpha^*(x'),\Psi_\beta(x)\} = \langle\beta|\rho_{\frac{3}{2}}^{-1}|\alpha\rangle\delta(x'-x),$$
$$\rho_{\frac{3}{2}}^{-1} = 1+\frac{p^2-(\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{p})^2}{2m^2}, \quad (2.22)$$

and all other anticommutators vanish. The time dependent anticommutation relations are

$$\{\psi_\alpha^*(x',t'),\psi_\beta(x,t)\} = \langle\beta|(\partial_t-iH_{\frac{3}{2}})\rho_{\frac{3}{2}}^{-1}|\alpha\rangle D(x-x',t-t'), \quad (2.23)$$

where $D(x-x',t-t')$ is the invariant D function.[9] Equation (2.23) can be shown to agree with the covarient spinor relations of Fierz.[2]

Spin 5/2

For $s=5/2$, one obtains

$$i\partial_t\Psi_{\frac{5}{2}} = H_{\frac{5}{2}}\Psi_{\frac{5}{2}},$$

$$H_{\frac{5}{2}} = \rho_1\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{p}\{1+M[p^2-(\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{p})^2]\} +\rho_3 m\{1+N[p^2-(\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{p})^2]\},$$

$$M = \frac{1}{\lambda}\left[\left(\frac{35}{3}p^4+\frac{115}{4}m^2p^2+\frac{50}{3}m^4\right) -25\left(p^2+\frac{17}{12}m^2\right)(\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{p})^2\right],$$

$$N = \frac{1}{\lambda}\left[\left(\frac{23}{4}p^4+\frac{135}{4}m^2p^2+\frac{25}{2}m^4\right) -\frac{25}{12}\left(5p^2+\frac{13}{2}m^2\right)(\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{p})^2\right], \quad (2.24)$$

$$\lambda = \frac{64}{25}p^6+\frac{484}{25}m^2p^4+40m^4p^2+25m^6,$$

$$\Sigma_i^{5/2} = \tfrac{2}{5}J_i^{5/2}.$$

Again it can be verified that $(H_{\frac{5}{2}})^2 = p^2+m^2$. The substitution $\boldsymbol{\Sigma}^{5/2}\rightarrow\boldsymbol{\sigma}$ again gives $H_{\frac{5}{2}}\rightarrow H_{\frac{1}{2}}$, but $\boldsymbol{\Sigma}^{5/2}\rightarrow\boldsymbol{\Sigma}^{3/2}$ does not give $H_{\frac{5}{2}}\rightarrow H_{\frac{3}{2}}$.

The charge is found to be

$$C_{\frac{5}{2}} = e\int d^3x\Psi^*_{\frac{5}{2}}\rho_{\frac{5}{2}}\Psi_{\frac{5}{2}},$$

$$\rho_{\frac{5}{2}} = 1-\frac{[S+T(\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{p})^2][p^2-(\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{p})^2]}{288\lambda^2[(4/9)p^2+m^2]},$$

$$S = \tfrac{1}{5}(4096p^{12}+62948m^2p^{10}+394468m^4p^8 +1435521m^6p^6+2754280m^8p^4 +2548750m^{10}p^2+900000m^{12}), \quad (2.25)$$

$$T = \tfrac{1}{3}(4096p^{10}+14468m^2p^8+151748m^4p^6 +505001m^6p^4+619000m^8p^2+258750m^{10}),$$

and the energy is

$$E_{\frac{5}{2}} = \int d^3x\Psi_{\frac{5}{2}}^* H_{\frac{5}{2}}\rho_{\frac{5}{2}}\Psi_{\frac{5}{2}}. \quad (2.26)$$

The anticommutation relations satisfied by the field operators are

$$\{\Psi_\alpha^*(x'),\Psi_\beta(x)\} = \langle\beta|\rho_{\frac{5}{2}}^{-1}|\alpha\rangle\delta(x'-x), \quad (2.27)$$

[9] G. Wentzel, *Quantum Theory of Fields* (Interscience Publishers, Inc., New York, 1949).

and

$$\{\Psi_\alpha^*(x',t'),\Psi_\beta(x,t)\} = \langle\beta|(\partial_t - iH_{\frac{1}{2}})\rho_{\frac{1}{2}}^{-1}|\alpha\rangle D(x-x', t-t'). \quad (2.28)$$

3. NONRELATIVISTIC LIMITS

Equations (1.9) and (1.10) for a DFP particle in an electromagnetic field cannot be reduced in closed form for arbitrary fields because the equations corresponding to (2.12) are differential equations containing the field vectors as coefficients. Their solution can be obtained only for special cases such as a constant magnetic field. It is, however, possible to expand the equations in powers of $1/m$ and by retaining terms of a given order one can obtain the reduced equations in the nonrelativistic limit. In this way the moments of the particles may be found. The expansion has been carried out to order $1/m$ for arbitrary half-integral spin particles yielding the magnetic moments of these particles. For the cases of spin 3/2 and spin 5/2, the expansion has been carried to order $(1/m)^2$ and the electric quadrupole moments of these particles have been obtained.

Magnetic Moments[10]

To terms of order $1/m$, Eq. (1.10) is identical to Eq. (1.4). Applying (1.4) to (1.9), one obtains Eq. (1.2) with ∂_μ replaced by D_μ. It is clear that the parameter A does not appear in these equations and hence the magnetic moments cannot depend on A. Again, Eqs. (2.4) through (2.12) hold with

$$H_{\frac{1}{2}} = e\varphi + \rho_1 \boldsymbol{\sigma}\cdot\boldsymbol{\Pi} + \rho_3 m,$$

where φ is the electrostatic potential and $\Pi_j = -iD_j$ is the gauge invariant momentum operator. For $H_{n-\frac{1}{2}}$ in (2.12), one now uses the expansion of the spin $n-1/2$ Hamiltonian in powers of $1/m$. This expansion starts with the term $\rho_3 m$ which can be seen by inspecting the equations in the zeroth order in $1/m$. To that order, the right hand side of Eq. (2.12) for $n=1$ vanishes and hence only the first term on the right hand side of Eq. (2.6) contributes to the zeroth order $H_{\frac{3}{2}}$ which consequently starts with the term $\rho_3 m$. Continuing, one finds that the right-hand side of (2.12) also vanishes for $n=2$ in the zeroth order and hence also $H_{\frac{5}{2}}$ starts with $\rho_3 m$ and so on.

Therefore, Eq. (2.12) becomes, to order $1/m$,

$$\langle n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}| = -\frac{i\rho_2}{2m}\langle n\tfrac{1}{2}n-\tfrac{1}{2}m_{n-\frac{1}{2}}|\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}|n\tfrac{1}{2}n+\tfrac{1}{2}m_{n+\frac{1}{2}}\rangle \times\langle n\tfrac{1}{2}n+\tfrac{1}{2}m_{n+\frac{1}{2}}|. \quad (3.1)$$

Substituting this into Eq. (2.6) for $l=n$, one obtains with $s=n+1/2$

$$i\partial_t\langle sm_s| = \langle s-\tfrac{1}{2}\tfrac{1}{2}sm_s|e\varphi+\rho_1\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}+\rho_3 m + \frac{1}{2m}\rho_3\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}P_{s-1}\boldsymbol{\sigma}\cdot\boldsymbol{\Pi}|s-\tfrac{1}{2}\tfrac{1}{2}sm_s'\rangle\langle sm_s'|, \quad (3.2)$$

where the projection operator P_{s-1} is

$$P_{s-1} = (s-\tfrac{1}{2}-\boldsymbol{\sigma}\cdot\mathbf{l})/2s, \quad (3.3)$$

and the l_i are the angular momentum $s-1/2$ matrices. With the help of this operator and the easily verified relation

$$\langle s-\tfrac{1}{2}\tfrac{1}{2}sm_s|\sigma_i|s-\tfrac{1}{2}\tfrac{1}{2}sm_s'\rangle = \frac{1}{s}\langle m_s|J_i^s|m_s'\rangle \equiv \langle m_s|\Sigma_i^s|m_s'\rangle, \quad (3.4)$$

where the J_i^s are the angular momentum s matrices, Eq. (3.2) becomes

$$i\partial_t\Psi_s = \left\{e\varphi+\rho_1\boldsymbol{\Sigma}^s\cdot\boldsymbol{\Pi}+\rho_3 m + \frac{\rho_3}{2m}[\Pi^2 + i\boldsymbol{\Sigma}^s\cdot\boldsymbol{\Pi}\times\boldsymbol{\Pi} - (\boldsymbol{\Sigma}^s\cdot\boldsymbol{\Pi})^2] + \text{higher order terms}\right\}\Psi_s, \quad (3.5)$$

where Ψ_s is the $2(2s+1)$ component spin s wave function. Noting that $\boldsymbol{\Pi}\times\boldsymbol{\Pi} = ie\mathbf{H}$ where $\mathbf{H}$ is the magnetic field vector, and diagonalizing Eq. (3.5) with respect to positive and negative energy states by means of a Foldy-Wouthuysen transformation,[11,12] one obtains

$$i\partial_t\Psi_s = H_s\Psi_s,$$

$$H_s = e\varphi + \rho_3\left[m + \frac{\Pi^2}{2m} - \frac{e}{2m}\boldsymbol{\Sigma}^s\cdot\mathbf{H}\right] + \text{higher order terms}. \quad (3.6)$$

With the definition (3.4) of $\boldsymbol{\Sigma}^s$, the magnetic moment of a DFP particle of half-integral spin s is seen to be $(1/s)(e\hbar/2mc)$ in conventional units.

Quadrupole Moments

Spin 3/2

Equations (1.9) and (1.10) will now be reduced for $s=3/2$ $(n=1)$ and terms of order $(1/m)^2$ will be retained. By means of Eq. (1.10) and the time part $(\kappa=4)$ of Eq. (1.9), one can eliminate the function Ψ_4 from the space part $(\kappa=i=1, 2, 3)$ of Eq. (1.9). The

[10] The results of this section were previously reported at the 1954 Detroit meeting of the American Physical Society by K. M. Case, Phys. Rev. **94**, 1442(A) (1954).

[11] L. L. Foldy and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).
[12] K. M. Case, Phys. Rev. **95**, 1323 (1954).

latter becomes then, to order $(1/m)^2$,

$$\langle 1m_1\tfrac{1}{2}m_{\frac{1}{2}}|P_{\frac{3}{2}}D_4-i\rho_1\boldsymbol{\sigma}\cdot\mathbf{D}+(1+P_{\frac{1}{2}})\rho_3 m+P_{\frac{1}{2}}i\rho_1(\boldsymbol{\sigma}+2\mathbf{l})\cdot\mathbf{D}$$
$$-P_{\frac{1}{2}}\frac{e}{m}(\boldsymbol{\sigma}+2\mathbf{l})\cdot\left[\frac{A+1}{A+\frac{1}{2}}\rho_3\mathbf{H}+2(A+1)\rho_2\mathbf{E}\right]$$
$$+P_{\frac{1}{2}}\frac{e}{m}\frac{A+1}{A+\frac{1}{2}}(\boldsymbol{\sigma}+\mathbf{l})\cdot(\rho_3\mathbf{H}+\rho_2\mathbf{E})$$
$$+\frac{e}{3m^2}\left(\frac{A}{A+\frac{1}{2}}+P_{\frac{1}{2}}\frac{A+2}{A+\frac{1}{2}}\right)[\mathbf{D}\cdot(i\rho_1\mathbf{H}+\mathbf{E})$$
$$-\mathbf{l}\cdot\mathbf{D}\mathbf{l}\cdot(i\rho_1\mathbf{H}+\mathbf{E})+i\mathbf{l}\cdot\mathbf{D}\times(i\rho_1\mathbf{H}+\mathbf{E})]|1m_1'\tfrac{1}{2}m_{\frac{1}{2}}'\rangle$$
$$\times\langle 1m_1'\tfrac{1}{2}m_{\frac{1}{2}}'|=0, \quad (3.7)$$

where $P_{\frac{3}{2}}$ and $P_{\frac{1}{2}}$ are the projection operators for the angular momentum 3/2 and 1/2 states and the l_i are the angular momentum unity matrices. The part of Eq. (3.7) which transforms according to $D_{\frac{3}{2}}$ is

$$D_4\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|=\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|\mathcal{O}|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}'\rangle\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}'|$$
$$+\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|\mathcal{O}|1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}\rangle\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|, \quad (3.8)$$

where the operator $\mathcal{O}$ contains those terms of (3.7) which are not preceded by $P_{\frac{1}{2}}$. The part of (3.7) which transforms according to $D_{\frac{1}{2}}$ does not contain any time derivatives and is therefore not an equation of motion. It can be solved for $\langle 1\frac{1}{2}\frac{1}{2}m_{\frac{1}{2}}|$, and the result to order $(1/m)^2$ is

$$\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|=\langle 1\tfrac{1}{2}\tfrac{1}{2}m_{\frac{1}{2}}|\frac{\rho_2}{3m}(1-\boldsymbol{\sigma})\cdot\mathbf{D}$$
$$+\frac{2}{9m^2}(\boldsymbol{\sigma}+2\mathbf{l})\cdot\mathbf{D}(1-\boldsymbol{\sigma})\cdot\mathbf{D}+\frac{e}{6m^2}\frac{A+1}{A+\frac{1}{2}}(1-\boldsymbol{\sigma})$$
$$\cdot(\mathbf{H}-i\rho_1\mathbf{E})|1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}\rangle\langle 1\tfrac{1}{2}\tfrac{3}{2}m_{\frac{3}{2}}|. \quad (3.9)$$

Eliminating $\langle 1\frac{1}{2}\frac{1}{2}m_{\frac{1}{2}}|$ from (3.8) by means of (3.9), one obtains Eq. (3.5) for $s=3/2$ with the additional terms

$$\frac{ie}{2m^2}[\boldsymbol{\Pi}\cdot\mathbf{E}+i\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\boldsymbol{\Pi}\times\mathbf{E}-\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\boldsymbol{\Pi}\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{E}]$$
$$+\rho_1/m^2\text{ terms}+\text{higher order terms.} \quad (3.10)$$

The parameter A has notably again disappeared. The ρ_1/m^2 terms do not contribute terms of order $(1/m)^2$ to the diagonalized Hamiltonian. The Foldy-Wouthuysen diagonalization yields the Hamiltonian (3.6) for $s=3/2$ with the additional terms

$$\frac{e}{m^2}\left\{\frac{11}{72}\nabla\cdot\mathbf{E}-\frac{i}{8}\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\nabla\times\mathbf{E}\right.$$
$$-\frac{5}{8}\left[\frac{1}{2}(\Sigma_i^{\frac{3}{2}}\Sigma_j^{\frac{3}{2}}+\Sigma_j^{\frac{3}{2}}\Sigma_i^{\frac{3}{2}})-\frac{5}{9}\delta_{ij}\right]\nabla_iE_j$$
$$\left.+\frac{i}{2}\mathbf{E}\cdot\boldsymbol{\Pi}-\frac{i}{2}\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{E}\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\boldsymbol{\Pi}+\frac{1}{12}\boldsymbol{\Sigma}^{\frac{3}{2}}\cdot\mathbf{E}\times\boldsymbol{\Pi}\right\}. \quad (3.11)$$

The gradient operator ∇ operates here only on the electric field vector $\mathbf{E}$ which follows it and not on the wave function. The first term in (3.11) is analogous to the Darwin term of the spin 1/2 theory but it has here the opposite sign. The third term is the quadrupole moment interaction. From it the electric quadrupole moment (as defined by Ramsey[13]) of the spin 3/2 DFP particle is found to be

$$Q_{\frac{3}{2}}=+(5/3)(\hbar/mc)^2. \quad (3.12)$$

Spin 5/2

Equations (1.9) and (1.10) for $n=2$ differ from the $n=1$ equations only in that the wave function has an additional vector index which is symmetrical with the first. The time components of this spin-tensor can be eliminated exactly as in the spin 3/2 case, and one obtains an equation identical to (3.7) except that the wave function now transforms according to $D_1\times D_1\times D_{\frac{1}{2}}$ and must therefore be written

$$\langle 1m_1 1\bar{m}_1\tfrac{1}{2}m_{\frac{1}{2}}|. \quad (3.13)$$

The operator of (3.7) does not act on $\bar{m}_1$. The symmetry of the vector indices of ψ means that the function (3.13) does not contain a part which transforms according to $D_1\times D_{\frac{1}{2}}$ and hence one must require that

$$\langle 111\mathfrak{m}_1|1m_1 1\bar{m}_1\rangle\langle 1m_1 1\bar{m}_1\tfrac{1}{2}m_{\frac{1}{2}}|=0, \quad (3.14)$$

where $\langle 111\mathfrak{m}_1|1m_11\bar{m}_1\rangle$ is the Wigner coefficient which picks out the $D_1\times D_{\frac{1}{2}}$ part of (3.13). With this new wave function and the requirement (3.14), Eqs. (3.8) and (3.9) again hold. The latter can be used to eliminate all but the $D_{\frac{5}{2}}$ function. The result is Eq. (3.5) for $s=5/2$ with the additional terms

$$\frac{ie}{2m^2}\left(1-\frac{1}{12}\frac{A}{A+\frac{1}{2}}\right)$$
$$\times[\boldsymbol{\Pi}\cdot\mathbf{E}+i\boldsymbol{\Sigma}^{5/2}\cdot\boldsymbol{\Pi}\times\mathbf{E}-\boldsymbol{\Sigma}^{5/2}\cdot\boldsymbol{\Pi}\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{E}]$$
$$+\rho_1/m^2\text{ terms}+\text{higher order terms.} \quad (3.15)$$

The parameter A no longer drops out. Diagonalization gives the Hamiltonian (3.6) for $s=5/2$ with the additional terms

$$\frac{e}{m^2}\left\{\frac{1}{6}\left(\frac{5}{4}-\frac{2}{15}\frac{A}{A+\frac{1}{2}}\right)\nabla\cdot\mathbf{E}+\frac{i}{2}\left(\frac{3}{4}-\frac{1}{15}\frac{A}{A+\frac{1}{2}}\right)\boldsymbol{\Sigma}^{5/2}\cdot\nabla\times\mathbf{E}\right.$$
$$-\frac{1}{8}\left(5-\frac{1}{3}\frac{A}{A+\frac{1}{2}}\right)\left[\frac{1}{2}(\Sigma_i^{5/2}\Sigma_j^{5/2}+\Sigma_j^{5/2}\Sigma_i^{5/2})-\frac{7}{15}\delta_{ij}\right]\nabla_iE_j$$
$$+\frac{i}{2}\left(1-\frac{1}{12}\frac{A}{A+\frac{1}{2}}\right)\mathbf{E}\cdot\boldsymbol{\Pi}+\frac{1}{4}\left(1-\frac{1}{10}\frac{A}{A+\frac{1}{2}}\right)\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{E}\times\boldsymbol{\Pi}$$
$$\left.-\frac{i}{2}\left(1-\frac{1}{12}\frac{A}{A+\frac{1}{2}}\right)\boldsymbol{\Sigma}^{5/2}\cdot\mathbf{E}\boldsymbol{\Sigma}^{5/2}\cdot\boldsymbol{\Pi}\right\}. \quad (3.16)$$

[13] N. F. Ramsey, *Nuclear Moments* (John Wiley and Sons, Inc., New York, 1953).

From the third term the electric quadrupole moment of the spin 5/2 DFP particle is found to be

$$Q_{\frac{5}{2}}=+\left[\frac{28}{15}+\frac{1}{15}\frac{1}{A+\frac{1}{2}}\right]\left(\frac{\hbar}{mc}\right)^2. \tag{3.17}$$

As a function of A it is seen that $Q_{\frac{5}{2}}$ is singular at the forbidden value $A=-1/2$ but approaches the value $+(28/15)(\hbar/mc)^2$ quite rapidly as one moves away from that value.

CONCLUSIONS

It has been shown that the magnetic moment of a DFP particle of half-integral spin $s>1/2$ is uniquely determined to be $(1/s)(e\hbar/2mc)$. In general, the values of higher moments must be expected to depend on the parameter A. The spin 3/2 quadrupole moment has however the unique value $+(5/3)(\hbar/mc)^2$. The spin 5/2 quadrupole moment depends on the choice of A.

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N-V Potential in the Lee Model

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An investigation is made of states in the Lee model representing a "physical" V-particle bound to an N-particle. It is found that the energy eigenvalue problem can be reduced to a single transcendental equation involving the parity, energy, renormalized coupling, and N-V separation. Only states of odd parity are considered. For such states this equation leads to a single-valued real potential if the renormalized coupling constant is less than the critical value for the appearance of the so-called "ghost" states of the free physical V-particle. For stronger couplings the potential becomes many-valued with half its real branches corresponding to the presence of "normal" V's, and half to "ghost" V's. For still stronger coupling no real energies exist. It is shown that complex energies appear, at least in the case where the virtual θ particles are assumed nonrelativistic and the N-V separation is sufficiently small. A possible reason for the appearance of these difficulties is suggested.

I. INTRODUCTION

In a recent paper, Lee[1] has presented an extremely interesting example of a quantum field theory in which mass and coupling-constant renormalization can be carried out exactly, i.e., without the use of perturbation theory. In the Lee model there are three fictitious, chargeless, spinless particles: N and V are heavy fermions and θ is a light boson. The Hamiltonian[2] for the model is

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}'+\delta\mathcal{H},$$

$$\mathcal{H}_0=m\int d^3\mathbf{p}\psi_V^*(\mathbf{p})\psi_V(\mathbf{p})+m\int d^3\mathbf{p}\psi_N^*(\mathbf{p})\psi_N(\mathbf{p})+\int d^3\mathbf{k}\omega a^*(\mathbf{k})a(\mathbf{k}),$$

$$\delta\mathcal{H}=-\delta m\int d^3\mathbf{p}\psi_V^*(\mathbf{p})\psi_V(\mathbf{p}),$$

$$\mathcal{H}'=-g_0(2\pi)^{-\frac{3}{2}}\int d^3\mathbf{p}d^3\mathbf{p}'d^3\mathbf{k}\delta(\mathbf{p}'+\mathbf{k}-\mathbf{p})\times f(k)(2\omega)^{-\frac{1}{2}}[\psi_V^*(\mathbf{p})\psi_N(\mathbf{p}')a(\mathbf{k})+\psi_V(\mathbf{p})\psi_N^*(\mathbf{p}')a^*(\mathbf{k})], \quad (1)$$

where g_0 is the unrenormalized coupling-constant, $f(k)$ a high-momentum cut-off function, $\omega=\omega(k)$ the energy of a free θ of momentum $\mathbf{k}$. The field operators ψ_V,ψ_N,a obey the usual commutation (anticommutation) relations for boson (fermion) unrenormalized field operators in a Schrödinger representation. The mass of the N and of the "physical" V is m; $m-\delta m$ is the mass of the "bare" V; and μ is the mass of θ. There is no distinction between "bare" and "physical" N's, θ's, or vacuum. We use units with $\hbar=c=\mu=1$.

In performing the renormalization, one obtains the relations

$$|\gamma/\gamma_0|=1-(\gamma/\gamma_c), \quad (2)$$

$$\delta m=-\mathcal{B}\gamma/1-(\gamma/\gamma_c), \quad (3)$$

where

$$\gamma_0\equiv(g_0/2\pi)^2 \quad \gamma\equiv(g/2\pi)^2$$

(g being the renormalized coupling constant), and

$$\gamma_c^{-1}=\int_0^\infty\frac{k^2f^2(k)dk}{\omega^3}, \quad \mathcal{B}=\int_0^\infty\frac{k^2f^2(k)dk}{\omega^2}. \quad (4)$$

Now (2) leads to a contradiction if $\gamma>\gamma_c$. The introduction of an indefinite metric with metric operator

$$\eta=\begin{cases}\exp\left(i\pi\int d^3\mathbf{p}\psi_V^*(\mathbf{p})\psi_V(\mathbf{p})\right) & (\gamma\geq\gamma_c)\\ 1 & (\gamma<\gamma_c)\end{cases}$$

[1] T. D. Lee, Phys. Rev. **95**, 1329 (1954).
[2] G. Källén and W. Pauli, Kgl. Danske Videnskab Selskab Mat.-fys. Medd. **30**, No. 7 (1955). (The notation of this paper will be adhered to as closely as possible.)