Representation of Charge Conjugation for Dirac Fields*

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(Received February 24, 1956)

Explicit representations of the charge conjugation transformation for fermion fields are derived in a form which does not make explicit reference to the expansion of the fields in a complete orthonormal set of space and spin functions. It is shown that the different forms must be equivalent if one is working in an irreducible representation of the Wigner-Jordan anticommutation rules, but it has not been found possible to prove the equivalence directly, thus suggesting that the representations are not equivalent if the representation of the anticommutation relations is not irreducible. The results are applied to the clarification of an apparent paradox concerning commutability of the charge conjugation and space inversion transformations.

THE transformation of charge conjugation as applied to quantized fields has been studied extensively of late[1] since it may represent one of the few exact symmetry principles in nature. The present note deals with the construction of explicit representations for this transformation in the case of Dirac fields. The final forms obtained are characterized by the fact that they make no explicit reference to an expansion of the fields in a complete orthonormal set of functions. We have not been able to prove directly the equivalence of different representations which we obtain, but only indirectly on the assumption that one is working with an irreducible representation of the Wigner-Jordan anticommutation relations. This suggests that the different representations may not be equivalent if the representation of the anticommutation relations is not irreducible. We apply our results to the resolution of an apparent paradox concerning the commutation of the space inversion and charge conjugation transformations. The explicit representations we obtain may also be of value in other applications.

CHARGE CONJUGATION TRANSFORMATION

We write the Dirac equation in the form

$$i\partial\psi/\partial t=(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})\psi, \tag{1}$$

where $\mathbf{p}=-i\nabla$, β and α_i are the usual Dirac matrices, and the units are chosen so that $\hbar=c=1$. With $\psi(\mathbf{r},t)$ considered an operator and its Hermitian conjugate designated $\psi^\dagger(\mathbf{r},t)$, we have for the equation satisfied by $\psi^\dagger$:

$$-i\partial\psi^\dagger/\partial t=(\beta^* m-\boldsymbol{\alpha}^*\cdot\mathbf{p})\psi^\dagger, \tag{2}$$

where the asterisk on the Dirac matrices designates their complex (not Hermitian) conjugates. To each solution of (1) we can associate another solution,

$$\psi_c(\mathbf{r},t)=\kappa\psi^\dagger(\mathbf{r},t), \tag{3}$$

and to each solution $\psi^\dagger$ of (2) another solution

$$\psi_c^\dagger(\mathbf{r},t)=\kappa^*\psi(\mathbf{r},t), \tag{4}$$

provided κ is a matrix such that

$$\begin{gathered}\kappa\beta^*\kappa^{-1}=\kappa\beta^T\kappa^{-1}=-\beta,\\ \kappa\alpha_i^*\kappa^{-1}=\kappa\alpha_i^T\kappa^{-1}=\alpha_i,\\ \kappa=\kappa^T,\\ \kappa\kappa^\dagger=\kappa\kappa^*=1,\end{gathered} \tag{5}$$

where the superscript T represents the transpose of the matrix. One can prove that a matrix κ with these properties exists for any irreducible representation of the Dirac matrices, and is unique to within a multiplicative factor of modulus unity for any irreducible representation. The transformation

$$\begin{gathered}\psi\to\psi_c=\kappa\psi^\dagger,\\ \psi^\dagger\to\psi_c^\dagger=\kappa^*\psi,\end{gathered} \tag{6}$$

is called charge conjugation. In the space of solutions of Eqs. (1) or (2) it is not a unitary nor even a linear transformation.

The operators ψ and $\psi^\dagger$ are assumed to satisfy the Wigner-Jordan anticommutation relations:

$$\begin{gathered}\psi_\lambda(\mathbf{r},t)\psi_\mu^\dagger(\mathbf{r}',t)+\psi_\mu^\dagger(\mathbf{r}',t)\psi_\lambda(\mathbf{r},t)=\delta_{\mu\lambda}\delta(\mathbf{r}-\mathbf{r}'),\\ \psi_\lambda(\mathbf{r},t)\psi_\mu(\mathbf{r}',t)+\psi_\mu(\mathbf{r}',t)\psi_\lambda(\mathbf{r},t)\\ =\psi_\lambda^\dagger(\mathbf{r},t)\psi_\mu^\dagger(\mathbf{r}',t)+\psi_\mu^\dagger(\mathbf{r}',t)\psi_\lambda^\dagger(\mathbf{r},t)=0,\end{gathered} \tag{7}$$

and we shall assume in particular that we are dealing with an irreducible representation of these relations. Now one can easily verify that ψ_c and $\psi_c^\dagger$ satisfy the same anticommutation relations as to ψ and $\psi^\dagger$, respectively. This suggests that there exists a unitary transformation U_c such that

$$\begin{gathered}U_c\psi U_c^{-1}=\psi_c=\kappa\psi^*,\\ U_c\psi^\dagger U_c^{-1}=\psi_c^\dagger=\kappa^*\psi.\end{gathered} \tag{8}$$

We shall now construct explicit representations of the transformation U_c.

* This work has been supported by the National Science Foundation and the U. S. Atomic Energy Commission.

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[1] A. Pais and R. Yost, Phys. Rev. 87, 871 (1952); L. Wolfenstein and D. G. Ravenhall, Phys. Rev. 88, 279 (1952); L. Michel, Nuovo cimento 10, 319 (1953); J. M. Jauch, Proceedings of Theoretical Seminar, Department of Physics, State University of Iowa, Winter 1953–54 (unpublished).

Let W be an Hermitian operator in the Hilbert space of solutions of Eq. (1) satisfying the two conditions:

$$W^2=1,\quad W\kappa=-\kappa W^*, \tag{9}$$

where W^* is the complex (not Hermitian) conjugate of the operator W. There exist many operators satisfying these conditions. Particular examples are β, $\Lambda=(\beta m+\boldsymbol{\alpha}\cdot\mathbf{p})/(m^2+p^2)^{\frac{1}{2}}$, and $i\alpha_1\alpha_2\alpha_3$. Let u_s be an orthonormal set of eigenfunctions of W belong to the eigenvalue $+1$, and let it be complete in the sense that any eigenfunction of W belonging to this eigenvalue may be expanded in the functions u_s. Define the functions v_s by

$$v_s=\kappa u_s^*. \tag{10}$$

Then the set of functions v_s form a complete orthonormal set of functions in which any eigenfunction of W belonging to the eigenvalue -1 can be expanded. To prove this, let ϕ be any eigenfunction of W belonging to the eigenvalue -1, whence we may easily verify that $\kappa\phi^*$ is an eigenfunction of W belonging to the eigenvalue $+1$ since

$$W(\kappa\phi^*)=-\kappa W^*\phi^*=-\kappa(W\phi)^*=\kappa\phi^*. \tag{11}$$

Hence $\kappa\phi^*$ can be expanded in the functions u_s:

$$\kappa\phi^*=\sum_s a_s u_s, \tag{12}$$

and from this we obtain:

$$\phi=(\kappa^{-1}\sum_s a_s u_s)^* =\kappa\sum_s a_s^* u_s^*=\sum_s a_s^* v_s. \tag{13}$$

The u_s and v_s together then form a complete orthonormal basis in the Hilbert space of solutions of (1) and in particular, we may expand any ψ as

$$\psi=\sum_s(a_s u_s+b_s^\dagger v_s), \tag{14}$$

where a_s, $a_s^\dagger$, b_s, $b_s^\dagger$ are the usual creation and destruction operators satisfying the familiar anticommutation relations:

$$a_s^\dagger a_{s'}+a_{s'}a_s^\dagger=\delta_{ss'},\quad a_s a_{s'}+a_{s'}a_s=a_s^\dagger a_{s'}^\dagger+a_{s'}^\dagger a_s^\dagger=0. \tag{15}$$

We then have

$$\psi_c=\kappa\psi^\dagger=\sum_s(a_s^\dagger v_s+b_s u_s) \tag{16}$$

and from (8), we then obtain

$$U_c a_s U_c^{-1}=b_s,\quad U_c b_s^\dagger U_c^{-1}=a_s^\dagger. \tag{17}$$

One can then use the procedure of Ravenhall and Wolfenstein[1] to obtain the general solution for U_c which apart from an arbitrary phase factor is given by

$$\begin{aligned}U_c&=\prod_s[i(1-a_s^\dagger a_s-b_s^\dagger b_s+a_s^\dagger b_s+b_s^\dagger a_s)]\\&=\prod_s\exp[\tfrac{1}{2}i\pi(1-a_s^\dagger a_s-b_s^\dagger b_s+a_s^\dagger b_s+b_s^\dagger a_s)]\\&=\exp[\tfrac{1}{2}i\pi\sum_s(a_s^\dagger a_s-b_s b_s^\dagger+a_s^\dagger b_s+b_s^\dagger a_s)].\end{aligned} \tag{18}$$

Now, one can easily derive that

$$A=\tfrac{1}{2}\int[\psi^\dagger W\psi-\psi W^*\psi^\dagger]d\mathbf{r}=\sum_s(a_s^\dagger a_s-b_s b_s^\dagger), \tag{19}$$

and

$$B=\tfrac{1}{2}\int[\psi\kappa^* W\psi-\psi^\dagger\kappa W^*\psi^\dagger]d\mathbf{r}=\sum_s(a_s^\dagger b_s+b_s^\dagger a_s), \tag{20}$$

and that A and B commute. Hence

$$U_c=\exp\tfrac{1}{2}i\pi(A+B)=\exp[i\pi B/2]\exp[i\pi A/2]. \tag{21}$$

It is clear that we obtain quite different expressions for A and B and hence for U_c depending on the choice of the operator W. We have not been able to prove directly that these different choices lead to equivalent transformations. However, if we have an irreducible representation of the Wigner-Jordan anticommutation relations, we may prove the equivalence indirectly. For if U_c and U_c' are two transformations obtained by different choices of W, then $U_c^{-1}U_c'$ commutes with ψ and $\psi^\dagger$ and hence must be a multiple of the identity. Hence U_c and U_c' can differ only in a multiplicative phase factor.

We can make use of the freedom in the choice of W to simplify the expression for U_c. In particular, the choice $W=\beta$ is particularly convenient, for if we introduce the adjoint to ψ by the definition

$$\bar{\psi}=\psi^\dagger\beta, \tag{22}$$

then we may write for A the simple form

$$A=\tfrac{1}{2}\int[\bar{\psi}\psi-\psi\bar{\psi}]d\mathbf{r}, \tag{23}$$

and by introducing the matrix

$$C=-\beta\kappa, \tag{24}$$

we may write B in the form:

$$B=\tfrac{1}{2}\int[\bar{\psi}C\bar{\psi}-\psi C^{-1}\psi]d\mathbf{r}. \tag{25}$$

The matrix C is then identical with that employed by Schwinger[2] and others, and can be defined abstractly by the relations:

$$C\gamma_\mu C^{-1}=-\gamma_\mu,\quad CC^\dagger=-CC^*=1, \tag{26}$$

with the γ_μ defined as usual. The charge conjugation transformation can then be written:

$$\psi_c=U_c\psi U_c^{-1}=C\bar{\psi},\quad \bar{\psi}_c=U_c\bar{\psi}U_c^{-1}=C^{-1}\psi. \tag{27}$$

This concludes our derivation of an explicit form for the charge conjugation transformation. In the remainder

[2] J. Schwinger, Phys. Rev. 74, 1439 (1948).

of this note, we employ this representation to clarify a question about the commutation of charge conjugation with the space inversion transformation.

SPACE INVERSION TRANSFORMATION

The space inversion transformation for the Dirac equation is defined as the following unitary transformation:

$$\psi(\mathbf{r},t)\rightarrow\psi_s(\mathbf{r},t)=U_s(\theta)\psi U_s^{-1}(\theta)=ie^{i\theta}\beta\psi(-\mathbf{r},t), \quad (28)$$

where θ is an arbitrary or undetermined phase factor. To obtain an explicit form for $U_s(0)$, one may make use of one of the theorems derived by Berger, Foldy, and Osborn,[3] from which one can see immediately that

$$U_s(0)=\exp[i\pi S/2]=\exp\left[\frac{i\pi}{2}\int\psi^\dagger(\mathbf{r},t)\beta\psi(-\mathbf{r},t)d\mathbf{r}\right]. \quad (29)$$

To obtain the additional phase factor $e^{i\theta}$ one requires a gauge transformation and hence

$$U_s(\theta)=\exp[i\theta Q]\exp[i\pi S/2], \quad (30)$$

where Q is the operator for the total charge:

$$Q=\int\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t)d\mathbf{r}. \quad (31)$$

Actually it will be more convenient to employ for S and Q the charge-symmetric expressions:

$$S=\tfrac{1}{2}\int[\psi^\dagger(\mathbf{r},t)\beta\psi(-\mathbf{r},t)-\psi(\mathbf{r},t)\beta^*\psi^\dagger(-\mathbf{r},t)]d\mathbf{r}$$
$$=\tfrac{1}{2}\int[\bar{\psi}(\mathbf{r},t)\psi(-\mathbf{r},t)-\psi(\mathbf{r},t)\bar{\psi}(-\mathbf{r},t)]d\mathbf{r}, \quad (32)$$

$$Q=\tfrac{1}{2}\int[\psi^\dagger\psi-\psi\psi^\dagger]d\mathbf{r}=\tfrac{1}{2}\int[\bar{\psi}\gamma_4\psi-\psi\gamma_4^T\bar{\psi}]d\mathbf{r}.$$

[3] Berger, Foldy, and Osborn, Phys. Rev. **87**, 1061 (1952).

One can then verify that

$$U_cSU_c^{-1}=S,\quad U_cQU_c^{-1}=-Q,\quad QS=SQ. \quad (33)$$

We now note that

$$U_cU_s(\theta)\psi(\mathbf{r},t)U_s^{-1}(\theta)U_c^{-1}=ie^{i\theta}\beta\kappa\psi^\dagger(-\mathbf{r},t), \quad (34)$$

while

$$U_s(\theta)U_c\psi(\mathbf{r},t)U_c^{-1}U_s^{-1}(\theta)=ie^{-i\theta}\beta\kappa\psi^\dagger(-\mathbf{r},t). \quad (35)$$

Thus it appears that U_c and U_s do not commute unless $\theta=n\pi$, and hence that one could obtain information about the phase factor θ from experiment. Thus if one had a system such as positronium and found that the stationary states of this system were simultaneous eigenstates of both charge conjugation parity and space parity, one might be tempted to conclude that θ is restricted to one of the values $n\pi$.

This apparent paradox is easily resolved. Using our explicit forms for U_c and $U_s(\theta)$ we can compute their commutator as follows: We have

$$U_c^{-1}[U_c,U_s(\theta)]=U_s(\theta)-U_c^{-1}U_s(\theta)U_c$$
$$=e^{i\theta Q}e^{i\pi S/2}-U_c^{-1}e^{i\theta Q}e^{i\pi S/2}U_c. \quad (36)$$

Then using (33) and multiplying on the left by U_c, we obtain

$$[U_c,U_s(\theta)]=2iU_cU_s(\theta)\sin(\theta Q). \quad (37)$$

Now the representation space (Hilbert space) can be decomposed into subspaces each associated with a given eigenvalue of the total charge Q of the system. In the subspace in which $Q=0$, Eq. (37) tells us that U_c and $U_s(\theta)$ always commute, independently of the value of θ. Thus one can learn nothing concerning the value of θ from experiments performed on systems such as positronium for which the total charge is zero. On the other hand, since U_c does not commute with the total charge it cannot be an observable in any subspace in which the total charge is not zero since this would violate the superselection principle for charge.[4] Thus experiments of the type envisaged above can never yield information about the phase factor θ.

[4] Wick, Wightman, and Wigner, Phys. Rev. **88**, 101 (1952).