energy are emitted within the same solid angle and are counted with equal efficiency. However, the method is applicable only to a source which has a simple decay scheme in which the beta spectrum has an allowed shape.

The values which we obtained are $\alpha_K=0.025\pm0.005$ and $\alpha_L=0.012\pm0.002$ in agreement with the results of other investigators.[1] An interpolation of the results of Rose, Goertzel, and Swift,[9] which include screening, gives $\alpha_K=0.031$ and $\alpha_L=0.012$.

CONCLUSION

Our results for the absolute electron energies, when combined with the best available absorption edge energies, are in agreement with the absolute gamma ray measurement of Muller, Hoyt, Klein, and DuMond. If one assumes their value for the gamma-ray energy, then our results serve as a check on the absorption edge energies and are in agreement to within our quoted uncertainty.

The agreement between the experimental and theoretical L subshell intensity ratios shows that the theoretical results may be used with confidence at least in the vicinity of $Z=80$, $E=400$ kev, and for an $E2$ transition.

ACKNOWLEDGMENTS

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[9] Rose, Goertzel, and Swift (privately distributed tables).

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Calculation of Electron-Deuteron Scattering Cross Sections*†

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Elastic and inelastic cross sections for electron-deuteron scattering with large momentum transfer have been investigated. The calculation has been performed in the first Born approximation. The neutron-proton interaction has been described by a phenomenological potential, and the nucleons have been represented by point charges and point magnetic moments. Finite size of nucleons causes major correction to these results.

I. INTRODUCTION

THE scattering of electrons with large momentum transfer has yielded some new and quite definite information about the charge distribution in a number of heavy nuclei. Experimental accuracy is improving to such an extent that this method holds promise of yielding new information even when applied to a relatively simple and well-understood nucleus, such as the deuteron. Some measurements[1] have already been made, in which the deuteron has been bombarded by high-energy electrons and the energies of the electrons scattered at large angles have been measured. A narrow elastic peak has been obtained, and a wide inelastic peak, corresponding to the breakup of the deuteron, has also been observed.

A few calculations[2] intended mainly for small momentum transfers have already been made. These calculations of elastic cross sections still give quite accurate results even at large momentum transfers. On the other hand, in the calculations of inelastic cross sections only the lowest electric and magnetic multipole moments have been considered. These results cannot be applied to the scattering with large momentum transfer when the contribution of higher multipole moments is quite important. Thus, it is necessary to perform a calculation that accounts for all multipoles.

Since the interaction between electron and nucleus is of electromagnetic nature, the matrix elements involved are similar to those used in calculating the photodisintegration of the deuteron. Calculations[3] and experimental data are plentiful in this case. The results cannot, however, be easily applied to our case, since in the photoprocess we have only real (transverse) photons, while in the electrodisintegration the main contribution comes from the longitudinal part of the

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[1] J. A. McIntyre and R. Hofstadter, Phys. Rev. **98**, 158 (1955); J. A. McIntyre (private communication).

[2] H. A. Bethe and R. Peierls, Proc. Roy. Soc. (London) **A148**, 146 (1935); B. Peters and C. Richman, Phys. Rev. **59**, 804 (1941); M. E. Rose, Phys. Rev. **73**, 282 (1948); J. H. Smith, Ph. D. dissertation, Cornell University, 1951 (unpublished); Phys. Rev. **95**, 271 (1954); Thie, Mullin, and Guth, Phys. Rev. **87**, 962 (1952); L. I. Schiff, Phys. Rev. **92**, 988 (1953).

[3] Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. **95**, 1635 (1954); J. M. Berger, Phys. Rev. **94**, 1698 (1954); L. Hulthén and B. C. H. Nagel, Phys. Rev. **90**, 62 (1953); H. Feshbach and J. Schwinger Phys. Rev. **84**, 194 (1951); L. I. Schiff, Phys. Rev. **78**, 733 (1950); J. F. Marshall and E. Guth, Phys. Rev. **78**, 738 (1950).

electromagnetic field. Moreover, the lowest order process in photodisintegration is the absorption of the photon, and a large momentum transfer is necessarily accompanied by a large energy transfer. In electrodisintegration, such a large momentum transfer usually causes a much smaller transfer of energy.

In our calculation, therefore, we have treated the deuteron nonrelativistically, while the electron may be considered extremely relativistic. Also, since the charge of the deuteron is small, we have treated the problem in the first Born approximation. Thus the wave functions for incident and scattered electrons have been represented by plane waves and the deflection of the electrons has generated Møller potentials[4] ϕ and $\mathbf{A}$ to act upon the charge and current distributions in the deuteron:

$$\phi(r) = -4\pi e a_0 (q^2-\Delta E^2)^{-1}\exp(i\mathbf{q}\cdot\mathbf{r}),$$
$$\mathbf{A}(r) = 4\pi e \mathbf{a}(q^2-\Delta E^2)^{-1}\exp(i\mathbf{q}\cdot\mathbf{r}),$$

where $\mathbf{q}$ and ΔE are the momentum and energy losses of the electron; $\mathbf{a}$ and a_0 are the matrix elements of the Dirac $\boldsymbol{\alpha}$ and unit operators between initial and final electron states. Also, we have chosen the system of units where $\hbar = c = M = 1$ (M is the mass of the nucleon). Assuming, in addition, that exchange currents can be neglected and the nucleons can be represented by point charges and point magnetic moments, the perturbation upon the deuteron becomes equal to

$$\sum_{k=1,2}\{q_k\phi(\mathbf{r}_k)+\tfrac{1}{2}iq_k[\mathbf{A}(\mathbf{r}_k)\cdot\nabla_k+\nabla_k\cdot\mathbf{A}(\mathbf{r}_k)] -\tfrac{1}{2}ie\mu_k\boldsymbol{\sigma}_k\cdot[\nabla_k\times\mathbf{A}(\mathbf{r}_k)]\}, \quad (1)$$

where $\mathbf{r}_k$ is the position of the kth nucleon, q_k is its charge and μ_k its magnetic moment in multiples of a nuclear magneton. Thus, with the preceding assumptions the calculation of cross sections is reduced to the calculation of the matrix element of (1) between initial and final states of the deuteron.

II. ELASTIC CROSS SECTION

Our calculation has been performed in the laboratory system, so that the deuteron is at rest initially. Taking (1) as a perturbation and eliminating the coordinate of the mass center of the deuteron, we obtain the elastic cross section:

$$d\sigma_e = \tfrac{1}{4}e^4[p_0^2\sin^4(\tfrac{1}{2}\theta)]^{-1}[1+p_0\sin^2(\tfrac{1}{2}\theta)]^{-1}d\Omega_p \times|\langle f|[a_0-\tfrac{1}{2}\nabla\cdot\mathbf{a}+\tfrac{1}{2}i\mu_p\boldsymbol{\sigma}_p\cdot(\mathbf{q}\times\mathbf{a})]e^{\frac{1}{2}i\mathbf{q}\cdot\mathbf{r}} +\tfrac{1}{2}i\mu_n\boldsymbol{\sigma}_n\cdot(\mathbf{q}\times\mathbf{a})e^{-\frac{1}{2}i\mathbf{q}\cdot\mathbf{r}}|0\rangle|^2, \quad (2)$$

where $\frac{1}{2}\mathbf{r}$ is the position of the proton with respect to the mass center; $\mathbf{p}_0$ is the momentum of the incident electron; $\mathbf{p}$ is the momentum of the scattered electrons; $d\Omega_p$ is the element of solid angle into which the electron has been scattered; θ is the scattering angle, $\cos\theta = \mathbf{p}_0\cdot\mathbf{p}/p_0 p$. For the elastic scattering, the initial state $|0\rangle$ and final state $|f\rangle$ of the deuteron are ground states. The wave functions for the ground state in the presence of tensor forces can be written in the form

$$\phi_m = (4\pi)^{-\frac{1}{2}}r^{-1}[u(r)+8^{-\frac{1}{2}}S_{np}w(r)]\chi_m,\ m=0,\pm1,$$

where χ_m is a triplet spin function and S_{np} is the conventional tensor operator:

$$S_{np} = 3r^{-2}(\boldsymbol{\sigma}_n\cdot\mathbf{r})(\boldsymbol{\sigma}_p\cdot\mathbf{r})-(\boldsymbol{\sigma}_n\cdot\boldsymbol{\sigma}_p).$$

Substituting this wave function into the matrix element of (2), averaging over initial directions of spin of the deuteron and the electron of positive energy, and summing over final directions (since the spins are not determined experimentally), we obtain the following for the elastic cross section:

$$d\sigma_e = \tfrac{1}{4}e^4\cos^2(\tfrac{1}{2}\theta)[p_0^2\sin^4(\tfrac{1}{2}\theta)]^{-1} \times[1+p_0\sin^2(\tfrac{1}{2}\theta)]^{-1}d\Omega_p\cdot F^2, \quad (3)$$

where

$$F^2 = \left[\int(u^2+w^2)j_0(\tfrac{1}{2}qr)dr\right]^2 + \left[\int 2w(u-8^{-\frac{1}{2}}w)j_2(\tfrac{1}{2}qr)dr\right]^2 + \tfrac{2}{3}(\tfrac{1}{2}q)^2[(2/\cos^2(\tfrac{1}{2}\theta))-1] \times\left[\int\{[(\mu_p+\mu_n)(u^2+w^2)-\tfrac{3}{2}(\mu_p+\mu_n-\tfrac{1}{2})w^2]j_0(\tfrac{1}{2}qr) +2^{-\frac{1}{2}}w[(\mu_p+\mu_n)(u+2^{-\frac{1}{2}}w)+3\times8^{-\frac{1}{2}}w]j_2(\tfrac{1}{2}qr)\}dr\right]^2. \quad (4)$$

The first term here comes from the spherically symmetric part of the charge distribution in the deuteron, the second is a "quadrupole term," and the last is a "magnetic moment term." The first two terms in this expression have been given previously by Schiff.[2]

We can estimate the quadrupole term rather easily since we know the values of the functions u and w outside the range of nuclear forces. u is determined quite accurately by the experimental value of triplet effective range, and w is known roughly from the quadrupole moment of the deuteron.[5] In this way it can be shown that the quadrupole term is approximately equal to $(8/9)(\frac{1}{2}q)^4Q^2$ (where $Q=0.274\times10^{-26}$ cm^2 is the quadrupole moment of the deuteron) for small recoil momenta; it rapidly reaches a maximum value ≈ 0.002 and remains of this order of magnitude for moderate recoil momenta ($q<3\times10^{13}$ cm^{-1}). Thus,

[4] C. Møller, Z. Physik **70**, 786 (1931); W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, Oxford, 1954), third edition, p. 231.

[5] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 106.

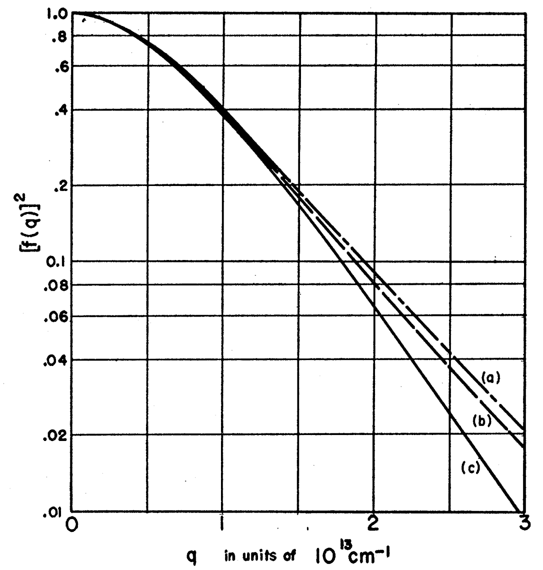

FIG. 1. The square of the Fourier transform of the charge distribution in the deuteron f^2 if neutron-proton interaction is described by (a) a Hulthén potential corresponding to triplet effective range $r_t = 1.7\times10^{-13}$ cm; (b) a Hulthén potential corresponding to $r_t = 1.8\times10^{-13}$ cm; (c) a Hulthén potential outside the hard core of radius 0.65×10^{-13} cm, $r_t = 1.7\times10^{-13}$ cm.

neglecting this and other small terms, we have

$$F^2 \approx \{1+\tfrac{2}{3}(\tfrac{1}{2}q)^2(\mu_p+\mu_n)^2[(2/\cos^2(\tfrac{1}{2}\theta))-1]\}f^2, \quad (5)$$

where

$$f=\int (u^2+w^2)j_0(\tfrac{1}{2}qr)dr$$

is the Fourier transform of the charge distribution in the deuteron. Since w^2 accounts only for about 4% of the charge distribution and its Fourier transform is not likely to become negative, we can disregard the ^3D part in the charge distribution without incurring an error larger than 4%. The tail of u^2 is determined by the effective range and binding energy and we are uncertain about the behavior of the charge distribution u^2 only within the range of nuclear forces. Since $j_0(\frac{1}{2}qr)$ is a decreasing function within the range of nuclear forces, we decrease the form factor f^2 by pushing the charge distribution outward. Thus the experimental form factor can be expected to fall between the extremes that correspond to a smooth charge distribution such as is caused by the Hulthén potential, and to a pushed-out charge distribution caused by a potential with a large hard core of radius $\approx 0.7\times10^{-13}$ cm. Such form factors have been illustrated in Fig. 1.

III. DEUTERON BREAKUP

Since the deuteron is a weakly bound structure, the impact of a high-energy electron is likely to break it up. The energy distribution of electrons scattered at large angles shows not only an elastic peak, but also a considerable inelastic peak. In analogy with (2), the expression for the inelastic cross section is

$$d\sigma_{\text{in}}=(4\pi)^{-3}e^4[p_0^2\sin^4(\tfrac{1}{2}\theta)]^{-1}kdpd\Omega_pd\Omega_k \times|\langle f|[a_0-i\nabla\cdot\mathbf{a}+\tfrac{1}{2}i\mu_p\boldsymbol{\sigma}_p\cdot(\mathbf{q}\times\mathbf{a})]e^{\frac{1}{2}i\mathbf{q}\cdot\mathbf{r}} +\tfrac{1}{2}i\mu_n\boldsymbol{\sigma}_n\cdot(\mathbf{q}\times\mathbf{a})e^{-\frac{1}{2}i\mathbf{q}\cdot\mathbf{r}}|0\rangle|^2, \quad (6)$$

where, in addition to previous notation, $\mathbf{k}$ represents the final momentum of the proton with respect to the mass center of the recoiling deuteron, so that the wave function for the final state is $e^{i\mathbf{k}\cdot\mathbf{r}}+$ the incoming wave.

Using this expression later on, we shall compute the energy spectrum of the electrons scattered at a given angle. We shall find that the inelastic peak is fairly narrow. Therefore, over the energy range from which the bulk of the contribution to the cross section comes, the recoil momentum q does not vary much. Considering $\mathbf{q}$ constant, the interaction in (6) does not depend explicitly upon the disintegration energy k^2, and we can quickly estimate the total cross section to be expected, using the closure property for the final wave functions of the deuteron. The total (elastic plus all inelastic) cross section is found to be nearly independent of the wave function of the deuteron in the ground state. Subtracting the value of the elastic cross section [(3) and (5)] from the value of the total cross section computed in this way, we obtain the total inelastic cross section, which is approximately

$$d\sigma_{\text{in}}\approx\tfrac{1}{4}e^4\cos^2(\tfrac{1}{2}\theta)[p_0^2\sin^4(\tfrac{1}{2}\theta)]^{-1}[1+p_0\sin^2(\tfrac{1}{2}\theta)]^{-1}d\Omega_p \times\{(1-f^2)+(\tfrac{1}{2}q)^2[(2/\cos^2(\tfrac{1}{2}\theta))-1](\mu_p^2+\mu_n^2)\} \quad (7)$$

for large values of the recoil momentum. For small values of recoil momentum ($q<\alpha\approx\frac{1}{4}\times10^{13}$ cm^{-1}, where α^{-1} is the "size" of the deuteron), the terms retained in (7) become about as small as the terms neglected. Comparison of our expressions for the elastic and inelastic cross sections shows that as the recoil momentum increases, the inelastic cross section increases at the expense of the elastic, and the magnetic spin terms assume more importance in the inelastic cross section than in the elastic. Using the approximation that leads to Eq. (7), we can also estimate the portion of the inelastic cross section that is caused by transitions from the ground state (considered spherically symmetric) to all final states of given angular momentum and spin. The result is independent of the particular complete set of radial wave functions chosen and depends only upon the wave function of the ground state and the magnitude of q. The calculation shows that states of high angular momentum contribute appreciably to the total cross section.

Thus, in calculating the energy spectrum for the inelastically scattered electrons, we start by taking a plane wave for the final nucleon wave function (neglecting incoming waves), and later correct for the neutron-proton interaction in the sates of successively increasing angular momentum. To simplify the calculation, we assume that the ground state has the Hulthén form:

$$\phi_m=(4\pi)^{-\frac{1}{2}}r^{-1}(2\alpha)^{\frac{1}{2}}(1-\alpha r_t)^{-\frac{1}{2}}u_g\chi_m, \tag{8}$$

where $u_g=e^{-\alpha r}-e^{-\gamma r}$, and γ is determined by the triplet effective range r_t. Taking plane waves for the final state makes the calculation of the inelastic cross section (6) straightforward. We perform the integrals involved, do the spin sums, and then integrate over the directions of disintegration ($d\Omega_k$) to obtain

$$d\sigma_{\text{in}}=(4\pi)^{-1}e^4\cos^2(\tfrac{1}{2}\theta)[p_0^2\sin^4(\tfrac{1}{2}\theta)]^{-1}2\alpha(1-\alpha r_t)^{-1}\times q^{-2}k^{-1}dp\,d\Omega_p O^2, \tag{9}$$

where

$$\begin{aligned}O^2=&\left\{\frac{1}{z^2-1}+\frac{1}{z_1^2-1}-\frac{2}{z_1-z}[Q_0(z)-Q_0(z_1)]\right\}\\ &\times\{1+[k^2+2(\tfrac{1}{2}q)^2(\mu_p^2+\mu_n^2)]\\ &\times\tfrac{1}{2}[(2/\cos^2(\tfrac{1}{2}\theta))-1-z^2k^2]\}\\ &-\left\{\frac{z}{z^2-1}+\frac{z_1}{z_1^2-1}-\frac{z_1+z}{z_1-z}[Q_0(z)-Q_0(z_1)]\right\}2zk^2\\ &-\left\{2+\frac{1}{z^2-1}+\frac{1}{z_1^2-1}-\frac{2zz_1}{z_1-z}[Q_0(z)-Q_0(z_1)]\right\}\\ &\times k^2\{\tfrac{1}{2}[(2/\cos^2(\tfrac{1}{2}\theta))-1-z^2k^2]-z^2k^2\}\\ &-\frac{(z_1-z)^2}{z_1z(z_1+z)}[Q_0(z)+Q_0(z_1)]\\ &\times(-\tfrac{1}{3}\mu_p\mu_n)(\tfrac{1}{2}q)^2[(2/\cos^2(\tfrac{1}{2}\theta))-1-z^2k^2],\end{aligned} \tag{10}$$

where $Q_0(z)=\coth^{-1}z$ is a Legendre function of the second kind, and z and z_1 are the abbreviations for

$$\begin{aligned}z&=[\alpha^2+(\tfrac{1}{2}q)^2+k^2]/qk,\\ z_1&=[\gamma^2+(\tfrac{1}{2}q)^2+k^2]/qk.\end{aligned} \tag{11}$$

Examination of this expression shows that the terms coming entirely from the convection current $(\mathbf{a}\cdot\nabla)$ are fairly small for moderate values of the disintegration energy k^2. The cross term between the convection current and the electric charge (a_0) becomes small because of the averaging over the directions of the nucleons, since $\mathbf{a}\cdot\mathbf{q}=-\Delta E a_0$.

However, in calculating O^2 in (10) with unperturbed final nucleon wave functions we have committed a large error. Since the neutron-proton interaction is felt very strongly in the final S states, we have to replace the erroneously calculated S contributions to O^2 in (10) by the contributions obtained using the correct final S wave functions. Neglecting the convection current $(\mathbf{a}\cdot\nabla)$, the ^3S contribution towards O^2 is

$$\{1+\tfrac{2}{3}(\tfrac{1}{2}q)^2(\mu_p+\mu_n)^2[(2/\cos^2(\tfrac{1}{2}\theta))-1-z^2k^2]\}A_{gt}^2,$$

and the ^1S part is

$$\tfrac{1}{3}(\tfrac{1}{2}q)^2(\mu_p-\mu_n)^2[(2/\cos^2(\tfrac{1}{2}\theta))-1-z^2k^2]A_{gs}^2,$$

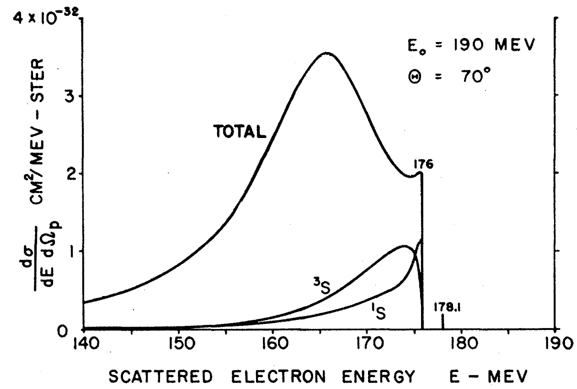

FIG. 2. Energy spectrum of 190-Mev electrons scattered inelastically at 70°. The total inelastic cross section is plotted, and the contributions to this cross section from the transitions to final S states are also indicated. The size of the elastic cross section ($=29.7\times10^{-32}$ cm²/steradian for the Hulthén potential) at 178.1 Mev is not indicated.

where

$$A_{gt(s)}=q\int u_{t(s)}{}^*j_0(\tfrac{1}{2}qr)u_g dr.$$

In previous calculation of these parts we have effectively used the unperturbed S wave function $\sin(kr)$ instead of the correct $u_{t(s)}$. The "correct" wave functions have been obtained assuming that the neutron-proton potential is an Eckart potential of the type[6]

$$V(r)=-\frac{2\lambda^2}{\cosh^2(\lambda r-\theta')}. \tag{12}$$

Then the wave functions in the S states are

$$\begin{aligned}u(r)=&(k^2+\lambda^2\tanh^2\theta')^{-\frac{1}{2}}(k^2+\lambda^2)^{-\frac{1}{2}}\\ &\times\{[(k^2-\lambda^2\tanh\theta')\sin(kr)+(1+\tanh\theta')k\lambda\cos(kr)]\\ &-\lambda[1-\tanh(\lambda r-\theta')][-\lambda\tanh\theta'\sin(kr)+k\cos(kr)]\},\end{aligned}$$

and the phase shifts satisfy the "shape-independent approximation" formula[7] exactly for all energy values. The wave function for the bound state is $(\theta_t'>0)$

$$\begin{aligned}u_g=\exp(-\lambda\tanh\theta_t' r)\{1-(1+\tanh\theta_t')^{-1}\\ \times[1-\tanh(\lambda_t r-\theta_t')]\}.\end{aligned}$$

The parameters λ, θ' have been adjusted to yield the experimental scattering lengths and effective ranges for triplet and singlet scattering, and the integrals A_{gt} and A_{gs} have been performed with the help of an additional approximation,

$$1-\tanh(\lambda r-\theta')\approx(1+\tanh\theta')[1+(1+\tanh\theta')\lambda r]e^{-2\lambda r}.$$

The corresponding contributions (^3S and ^1S) towards the inelastic cross section are illustrated in Figs. 2 and 3 for two combinations of incident electron energy E_0

[6] V. Bargmann, Revs. Modern Phys. **21**, 488 (1949); C. Eckart, Phys. Rev. **34**, 1303 (1930).

[7] J. M. Blatt and V. F. Weisskopf, reference 5, p. 62.

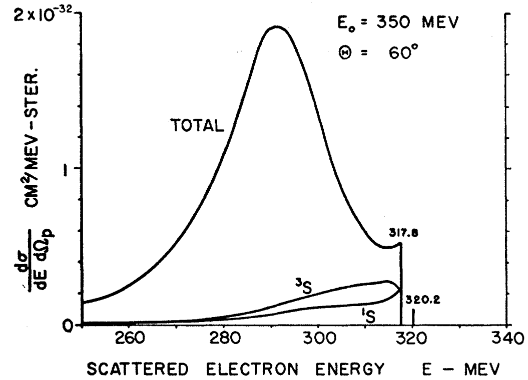

FIG. 3. Energy spectrum of 350-Mev electrons scattered inelastically at 60°. The total inelastic cross section is plotted, and the contributions to this cross section from the transitions to the final S states are also indicated. The size of the elastic cross section ($=6.77\times10^{-32}$ cm^2/steradian for the Hulthén potential) at 320.2 Mev is not indicated.

and scattering angle θ. The total cross section indicated in each of these figures is the cross section calculated by (9) and (10) after the correction for the interaction in S states has been made.

The Eckart potential (12) is, of course, only a special case of a potential consistent with the information derived from neutron-proton scattering at low energies. For this reason the electric monopole part (^3S) has been checked by recalculating it for a potential with a hard core. Also, the effect of tensor forces has been investigated by computing the matrix element for the transition to final 1α states[8] and comparing it with the matrix element for the transitions to the ^3S states calculated previously. In both cases the change in the electric monopole cross section is small for small disintegration energy, since the wave functions then are adequately described by the shape-independent approximation. The change increases percentagewise for larger disintegration energies, but is still unimportant in computing the total cross section since the S part itself then becomes negligible.

The effect of neutron-proton interaction in the final states of higher angular momentum ($l>0$) is considerably smaller than in the S states, since the interaction is shielded by the centrifugal potential $l(l+1)r^{-2}$. Thus we have estimated it using the Born approximation to obtain the final wave functions and representing the neutron-proton interaction by a Yukawa potential,

$$V(r)=-cr^{-1}\exp(-\beta r), \qquad (13)$$

of such strength and range[9] that it would yield the experimental scattering length and effective range if acting in the ^3S states. The calculation has been performed in momentum space so that to the wave function for the ground state,

$$\phi_g(\mathbf{r})=r^{-1}(e^{-\alpha r}-e^{-\gamma r}),$$

corresponds

$$\psi_g(\boldsymbol{\kappa})=(2\pi)^{-\frac{3}{2}}4\pi\left(\frac{1}{\alpha^2+\kappa^2}-\frac{1}{\gamma^2+\kappa^2}\right)$$

in momentum space. The wave function for the final state in the Born approximation is

$$\psi(\boldsymbol{\kappa})=(2\pi)^{\frac{3}{2}}\left[\delta(\boldsymbol{\kappa}-\mathbf{k})+\frac{c}{2\pi^2}\frac{1}{\kappa^2-k^2+i\epsilon}\frac{1}{\beta^2+|\boldsymbol{\kappa}-\mathbf{k}|^2}\right].$$

Then the matrix element

$$M=qk\int\phi^*(\mathbf{r})e^{\frac{1}{2}i\mathbf{q}\cdot\mathbf{r}}\phi_g(\mathbf{r})d\mathbf{r}/4\pi$$

$$=qk\int\psi^*(\boldsymbol{\kappa})\psi_g(\boldsymbol{\kappa}-\tfrac{1}{2}\mathbf{q})d\boldsymbol{\kappa}/4\pi,$$

can be expanded in spherical harmonics, corresponding to transitions into the states of appropriate angular momentum. Using the relation[10]

$$1/(z-\cos\vartheta)=\textstyle\sum_l(2l+1)P_l(\cos\vartheta)Q_l(z),$$

we see that

$$M=\sum_l(2l+1)P_l(\cos\vartheta)\times\left\{[Q_l(z)-Q_l(z_1)]\right.$$

$$\left.+\frac{c}{2\pi}\int_{-\infty}^{+\infty}\frac{d\kappa}{\kappa^2-k^2-i\epsilon}Q_l(z_2')[Q_l(z')-Q_l(z_1')]\right\}, \qquad (14)$$

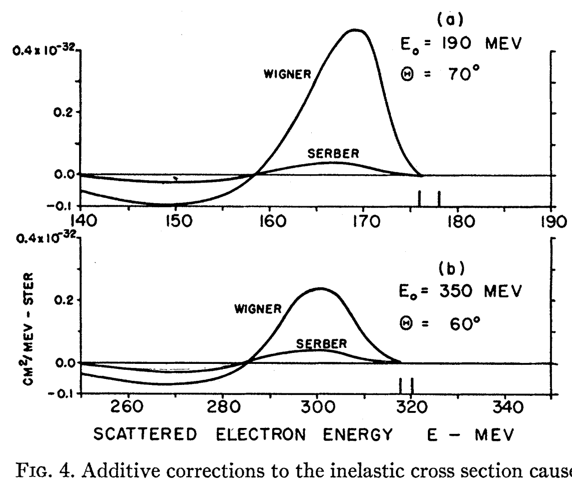

FIG. 4. Additive corrections to the inelastic cross section caused by neutron-proton interaction in the final states of angular momenta $l\geq1$, when the same potential is effective in states of all angular momenta (Wigner), and when the potential acts only in states of even angular momenta (Serber). (a) Correction to the total cross section of Fig. 1; (b) Correction to the total cross section of Fig. 2.

[8] For notation see J. M. Blatt and L. C. Biedenharn, Phys. Rev. **86**, 399 (1954).
[9] J. M. Blatt and J. D. Jackson, Phys. Rev. **76**, 18 (1949).
[10] E. W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge University Press, Cambridge, 1931), p. 58.

where $\cos\vartheta = \mathbf{k}\cdot\mathbf{q}/kq$, z and z_1 are abbreviations (11), and

$$z' = [\alpha^2 + (\tfrac{1}{2}q)^2 + \kappa^2]/q\kappa,$$
$$z_1' = [\gamma^2 + (\tfrac{1}{2}q)^2 + \kappa^2]/q\kappa,$$
$$z_2' = [\beta^2 + k^2 + \kappa^2]/2k\kappa.$$

The integral in (14) has been transformed into a proper integral over a finite range by introducing a new variable of integration:

$$x\begin{cases} = \kappa/k & \text{for } |\kappa| < k, \\ = k/\kappa & \text{for } |\kappa| > k. \end{cases}$$

The quadratures were then performed numerically. The resulting corrections to the inelastic cross section decrease very rapidly with increasing angular momentum, so that the calculation has been performed only for $l=1, 2, 3$; the magnitude of the correction to be expected from neutron-proton interaction in these final states is indicated in Fig. 4. Two possibilities have been considered: when the same potential (13) acts in states of all angular momenta $l=1, 2, 3$ (Wigner force); when it acts only in the state of even angular momentum $l=2$ (Serber force). The results obtained here are only approximate, since the Born approximation has been used and only the Yukawa-type interaction has been considered.

IV. DISCUSSION

In the preceding calculation we have shown that the elastic cross section is somewhat sensitive to the presence or absence of the hard core in the interaction and that the inelastic cross section is somewhat sensitive to the presence or absence of the neutron-proton interaction in the final ^3P state. However, the calculation has been made using bare nucleons to represent the proton and the neutron. Actually, since the wavelength associated with our recoil momentum is of the order of the meson Compton wavelength, the finite size of the nucleons should be felt. Since the deuteron is a weakly bound structure, the proton and neutron spend most of the time outside the range of nuclear forces, and we can roughly describe the charge and current distributions around each nucleon as if they were free. In this approximation previous expressions for the matrix elements remain unchanged except that the nominal values for the charges and magnetic moments have to be replaced by effective charges and effective magnetic moments that depend upon the recoil momentum q. Then numerical values for the cross sections can easily be obtained if the form factors for the proton and neutron are known separately. Assuming, as an example, that the effective charge of the neutron vanishes and the charge of the proton has an rms radius 0.7×10^{-13} cm, as favored by present experimental evidence,[11] and that the form factors for the magnetic moments are the same as the form factor for the proton charge, we find that both elastic and inelastic cross sections are to be multiplied by the factor $[1-\frac{1}{6}(q\times0.7\times10^{-13}\text{ cm})^2+\cdots]^2$. This factor lowers the elastic cross section more than does the introduction of a hard core.

However, spreading out the charges and magnetic moments of the nucleons accounts only for one-particle terms in Foldy's phenomenological theory.[12] The charge and current distribution terms that depend on the coordinates of both particles are significant only within the range of the nuclear forces, and some idea about the contribution of these terms can be obtained by expanding the exponential $\exp(i\mathbf{q}\cdot\mathbf{r}')$ in power series of q and taking only the lowest nonvanishing terms. Of these, the exchange moment operators, which contribute to the magnetic moments of nuclei, have been investigated in more detail.[13] They are not important for the elastic cross section, since their contribution to the magnetic moment of the deuteron is very small.[14] The contribution of exchange moments to the inelastic cross section has been evaluated following Berger,[3] and has been found to be small for the moderate disintegration energy considered.

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