A Note on Sommerfeld's Bremsstrahlung Formula

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Sommerfeld's bremsstrahlung formula is discussed by the method of angular momenta. The complete correspondence in the classical limit to the work of Landau and Lifshitz is shown.

IN his discussion of nonrelativistic bremsstrahlung[1] Sommerfeld has given, for the electric dipole case without retardation, a quite remarkable formula,[2] which provides in closed form an exact result for the total radiation loss. Subsequently, Landau and Lifshitz[3] gave an equivalent classical result for the radiation from particles travelling in Kepler orbits. Current consideration of the closely related problems of proton bremsstrahlung[4] and Coulomb excitation[5] has renewed interest in these results of Sommerfeld and of Landau and Lifshitz.

The methods by which Sommerfeld, and Landau and Lifshitz arrived at their results are, however, quite different: in the former case, parabolic coordinates are employed (i.e., the "summed field" method as it is frequently called), whereas in the latter case, an averaging over orbit eccentricities (impact parameters) is utilized. This latter method is a precise transcription, in the classical limit, of the quantum mechanical angular momentum method of partial waves. Now the circumstance that Coulomb excitation involves the irregular operator, to which the "summed field" technique has so far proved generally inapplicable, more or less forces one to use angular momentum techniques, and such techniques are useful also for the bremsstrahlung problem. In particular, if one does treat dipole bremsstrahlung in this way, the total energy loss so obtained is not at all clearly related to Sommerfeld's answer. It is the purpose of this note to give a straightforward proof of Sommerfeld's formula directly from the method of angular momenta. While this adds very little that is new, it is satisfying that the present method shows rather well the nature of the classical limit and its relation to the work of Landau and Lifshitz.[6]

For the electric dipole case, the matrix elements of the regular ($\mathbf{r}$) and irregular ($\mathbf{r}/r^3$) operators are essentially equivalent, since

$$[H,[H,\mathbf{r}]]=(Zze^2\hbar^2/M)(\mathbf{r}/r^3), \tag{1}$$

with H being the Hamiltonian containing the Coulomb interaction. For this case, then, the Coulomb excitation and bremsstrahlung results differ only trivially. Utilizing an expansion of the Coulomb "plane" wave, one readily finds that[7,8]

$$d\sigma=\frac{32}{3}z^2Z^2(e^2/\hbar c)^3k_1^{-3}k_2^{-1}\left(\frac{zA_1-ZA_2}{A_1+A_2}\right)^2\times\frac{d\omega}{\omega}b_0, \tag{2a}$$

$$b_0\equiv\sum_{l=1}^{\infty} l[(1,2;l-1)^2+(-1,2;l)^2], \tag{2b}$$

where we have used the definitions that

$$(m,n;l)\equiv\int_0^{\infty} dr r^{-n}F_l(\eta_1,k_1r)F_{l+m}(\eta_2,k_2r), \tag{3a}$$

$$k_1=(M/\hbar)v_{\text{initial}},\quad \eta_1=zZe^2/\hbar v_{\text{initial}}, \tag{3b}$$

with k_2, η_2 defined similarly for v_{final}.

The task now is to show directly that the sum for b_0 can be done exactly. That this must be possible is, of course, obvious from Sommerfeld's work. The $(m,n;l)$ are, in general, Appell functions, and, as such, difficult to manipulate. Fortunately, however, the $(\pm1,2;l)$ are reducible[9] to ordinary hypergeometric functions, namely,

$$(1,2;l-1)=k_2\left|1+\frac{i\eta_2}{l}\right|(0,1;l-1)-k_1\left|1+\frac{i\eta_1}{l}\right|(0,1;l), \tag{4a}$$

$$(-1,2;l)=k_1\left|1+\frac{i\eta_1}{l}\right|(0,1;l-1)-k_2\left|1+\frac{i\eta_2}{l}\right|(0,1;l), \tag{4b}$$

[1] A. J. F. Sommerfeld, *Atombau und Spektrallinien* (Ungar, New York, 1953), Vol. 2, Chap. 7.
[2] See reference 1, p. 527, Eq. (12).
[3] L. Landau and E. Lifshitz, *Classical Theory of Fields* (translated by M. Hamermesh) (Addison Wesley Press, Cambridge, 1951), p. 200 ff.
[4] C. Zupančič and T. Huus, Phys. Rev. **94**, 205 (1954); S. Drell and K. Huang, Phys. Rev. **99**, 686 (1955).
[5] Biedenharn, McHale, and Thaler, Phys. Rev. **100**, 376 (1955). References to the extensive literature may be found here. See also the forthcoming review article by Bohr *et al.*
[6] The corresponding problem of giving the classical analog to Sommerfeld's method is straightforward and is obtained directly from the classical limit. The result is even more immediate, however, if one uses the connection between the angle of deflection and the eccentricity (i.e., $\sin\frac{1}{2}\theta=\epsilon^{-1}$).
[7] See reference 1, p. 138; *Tables of Coulomb Wave Functions*, National Bureau of Standards, Applied Mathematics Series, No. 17 (U. S. Government Printing Office, Washington, D. C., 1952), Vol. 1.
[8] The result for the b_0 is derived for Coulomb excitation in reference 5. Bremsstrahlung actually leads to the integrals $(\pm1,-1;l)$ but by taking matrix elements of Eq. (1) for angular momentum wave functions the relation to the $(\pm1,2;l)$ is immediately obtained.
[9] See reference 5, Eq. (64). The reducibility of $(0,1;l)$ is due to Sommerfeld.

and

$$(0,1;l)=(-)^l(4\pi)^{-2}\left[\frac{4k_1k_2}{(k_1+k_2)^2}\right]^{l+1}e^{-\frac{1}{2}\pi|\eta_1-\eta_2|}$$
$$\times\left|\frac{k_1-k_2}{k_1+k_2}\right|^{-2l-2+i\eta_1+i\eta_2}$$
$$\times\frac{|\Gamma(l+1+i\eta_1)\Gamma(l+1+i\eta_2)|}{\Gamma(2l+2)}$$
$$\times{}_2F_1\left(l+1-i\eta_1,\, l+1-i\eta_2,\, 2l+2;\frac{-4k_1k_2}{(k_1-k_2)^2}\right). \quad (4c)$$

The sum can thus be put in the more tractable form (note that $k_1\eta_1=k_2\eta_2$)

$$b_0=\sum_{l+1}^{\infty} l\left\{\left[k_1^2+k_2^2+\frac{2k_1^2\eta_1^2}{l}\right][(0,1;l-1)^2+(0,1;l)^2]\right.$$
$$\left.-4k_1k_2\left|1+\frac{i\eta_1}{l}\right|\left|1+\frac{i\eta_2}{l}\right|(0,1;l-1)(0,1;l)\right\}. \quad (5)$$

Now the $(0,1;l)$ obey a rather simple three-term recursion formula[10]:

$$2k_1k_2(l+1)\left|1+\frac{i\eta_1}{l+1}\right|\left|1+\frac{i\eta_2}{l+1}\right|(0,1;l+1)$$
$$-(2l+1)\left[k_1^2+k_2^2+\frac{2k_1^2\eta_1^2}{l(l+1)}\right](0,1;l)$$
$$+2k_1k_2l\left|1+\frac{i\eta_1}{l}\right|\left|1+\frac{i\eta_2}{l}\right|(0,1;l-1)=0. \quad (6)$$

Consider now the function $Q(l)$:

$$Q(l)\equiv 2lk_1k_2\left|1+\frac{i\eta_1}{l}\right|\left|1+\frac{i\eta_2}{l}\right|(0,1;l)(0,1;l-1)$$
$$-l\left[k_1^2+k_2^2+\frac{2k_1^2\eta_1^2}{l^2}\right](0,1;l-1)^2. \quad (7)$$

If we use the finite difference operator, $\Delta Q(l)\equiv Q(l+1)-Q(l)$, then by virtue of Eq. (6) it is easily shown that

$$\Delta Q(l)=l\left\{\left[k_1^2+k_2^2+\frac{2k_1^2\eta_1^2}{l^2}\right][(0,1;l-1)^2+(0,1;l)^2]\right.$$
$$\left.-4k_1k_2\left|1+\frac{i\eta_1}{l}\right|\left|1+\frac{i\eta_2}{l}\right|(0,1;l-1)(0,1;l)\right\}. \quad (8)$$

[10] See reference 5, Eq. (63).

Since the right-hand side of Eq. (8) is just the summand of Eq. (5), it is clear that b_0 is exactly summable between *arbitrary* limits (not merely for 1 to ∞ as might conceivably be the case). We conclude that

$$b_0=[(k_1^2+k_2^2+2k_1^2\eta_1^2)(0,1;0)$$
$$-2k_1k_2|1+i\eta_1|\,|1+i\eta_2|(0,1;0)](0,1;1). \quad (9)$$

By using the properties of hypergeometric functions, this can be put precisely in the form of Sommerfeld's result, but for many purposes the present form is to be preferred.

It remains only to show the complete analogy of these results to the classical calculations of Landau and Lifshitz. In particular, one finds that

$$(0,1;l)\xrightarrow[\text{classical limit}]{}\tfrac{1}{4}\int_{-\infty}^{\infty}dt\,e^{-i\xi(\epsilon\sinh t+t)}. \quad (10)$$

The classical limit (i.e., $\hbar$ to $\rightarrow 0$) implies that $l\rightarrow\infty$, $\eta\rightarrow\infty$, $k_1/k_2\equiv\rho\rightarrow 1$, and $\eta_1-\eta_3\rightarrow\xi=$finite. In the notation of Landau-Lifschitz, one has $\xi=\omega/\omega_0$ and ϵ (the eccentricity of the Kepler orbit) is given by $\epsilon=(1+l^2/\eta^2)^{\frac{1}{2}}$. The integrals on the right-hand side of (10) are Bessel functions, and one may write, in accord with Landau and Lifshitz,

$$(0,1;l)\rightarrow\tfrac{1}{4}i\pi e^{-\pi|\xi|}H_{i\xi}(i\epsilon\xi). \quad (11)$$

Similarly one finds from Eq. (4) that

$$(\pm1,2;l)\rightarrow i\pi\frac{|\xi|}{4}\left(\frac{k}{\eta}\right)$$
$$\times e^{-\pi|\xi|}\left[\pm\frac{[\epsilon^2-1]^{\frac{1}{2}}}{\epsilon}H_{i\xi}(i\xi\epsilon)-i\xi H_{i\xi}'(i\xi\epsilon)\right]. \quad (12)$$

The sum for b_0 goes over into an integral in the classical limit, $l\Delta l\rightarrow\eta^2\epsilon d\epsilon$:

$$b_0\rightarrow\frac{-\pi^2k^2\xi^2}{8}e^{-2\pi|\xi|}$$
$$\times\int_1^{\infty}\epsilon d\epsilon\left\{\frac{\epsilon^2-1}{\epsilon^2}[H_{i\xi}(i\xi\epsilon)]^2-[H_{i\xi}'(i\xi\epsilon)]^2\right\}, \quad (13)$$

and the integrand, as a consequence of the Bessel equation, is just the derivative of $q\equiv xH_{i\xi}(x)H_{i\xi}'(x)$, with $x=i\xi\epsilon$. Here q is clearly the analog to the classical limit of Eq. (7). The correspondence between the quantum and classical calculations is complete upon noting that Eq. (6) in the classical limit becomes the Bessel equation.

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