# Use of the Boltzmann Equation for the Study of Ionized Gases of Low Density. I

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The Boltzmann equation is studied for the case of a low-density ionized gas in an externally applied electromagnetic field. Particle-particle collisions are neglected, but long-range collective interactions are included. In Part I the static problem is treated in detail. For this case the Boltzmann equation is solved using individual-particle orbits—an approach which emphasizes the physical basis of the solution.

## I. INTRODUCTION

THE conventional treatment of transport phenomena in nonuniform gases is that of Chapman and Enskog.[1] This is a procedure for solving the Boltzmann equation in a series of terms, the first of which describes a local Maxwellian velocity distribution at each point in space. The series expansion used converges rapidly when the mean free path for particle collisions is much smaller than all pertinent macroscopic dimensions of the gas. On the other hand, for a rarefied gas in which collision mean free paths are long, the method does not seem to be useful.

In the present paper, we wish to consider a method of solving the Boltzmann equation when particle "collisions" are negligible. We treat the case of a completely ionized gas in a strong, externally applied magnetic field. To make the treatment as simple as possible, we suppose there to be only two kinds of particles present—electrons and ions. Their respective masses will be denoted by $M_e$ and $M_i$ and their charges are assumed to be $-e$ and $+e$.[2]

There are various geophysical and astrophysical applications of the study of conducting fluids in electromagnetic fields.[3] These are usually treated by the use of hydrodynamic equations (the *hydromagnetic equations*). For conducting liquids and dense gases (i.e., gases for which the mean free paths for collisions are small compared to macroscopic dimensions), the application of these equations seems justified. On the other hand, there are astrophysical phenomena, as well as gaseous discharge phenomena, for which particle-particle collisions are of very little importance in determining the behavior of the gas. It is with such conditions that we shall be concerned.

In discussing ionized gases, it is necessary to be careful in defining the term "collision," since the particle motions are very strongly coupled through the long-range electromagnetic interactions. For this reason, it is customary to divide the interactions of a particle into long-range "collective" effects and short-range "particle-particle" interactions. The characteristic distance within which particle-particle collisions are important is the Debye radius,

$$R_D=[\theta/4\pi e^2n]^{\frac{1}{2}},$$

where $\theta$ is the temperature and $n$ is the gas density.[4]

The "collision time" $\tau$ is[4]

$$\frac{1}{\tau}\simeq\frac{M_e^{\frac{1}{2}}}{\theta}\frac{n}{M_e^2}\frac{8\sqrt{\pi}}{3}e^4\ln\left(\frac{R_D}{R_{\min}}\right),$$

where $R_{\min}$ is a minimum impact parameter. If $\tau$ is less than macroscopic periods of the motion, we expect a local Maxwellian velocity distribution to develop. In this case, the hydrodynamic or Chapman-Enskog methods should be applicable. For the problems of interest to us, however, we shall assume that $\tau$ is larger than other periods of the system and that particle collisions are negligible.

We must thus consider the two Boltzmann equations[5]

$$\begin{aligned}\frac{\partial f_i}{\partial t}+\mathbf{c}\cdot\nabla f_i+\frac{e}{M_i}\left[\mathbf{E}+\frac{\mathbf{c}}{C}\times\mathbf{B}\right]\cdot\nabla_c f_i=0,\\ \frac{\partial f_e}{\partial t}+\mathbf{c}\cdot\nabla f_e+\frac{(-e)}{M_e}\left[\mathbf{E}+\frac{\mathbf{c}}{C}\times\mathbf{B}\right]\cdot\nabla_c f_e=0.\end{aligned}\tag{1}$$

Here $f_e(\mathbf{r},\mathbf{c},t)$ and $f_i(\mathbf{r},\mathbf{c},t)$ are the respective electron and ion distribution functions, where $\mathbf{r}$ is a space and $\mathbf{c}$ a velocity coordinate. ($\nabla_c$ is the "del-operator" in velocity space.) $\mathbf{E}$ and $\mathbf{B}$ are the respective electric and magnetic field vectors.

Several velocity moments of the $f$'s will be of later

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[1] S. Chapman and T. G. Cowling, *Mathematical Theory of Non-Uniform Gases*, (Cambridge University Press, London, 1952).

[2] We consider only singly ionized particles. This permits us to ignore states of multiple ionization and the consequent introduction of several "types" of particles. There would be no inherent complication of the problem in such a generalization, however.

[3] See, for instance, H. Alfvén, *Cosmical Hydrodynamics* (Oxford University Press, New York, 1950).

[4] See, for instance, L. Spitzer, *The Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1955).

[5] See, for instance, reference 1. In the general case, the right-hand side is set equal to the rate of change of $f$ due to particle collisions.

importance to us:

$$n_e=\int f_e d^3c=\text{average electron density.}$$

$$\mathbf{v}_e=\int \mathbf{c} f_e d^3c=\text{average electron velocity.}$$

$$\mathbf{p}_e=M_e\int[\mathbf{c}-\mathbf{v}_e][\mathbf{c}-\mathbf{v}_e]f_e d^3c \tag{2}$$

$$=\text{electron pressure tensor.}$$

$$\mathbf{Q}_e=M_e\int[\mathbf{c}-\mathbf{v}_e][\mathbf{c}-\mathbf{v}_e][\mathbf{c}-\mathbf{v}_e]f_e d^3c$$

$$=\text{heat flux tensor, etc.}$$

In addition to these, there are the corresponding quantities for ions, for which we use the subscript $i$. We have also the electric current and charge density which are

$$\begin{aligned}\mathbf{j}&=e[n_i\mathbf{v}_i-n_e\mathbf{v}_e],\\ \epsilon&=e[n_i-n_e].\end{aligned} \tag{3}$$

Finally we have Maxwell's equations

$$\begin{aligned}\nabla\times\mathbf{B}&=\frac{4\pi}{C}\mathbf{j}+\frac{1}{C}\frac{\partial\mathbf{E}}{\partial t},\\ -\frac{1}{C}\frac{\partial\mathbf{B}}{\partial t}&=\nabla\times\mathbf{E},\\ \nabla\cdot\mathbf{B}&=0,\\ \nabla\cdot\mathbf{E}&=4\pi\epsilon,\end{aligned} \tag{4}$$

which are coupled to Eqs. (1) through Eqs. (3). The fields due to sources of current and charge external to the plasma are to be specified by the boundary conditions on (4) rather than by the charges and currents themselves.

For most of our discussion, it will not be necessary to specify the distinction between electrons and ions, so we shall, unless specifically stated otherwise, omit the subscripts "$e$" or "$i$." For instance, we write the Boltzmann equation simply as

$$\frac{\partial f}{\partial t}+\mathbf{c}\cdot\nabla f+\frac{e}{M}\left[\mathbf{E}+\frac{1}{C}\mathbf{c}\times\mathbf{B}\right]\cdot\nabla_c f=0. \tag{5}$$

We need then only give $e$ its correct sign and apply the appropriate subscript when we desire to be more detailed.

Our problem will be divided into two parts. The first is that of describing static solutions to Eq. (5). The second, which will be treated in more detail in the following paper, concerns the time-dependent solution of (5) for $f$ when the distribution differs only by an *infinitesimal* amount from the static distribution. Indeed, let

$$\begin{aligned}\mathbf{E}&=\mathbf{E}_0+\mathbf{E}',\\ \mathbf{B}&=\mathbf{B}_0+\mathbf{B}',\quad(\text{with }\mathbf{E}_0\cdot\mathbf{B}_0=0),\end{aligned} \tag{6}$$

where $\mathbf{E}_0$ and $\mathbf{B}_0$ are static and $\mathbf{E}'$ and $\mathbf{B}'$ are infinitesimal fluctuating quantities. Also, we abbreviate the Boltzmann operator by

$$\mathfrak{D}=\frac{\partial}{\partial t}+\mathbf{c}\cdot\nabla+\frac{e}{M}\left[\mathbf{E}_0+\frac{1}{C}\mathbf{c}\times\mathbf{B}_0\right]\cdot\nabla_c. \tag{7}$$

Then the solution to the "static" problem will be denoted by $f_0(\mathbf{c})$, which satisfies the Boltzmann equation

$$\mathfrak{D}f_0=0. \tag{8}$$

Returning to the general problem (5), we write

$$f(\mathbf{c})=f_0(\mathbf{c}-\mathbf{u})+f'. \tag{9}$$

Here $\mathbf{u}(\mathbf{r},t)$ is a "drift velocity" superimposed on the static distribution $f_0(\mathbf{c})$. In accordance with Eq. (6), we suppose both $\mathbf{u}$ and $f'$ to be infinitesimal and that $\mathbf{u}$ satisfies the equation of motion

$$\frac{\partial\mathbf{u}}{\partial t}=\frac{e}{M}\left[\mathbf{E}'+\frac{1}{C}\mathbf{u}\times\mathbf{B}_0\right]. \tag{10}$$

Equation (9) is next substituted into Eq. (5). We make use of Eqs. (6), (8), and (10) and linearize the resulting equation in *small* quantities to obtain

$$\mathfrak{D}f'=-\mathbf{u}\cdot\nabla f_0+\mathbf{c}\cdot(\nabla\mathbf{u})\cdot\nabla_c f_0-\frac{e}{MC}\mathbf{c}\times\mathbf{B}'\cdot\nabla_c f_0. \tag{11}$$

Equation (11) represents the basis of our discussion of the behavior of the system in the neighborhood of an equilibrium distribution (that is, $f_0$). It will be used to describe the dynamic and thermodynamic behavior of the gas in Part II.

It will be noticed that we are following a procedure often employed in hydrodynamics in the treatment of stability of flow (or the onset of turbulence). In such problems, the hydrodynamic equations are linearized about a state of static flow. If the linearized perturbation of the flow grows in time, the flow is said to be "unstable"—otherwise, "stable". Recently these methods have been extended by Chandrasekhar[6] to certain hydromagnetic problems involving the stability of a conducting liquid in an applied magnetic field. Astrophysical applications to the stability of ionized gases in magnetic and gravitational fields have been made by Chandrasekhar and Fermi,[7] by Kruskal and Schwarzschild,[8] and by Frieman *et al.*[9]

[6] See, for instance, S. Chandrasekhar, Phil. Mag. Ser. 7, **43**, 501 (1952) and **45**, 1177 (1954). Further references are given here.

[7] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 116 (1953).

[8] M. Kruskal and M. Schwarzschild, Proc. Roy. Soc. (London) **A223**, 348 (1954).

[9] Frieman, Bernstein, Kruskal, and Kulsrud, Revs. Modern Phys. (to be published).

These studies have all involved a hydrodynamic approach, which is not in general justifiable for our problems. It is well known, however, that the equations obtained by taking successive moments of Eq. (5) provide a formally rigorous hydrodynamics, involving an infinite series of coupled equations. This series of equations may be terminated at any stage by evaluating the appropriate moment from the solution $f'$ to Eq. (11). In a sense, this is purely formal, since if $f'$ is known (we assume that $f_0$ is specified as an initial condition) the problem is completely solved. From a practical point of view, however, this is often a useful approach—and, in this respect, similar to the Chapman-Enskog method.[10]

## II. RELATION TO PARTICLE EQUATIONS OF MOTION

For the sake of added physical clarity, we shall discuss the static solution by means of particle orbits. It is quite apparent that this may be done, the formal argument being as follows:

Abbreviate

$$\frac{e}{M}\left[\mathbf{E}+\frac{1}{C}\mathbf{c}\times\mathbf{B}\right]\equiv\mathbf{F} \tag{12}$$

in Eq. (5), and consider

$$\begin{aligned} d\mathbf{c}/dt&=\mathbf{F},\\ d\mathbf{r}/dt&=\mathbf{c}. \end{aligned} \tag{13}$$

Let the solution to these equations be

$$\begin{aligned} \mathbf{c}&=\mathbf{c}(\alpha_1\cdots\alpha_6,t),\\ \mathbf{r}&=\mathbf{r}(\alpha_1\cdots\alpha_6,t), \end{aligned} \tag{14}$$

where the $\alpha$'s are integration constants. Let us also suppose that these equations may be solved for the $\alpha$'s in terms of $r$, $c$, and $t$. Then

$$\alpha_i=\alpha_i(\mathbf{r},\mathbf{c},t)\,(i=1,2,\cdots 6). \tag{15}$$

Now, *any* function (possessing derivatives with respect to the $\alpha$'s),

$$f(\alpha_1\cdots\alpha_6),$$

of the $\alpha$'s (when the $\alpha$'s are replaced by the functions (15) of $(r,c,t)$ is a solution of the Boltzmann equation (5). Indeed, on substituting into Eq. (5), we obtain

$$\sum_{i=1}^{6}\frac{\partial f}{\partial\alpha_i}\left[\frac{\partial\alpha_i}{\partial t}+\mathbf{c}\cdot\nabla\alpha_i+\mathbf{F}\cdot\nabla_c\alpha_i\right]=\sum_{i=1}^{6}\frac{\partial f}{\partial\alpha_i}\frac{d\alpha_i}{dt}=0, \tag{16}$$

since the $\alpha_i$'s are *constants*. When the $\alpha_i$'s are not true constants, but adiabatic invariants, the resulting $f$ satisfies the Boltzmann equation to within the accuracy of the adiabatic theorem.

A useful set of $\alpha$'s are the initial position $\mathbf{r}_0$ and initial velocity $\mathbf{c}_0$. Then

$$f=f(\mathbf{r}_0(\mathbf{r},\mathbf{c},t),\ \mathbf{c}_0(\mathbf{r},\mathbf{c},t)) \tag{17}$$

satisfies Eq. (5).

Equations (16) and (17) hold in general. To obtain a static solution, it is evidently necessary to choose $f$ so that expression (17) is independent of $t$.

## III. THE SPECIFICATION OF THE PROBLEM

In order to be able to make detailed statements concerning the behavior of the gas, it is necessary to specify in some detail the physical properties of the static state. We shall assume:

(A) The gas is very nearly electrically neutral, or that $n_i\simeq n_e$. More precisely, we shall keep terms of order no higher than the first in $(n_i-n_e)$.

(B) The quantity

$$\eta\equiv\frac{\text{Average Larmor radius of particle orbits}}{\text{Dimensions of the system}}\ll 1.$$

This will be considered as an expansion parameter (to which we shall frequently refer) and we shall usually be interested in keeping no more than first-order terms in $\eta$. Assumption (B) implies a "strong" magnetic field and means that Larmor frequencies will be much greater than other characteristic frequencies of the system. This is apparent, on recalling that the Larmor radius $\simeq$thermal velocity/Larmor frequency.

(C) The quantity

$$\beta\simeq\frac{\text{Kinetic energy density in gas}}{\text{Magnetic field energy density}}\ll 1.$$

This is clearly compatible with our assumption of low gas density and strong applied magnetic field. Taking $\beta\ll 1$ simplifies considerably the solution of Maxwell's equations, since it implies that the magnetic field arising from the external sources will provide a reasonable starting approximation to the actual $\mathbf{B}$.

(D) The static electric field, $\mathbf{E}_0$, is small and its derivatives are negligible.

Of our four assumptions, (B) is by far the most important since it determines the general nature of our conclusions. Assumption (C) is convenient in obtaining explicit solutions of our equations. The remaining two assumptions are of less significance and might have been relaxed without invloving much additional complication.

## IV. THE STATIC DISTRIBUTION FUNCTION

In accordance with the theorem quoted in Sec. II, we consider first the orbits of a charged particle in a magnetic field. We keep first-order terms in $\eta$ in solving

$$\frac{\partial\mathbf{c}}{\partial t}=(e/M)\mathbf{E}_0+\mathbf{c}\times\boldsymbol{\omega}(\mathbf{r}), \tag{18}$$

[10] We are indebted to Dr. G. Chew, Dr. M. Goldberger, and Dr. F. Low for calling our attention to the similarity here to the Chapman-Enskog method.

where

$$\boldsymbol{\omega}(\mathbf{r})=(e/MC)\mathbf{B}_0(\mathbf{r}). \tag{19}$$

As explained in Appendix A, we introduce at each point in space a local coordinate system with orthogonal unit vectors

$$\hat{e}_1,\ \hat{e}_2,\ \text{and}\ \hat{e}_3$$

and "coordinates" $x_1$, $x_2$, and $x_3$. The $x$'s are lengths measured along the direction of the unit vectors. They are not true coordinates, since they are defined only locally. The unit vector $\hat{e}_1$ is in the direction of $\mathbf{B}$, and we have

$$\hat{e}_2/R_1=\partial\hat{e}_1/\partial x_1,\quad \hat{e}_3=\hat{e}_1\times\hat{e}_2.$$

Other relations involving these unit vectors are given in Appendix A.

In Appendix B we derive the motion of the charged particle. The dominating term in the motion perpendicular to $\mathbf{B}_0$ is the Larmor velocity,

$$\begin{aligned} V_2&=V_\perp\cos\int^t\omega dt',\\ V_3&=V_\perp\sin\int^t\omega dt',\\ \omega&=(e/MC)B_0(G). \end{aligned} \tag{20}$$

Here $B_0(G)$ is the value of the field $B_0$ at the position of the guiding center of the Larmor orbit. The guiding center moves along the field lines with a velocity $V_1$ determined from

$$dV_1/dt=V_\perp^2/2D. \tag{21}$$

[$1/D\equiv-(1/B)(\partial B/\partial x_1)$, as described in Appendix (A).]

Also, the quantity $V_\perp$ in Eqs. (20) satisfies

$$(d/dt)(V_\perp^2)=-V_1(V^2_\perp/D). \tag{22}$$

It is convenient to write a vector $\mathbf{V}$ as

$$\mathbf{V}\equiv V_1\hat{e}_1+V_2\hat{e}_2+V_3\hat{e}_3, \tag{23}$$

where *all* the unit vectors are defined at the guiding center position.

Then, (as is shown in Appendix B), to first order in $\eta$ the solution to Eq. (18) is

$$\mathbf{c}=\mathbf{V}+\mathbf{v}_D+\mathbf{v}_{E_0}-\hat{e}_1\frac{V_1V_3}{\omega R_1}+\hat{e}_1\frac{V_2V_3}{2\omega}\left(\frac{1}{R_3}-\frac{1}{R_2}\right)+\frac{1}{3\omega R_1}\left\{\hat{e}_2V_2V_3-\frac{\hat{e}_3}{2}(V_2^2-V_3^2)\right\}, \tag{24}$$

where

$$\begin{aligned} \mathbf{v}_{E_0}&=c\mathbf{E}_0\times\mathbf{B}_0/B_0^2,\\ \mathbf{v}_D&=\frac{\hat{e}_3}{\omega R_1}[V_1^2+\tfrac{1}{2}V_\perp^2]. \end{aligned} \tag{25}$$

Again our unit vectors are defined at the position of the guiding center. The position of the guiding center is $\mathbf{R}_c$, which is related to the particle position $\mathbf{r}$ by

$$\mathbf{r}=\mathbf{R}_c-(\mathbf{V}\times\hat{e}_1/\omega), \tag{26}$$

to first order in $\eta$.

We have now put the solution to the equation of motion in the desired form. To continue, we define a *guiding center* distribution function $F$ by

$$f(\mathbf{r},\mathbf{c})d^3rd^3c=F(V_1,V_2,V_3,R_{c1},R_{c2},R_{c3})d^3R_cd^3V. \tag{27}$$

Equation (27) is an expression for the one-to-one correspondence between particles and guiding centers.[11] To obtain the time dependence of $F$ using the method of Sec. II, we observe that

$$d^3rd^3c=\text{const},$$

following the dynamical motion (this is a consequence of Liouville's theorem). Next, we may show that

$$d^3R_cd^3V=\text{const}.$$

This follows on evaluation of the Jacobian of the transformation[12]

$$\begin{aligned} R_{c1}&=R_{c10}V_1\delta t,\\ R_{c2}&=R_{c20},\\ R_{c3}&=R_{c30}+v_D\delta t,\\ V_1&=V_{10}+(V_\perp^2/2D)\delta t,\\ V_2&=[V_{20}+V_{30}\omega\delta t][1-(V_{10}/2D)\delta t],\\ V_3&=[V_{30}-V_{20}\omega\delta t][1-(V_{10}/2D)\delta t], \end{aligned} \tag{28}$$

where $\delta t$ is an infinitesimal time interval. $\mathbf{V}_0$ and $\mathbf{R}_{c0}$ represent the values of $\mathbf{V}$ and $\mathbf{R}_c$ at the *beginning* of the interval $\delta t$. Equations (28) follow immediately from Eqs. (24) and (25), if we drop $\mathbf{v}_{E_0}$ or assume it to be in the $\hat{e}_3$ direction and thus included in $v_D$.

In the same interval $\delta t$, we have

$$\begin{aligned} \mathbf{r}_0&=\mathbf{r}-\mathbf{c}\delta t,\\ \mathbf{c}_0&=\mathbf{c}-(e/M)\mathbf{E}_0\delta t-\mathbf{c}\times\boldsymbol{\omega}(r)\delta t. \end{aligned} \tag{29}$$

Referring to Eq. (27), we see that during $\delta t$, $f$ changes according to Eqs. (29) while $F$ changes according to Eqs. (28). The differential equation for $F$ is determined from

$$\frac{\partial F}{\partial t}=\frac{1}{\delta t}[F(\delta t)-F(0)]. \tag{30}$$

It is apparent that if $F$ is not to oscillate at the Larmor frequency, it must depend only on $V_2$ and $V_3$ in the combination

$$V_2^2+V_3^2=V_\perp^2.$$

Also if $F$ is not to change due to the guiding center drift

[11] This approach to the distribution function using $F$ was begun in collaboration with M. L. Goldberger.

[12] The argument is given in more detail in Appendix A.

$v_D$, $F$ should *not* sepend upon $R_{c3}$. Thus,[13]

$$F=F(V_1,V_\perp^2,R_{c1},R_{c2}). \tag{31}$$

There is one final condition on $F$ due to motion in the "1" direction. In time $\delta t$

$$\begin{aligned} R_{c10}&=R_{c1}-V_1\delta t,\\ V_{10}&=V_1-(V_\perp^2/2D)\delta t,\\ V_{\perp 0}^2&=V_\perp^2+V_1(V_\perp^2/D)\delta t, \end{aligned} \tag{32}$$

which follows directly from Eqs. (28). Using Eq. (31) we have

$$\frac{\partial F}{\partial t}=\frac{1}{\delta t}\left\{F\left(V_1-\frac{V_\perp^2}{2D}\delta t,\ V_\perp^2+V_1\frac{V_\perp^2}{D}\delta t,\ R_{c1}-V_1\delta t,\ R_{c2}\right)-F(V_1,V_\perp^2,R_{c1},R_{c2})\right\}=0$$

or

$$\frac{V_\perp^2}{2D}\frac{\partial F}{\partial V_1}+V_1\frac{\partial F}{\partial R_{c1}}-V_1\frac{V_\perp^2}{D}\frac{\partial F}{\partial V_\perp^2}=0. \tag{33}$$

This represents the final restriction on the form of $F$. If $\mathbf{v}_{E_0}$ is not in the $\hat{e}_3$ direction, then the functional form of $F$ is further restricted. The resulting equations may be easily worked out as we have done above.

Dividing Eq. (33) by $V_1$, we see that this equation depends only on $V_1^2$. We thus take $F$ to be a function of $V_1^2$:

$$F=F(V_1^2,V_\perp^2,R_{c1},R_{c2}), \tag{34}$$

and satisfying Eq. (33).

From Eq. (26) we have the relations between the volume elements as

$$\begin{aligned} d^3R_c&=d^3r\left[1+\nabla\cdot\left(\frac{\mathbf{V}\times\hat{e}_1}{\omega}\right)\right]\\ &=d^3r\left[1-\frac{\mathbf{V}\cdot(\nabla\times\hat{e}_1)}{\omega}+\frac{V_3}{\omega}\nabla\cdot\hat{e}_2-\frac{V_2}{\omega}\nabla\cdot\hat{e}_3\right]. \end{aligned} \tag{35}$$

It is also necessary to express $F$ in terms of its value at $\mathbf{r}$ through [see Eq. (26)]

$$F(R_c)=F_r+\frac{V_3}{\omega}\frac{\partial F_r}{\partial R_{c2}}, \tag{36}$$

where $F_r$ means $F$ at the point $r$. These relations permit us to write Eq. (27) as

$$f(\mathbf{r},\mathbf{c})d^3c=\left[F(V_1^2,V_\perp^2,x_1,x_2)+\frac{V_3}{\omega}\frac{\partial F}{\partial x_2}\right]+\left[1-\frac{\mathbf{V}\cdot\nabla\times\hat{e}_1}{\omega}+\frac{V_3}{\omega}\nabla\cdot\hat{e}_2-\frac{V_2}{\omega}\nabla\cdot\hat{e}_3\right]d^2V. \tag{37}$$

Here we must keep only zero- and first-order terms of $\eta$.

[13] We are not seeking the most general static solution of Eq. (30), Eq. (31) being sufficiently general for our purposes.

## V. CONDITIONS ON THE MOMENTS OF THE STATIC DISTRIBUTION

The static distribution $f$ of the previous section is the distribution $f_0$ about which we shall linearize the time-dependent equations in Part II. Its general properties lead to conditions on the moments introduced in Eq. (2). For instance, we obtain directly from Eq. (37)

$$n\equiv\int fd^3c=\int Fd^3V\equiv N, \tag{38}$$

the guiding center density, to and including first order terms in $\eta$. For the higher moments, we introduce the notation

$$N\langle\phi\rangle_G\equiv\int\phi Fd^3V, \tag{39}$$

where $\phi(V)$ is some function of $\mathbf{V}$.

We may also easily express the higher moment expressions (2) in terms of the moments of $F$. For the $i$th component of the drift velocity at the point $\mathbf{r}$ we have, where $\hat{e}_i(\mathbf{r})$ is the appropriate value of $\hat{e}_i$ at the point $\mathbf{r}$.

$$\begin{aligned} n\mathbf{v}\cdot\hat{e}_i(r)\equiv nv_i=\int\hat{e}_i(r)\cdot\Big\{&\mathbf{V}+\hat{e}_1\frac{V_2V_3}{2\omega}\left(\frac{1}{R_3}-\frac{1}{R_2}\right)-\hat{e}_1\frac{V_1V_3}{\omega R_1}+\frac{\hat{e}_3}{\omega R_1}[V_1^2+\tfrac{1}{2}V_\perp^2]\\ &+\frac{1}{3\omega R_1}\left[\hat{e}_2V_2V_3-\frac{\hat{e}_3}{2}(V_2^2-V_3^2)\right]\Big\}\left[F_r+\frac{V_3}{\omega}\frac{\partial F_r}{\partial x_2}\right]\\ &\times\left[1-\frac{\mathbf{V}\cdot(\nabla\times\hat{e}_1)}{\omega}+\frac{V_3}{\omega}\nabla\cdot\hat{e}_2-\frac{V_2}{\omega}\nabla\cdot\hat{e}_3\right]d^3V. \end{aligned} \tag{40}$$

To evaluate this we recall that the components of $\mathbf{V}$ are defined in terms of the $\hat{e}_i(\mathbf{R}_c)$, so we must replace the $\hat{e}_i(\mathbf{r})$ in the right-hand side of (40) by

$$\hat{e}_i(r)=\hat{e}_i(\mathbf{R}_c)-\left(\frac{\mathbf{V}\times\hat{e}_1}{\omega}\right)\cdot\nabla\hat{e}_i(\mathbf{R}_c). \tag{41}$$

The integral is now readily evaluated to first order in $\eta$ to give

$$\mathbf{v}=\hat{e}_3(r)\left\{\frac{1}{\omega Mn}\frac{\partial}{\partial x_2}M(n\langle V_3^2\rangle_G)+\frac{1}{R_1\omega}[\langle V_1^2\rangle_G-\langle V_3^2\rangle_G]\right\}. \tag{42}$$

The pressure tensor is evaluated in the same manner from

$$p_{ij}=M\int\hat{e}_i(r)\cdot[\mathbf{c}-\mathbf{v}][\mathbf{c}-\mathbf{v}]\cdot\hat{e}_j(r)fd^3c. \tag{43}$$

To and including first-order terms in $\eta$, we find that

$$p_{11}=Mn\langle V_1^2\rangle_G,$$
$$p_{22}=p_{33}=Mn\langle V_3^2\rangle_G, \tag{44}$$
$$p_{12}=p_{13}=p_{23}=0.$$

Thus the pressure tensor is diagonal and has two different elements in our coordinate system.

Finally, to zero order in $\eta$, the heat flow tensor

$$\mathbf{Q}=0. \tag{45}$$

Our moments are still not arbitrary, but are further related by the differential equation (33). Indeed, taking moments of Eq. (33), we obtain

$$\frac{\partial p_{11}}{\partial x_1}+\frac{p_{11}-p_{33}}{D}=0,^{14}$$

and

$$\nabla\cdot\mathbf{Q}=0. \tag{46}$$

Any further detail must be obtained by making more specific assumptions about the distribution function $f$ (or $f_0$). This cannot be determined from general principles, but must involve the physical details of the plasma structure. Such arbitrariness is not a shortcoming of our analysis, since it represents an actual arbitrariness in the corresponding physical situation.

In Part II, where the nonstatic problem is studied, the static solution obtained here will represent an initial condition on the system with respect to which small perturbations are made.

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## APPENDIX A

We discuss here the differential geometry used above. The magnetic field is

$$\mathbf{B}=B\hat{e}_1. \tag{A-1}$$

In accordance with our assumption that $\beta\ll 1$, we take

$$\nabla\times\mathbf{B}=0,$$

or, using (A-1),

$$\nabla\times\hat{e}_1=\frac{1}{B}\hat{e}_1\times\nabla B. \tag{A-2}$$

Let $\delta x_1$ represent an infinitesimal displacement in the direction of $\hat{e}_1$. Then

$$\hat{e}_2/R_1=\partial\hat{e}_1/\partial x_1 \tag{A-3}$$

defines the unit vector $\hat{e}_2[\hat{e}_2\cdot\hat{e}_1=0]$ and the principal radius of curvature of the field lines. A third vector is

$$\hat{e}_3=\hat{e}_1\times\hat{e}_2. \tag{A-4}$$

Define

$$\frac{1}{D}\equiv-\frac{1}{B}\frac{\partial B}{\partial x_1}. \tag{A-5}$$

Using $\nabla\cdot\mathbf{B}=0$, we have also

$$\frac{1}{D}=\hat{e}_2\cdot\left(\frac{\partial\hat{e}_1}{\partial x_2}\right)+\hat{e}_3\cdot\left(\frac{\partial e_1}{\partial x_3}\right), \tag{A-6}$$

where $\delta x_2$ and $\delta x_3$ are infinitesimal displacements in the direction of $\hat{e}_2$ and $\hat{e}_3$, respectively. We also define

$$\hat{e}_2\cdot\left(\frac{\partial\hat{e}_1}{\partial x_2}\right)\equiv\frac{1}{R_2},\quad \hat{e}_3\cdot\left(\frac{\partial\hat{e}_1}{\partial x_3}\right)\equiv\frac{1}{R_3}. \tag{A-7}$$

Further differential relations may easily be worked out. In treating the equations of motion in Appendix B, it is assumed that the radius of torsion of the field lines is large enough to be neglected.

The unit vectors $\hat{e}_1$,$\hat{e}_2$ and $\hat{e}_3$ along with the displacements $\delta x_1$, $\delta x_2$ and $\delta x_3$ define a rectangular coordinate system in the neighborhood of each point. They are not true coordinates, since they are defined only in an infinitesimal region. For instance

$$\frac{\partial^2}{\partial x_1\partial x_2}-\frac{\partial^2}{\partial x_2\partial x_1}\neq 0,$$

in general.

The $\nabla$ operator is

$$\nabla=\hat{e}_1\frac{\partial}{\partial x_1}+\hat{e}_2\frac{\partial}{\partial x_2}+\hat{e}_3\frac{\partial}{\partial x_3}, \tag{A-8}$$

consistent with the divergence and Stokes' theorems. We obtain, for instance

$$\nabla\times\hat{e}_1=\frac{\hat{e}_3}{R_1}. \tag{A-9}$$

In connection with Eq. (28) and the discussion preceding this equation, we may now show the constancy of the volume element

$$d^3R_cd^3V$$

in more detail. The motion of $\mathbf{R}_c$ is given by

$$\mathbf{R}_c=\mathbf{R}_{c0}+\mathbf{v}_D\delta t+\hat{e}_1(0)V_{10}\delta t. \tag{A-10}$$

Here $\hat{e}_i(0)$ is the value of $\hat{\mathbf{e}}_i$ at $t=0$. $\hat{e}_i(\delta t)$ is the corresponding value at $\mathbf{R}_c$ at time $\delta t$. Our volume element is

[14] This will be recognized as $(\nabla\cdot\mathbf{p})\cdot\hat{e}_1=0$, which is obviously necessary for our equilibrium solution.

[15] A development similar to that presented here will be published separately by Chew, Low, and Goldberger.

[16] Part II is written in collaboration with Dr. K. A. Brueckner.

constructed from

$$\Delta\mathbf{R}_c=\sum_{i=1}^{3}\left\{\frac{\partial\mathbf{R}_{c0}}{\partial R_{ci0}}+V_{10}\delta t\frac{\partial\hat{e}_1(0)}{\partial R_{ci0}}\right\}\Delta R_{ci0}. \quad \text{(A–11)}$$

(The contribution from $\mathbf{v}_D$ is of higher order than the terms which we are keeping.) Then,

$$\begin{aligned}\Delta R_{c1}&=\hat{e}_1(\delta t)\cdot\Delta\mathbf{R}_c=\Delta R_{c10}+\cdots,\\ \Delta R_{c2}&=\hat{e}_2(\delta t)\cdot\Delta\mathbf{R}_c=\left[1+\frac{V_{10}}{R_2}\delta t\right]\Delta R_{c20}+\cdots,\\ \Delta R_{c3}&=\left[1+\frac{V_{10}}{R_3}\delta t\right]\Delta R_{c30}+\cdots.\end{aligned} \quad \text{(A–12)}$$

For evaluating the Jacobian determinant only the terms on the principal diagonal are needed. These are just the terms explicitly written above. Using Eq. (28) in the form given for the velocities, we obtain immediately

$$d^3R_cd^3V=\left[1+\frac{V_{10}}{R_2}\delta t\right]\left[1+\frac{V_{10}}{R_3}\delta t\right]\times\left[1-\frac{V_{10}}{2D}\delta t\right]^2 d^3R_{c0}d^3V_0=d^3R_{c0}d^3V_0 \quad \text{(A–13)}$$

to first order in $\delta t$.

## APPENDIX B

We now discuss the solution of Eq. (18) to first order in $\eta$:

$$\frac{d\mathbf{c}}{dt}=\frac{e}{M}\mathbf{E}_0+\mathbf{c}\times\boldsymbol{\omega}(\mathbf{r}). \quad \text{(B–1)}$$

We introduce the "Larmor velocity"

$$\mathbf{v}_L=[\hat{e}_2'V_2+\hat{e}_3'V_3], \quad \text{(B–2)}$$

where $\hat{e}_2'$ and $\hat{e}_3'$ differ from $\hat{e}_2$ and $\hat{e}_3$ by terms of the first order in $\eta$. Explicit forms will be given later for $\hat{e}_2'$ and $\hat{e}_3'$. The *position* of the *guiding center* is defined to be

$$\mathbf{R}_c=\mathbf{r}+(\mathbf{v}_L\times\hat{e}_1/\omega). \quad \text{(B–3)}$$

Also [from (B–3)]

$$\boldsymbol{\omega}(\mathbf{r})=\boldsymbol{\omega}(\mathbf{R}_c)-\left(\frac{\mathbf{v}_L\times\hat{e}_1}{\omega}\right)\cdot\nabla\boldsymbol{\omega}(R_c) \quad \text{(B–4)}$$

to first order. We henceforth write for brevity

$$\boldsymbol{\omega}=\boldsymbol{\omega}(\mathbf{R}_c),$$

and also suppose $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$ are evaluated (as prescribed in Appendix A) at the position $\mathbf{R}_c$. Then Eq. (B–1) becomes

$$\frac{d\mathbf{c}}{dt}=\frac{e}{M}\mathbf{E}_0+\mathbf{c}\times\boldsymbol{\omega}-\mathbf{c}\times\left[\left(\frac{\mathbf{v}_L\times\hat{e}_1}{\omega}\right)\cdot\nabla\right]\cdot\boldsymbol{\omega}. \quad \text{(B–5)}$$

Set

$$\mathbf{c}=\mathbf{v}_L+v_1\hat{e}_1+\mathbf{v}_{E_0}+\mathbf{v}_D+\mathbf{v}_L', \quad \text{(B–6)}$$

with

$$\mathbf{v}_{E_0}=c\mathbf{E}_0\times B_0/B_0^2. \quad \text{(B–7)}$$

(We recall that $\mathbf{E}_0$ is considered to be small and that we can neglect its derivatives.) Now,

$$\frac{d}{dt}(v_1\hat{e}_1)=\frac{\partial v_1}{\partial t}\hat{e}_1+\frac{v_1V_1}{R_1}\hat{e}_2\simeq\frac{\partial v_1}{\partial t}\hat{e}_1+\frac{V_1^2}{R_1}\hat{e}_2, \quad \text{(B–8)}$$

where $V_1$ is the *nonoscillatory* part of $v_1$. (That is, the difference $v_1-V_1$ will turn out to be smaller than terms we need keep in Eq. (B–8).)

Making use of the differential relationships of Appendix (A) and using Eq. (B–6), we finally reduce Eq. (B–5) to

$$\begin{aligned}&\left[\frac{dv_1}{dt}-\left(\frac{V_2^2}{R_3}+\frac{V_3^2}{R_2}\right)\right]\hat{e}_1+\frac{V_1^2}{R_1}\hat{e}_2+\frac{d\mathbf{v}_L}{dt}+\frac{d\mathbf{v}_D}{dt}+\frac{d\mathbf{v}_L'}{dt}\\ &=-V_1\frac{\mathbf{v}_L}{2D}+\mathbf{v}_L\times\boldsymbol{\omega}-\frac{V_1}{2}\left(\frac{1}{R_3}-\frac{1}{R_2}\right)(V_2\hat{e}_2-V_3\hat{e}_3)\\ &\quad-\hat{e}_2\frac{V_\perp^2}{2R_1}+\frac{1}{R_1}\left[V_2V_3\hat{e}_3+\frac{\hat{e}_2}{2}(V_2^2-V_3^2)\right]\\ &\quad+\mathbf{v}_D\times\boldsymbol{\omega}+\mathbf{v}_L'\times\boldsymbol{\omega}. \end{aligned} \quad \text{(B–9)}$$

Here we have replaced $\hat{e}_2'$ and $\hat{e}_3'$ by $\hat{e}_2$ and $\hat{e}_3$, respectively, in all terms which involve field radii of curvature, and set

$$V_\perp^2=V_2^2+V_3^2. \quad \text{(B–10)}$$

Now, take

$$\begin{aligned}\mathbf{v}_D&=\frac{\hat{e}_3}{R_1\omega}\left[V_1^2+\frac{V_\perp^2}{2}\right],\\ \frac{dV_1}{dt}&=\frac{V_\perp^2}{2D},\\ \frac{dv_1}{dt}&=\frac{V_2^2}{R_3}+\frac{V_3^2}{R_2}.\end{aligned} \quad \text{(B–11)}$$

Also, define $\mathbf{v}_L$ by

$$\frac{d\mathbf{v}_L}{dt}=-\frac{V_1\mathbf{v}_L}{2D}+\mathbf{v}_L\times\boldsymbol{\omega}-\frac{V_1}{2}\left(\frac{1}{R_3}-\frac{1}{R_2}\right)(V_2\hat{e}_2-V_3\hat{e}_3). \quad \text{(B–12)}$$

Now, by Eq. (B–2)

$$\frac{d\mathbf{v}_L}{dt} \simeq \hat{e}_2' \frac{dV_2}{dt} + \hat{e}_3' \frac{dV_3}{dt} - \frac{V_1V_2}{R_1}\hat{e}_1, \quad \text{(B–13)}$$

if we neglect the torsion, $\partial\hat{e}_3/\partial x_1$. Choosing

$$\hat{e}_3' = \hat{e}_3 - \hat{e}_1(V_1/\omega R_1), \quad \hat{e}_2' = \hat{e}_2, \quad \text{(B–14)}$$

Eq. (B–13) reduces to

$$\frac{d\mathbf{v}_L}{dt} = \hat{e}_2 \frac{dV_2}{dt} + \hat{e}_3 \frac{dV_3}{dt}. \quad \text{(B–15)}$$

($dV_3/dt \simeq -\omega V_2$, to first order in $\eta$.) Substituting Eq. (B–15) into Eq. (B–12), we obtain coupled differential equations for $dV_2/dt$ and $dV_3/dt$. By means of the adiabatic theorem in mechanics, we can show that, keeping first-order terms in $\eta$, the solution is just Eqs. (20) with $V_\perp$ given by Eq. (22).

From Eqs. (B–11), we easily obtain

$$v_1 - V_1 = \left(\frac{V_2V_3}{2\omega}\right)\left(\frac{1}{R_2} - \frac{1}{R_3}\right). \quad \text{(B–16)}$$

We have now defined all quantities except $\mathbf{v}_L'$ in Eq. (B–9). Substituting, this becomes

$$\mathbf{v}_L' = \mathbf{v}_L' \times \boldsymbol{\omega} + \frac{1}{R_1}\left[V_2V_3\hat{e}_3 + \frac{\hat{e}_2}{2}(V_2^2 - V_3^2)\right], \quad \text{(B–17)}$$

which has the solution (to order $\eta$)

$$\mathbf{v}_L' = \frac{1}{3\omega R_1}\left\{\hat{e}_2V_2V_3 - \frac{\hat{e}_3}{2}(V_2^2 - V_3^2)\right\}. \quad \text{(B–18)}$$

Thus we have all the quantities in Eq. (B–6), which is just Eq. (24).

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# Use of the Boltzmann Equation for the Study of Ionized Gases of Low Density. II*

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The Boltzmann equation for ionized gases of low density in an external magnetic field is used to obtain approximate solutions in the nonstatic case. The Boltzmann and Maxwell equations are linearized by assuming small deviations from a static solution. It is shown that in the limit of a strong magnetic field ($\eta \ll 1$, as defined in the text), the motion transverse to the magnetic field is described by the conventional hydrodynamic equations. The variation along field lines is described by a one-dimensional (i.e., one space dimension and one velocity dimension) Boltzmann equation. Several applications are given, including an analysis of the Kruskal-Schwarzschild gravitational instability of a plasma.

## I. INTRODUCTION

IN Part I[1] we discussed on rather general grounds the behavior of an ionized gas of low density in a strong magnetic field. The properties of the static state were treated in detail. In the present paper we study further the dynamic and thermodynamic behavior of the gas.

We recall a few of the basic equations from Part I. The dynamical properties were described by the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{c}\cdot\nabla f + \frac{e}{M}\left[\mathbf{E} + \frac{1}{C}\mathbf{c}\times\mathbf{B}\right]\cdot\nabla_c f = 0, \quad (1)$$

which applies to either electrons or ions on giving $e$ its proper sign and on assigning the appropriate subscripts $e$ or $i$. The first four moments of $f$ were written as

$$\begin{aligned} n &\equiv \int f d^3c, \\ \mathbf{v} &\equiv \frac{1}{n}\int \mathbf{c} f d^3c, \\ \mathbf{p} &\equiv M\int (\mathbf{c}-\mathbf{v})(\mathbf{c}-\mathbf{v}) f d^3c, \\ \mathbf{Q} &\equiv M\int (\mathbf{c}-\mathbf{v})(\mathbf{c}-\mathbf{v})(\mathbf{c}-\mathbf{v}) f d^3c, \end{aligned} \quad (2)$$

* A development closely paralleling in many respects that given here has been found by Chew, Goldberger, and Low. This treatment is to be published separately.

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[1] K. M. Watson, preceding paper [Phys. Rev. **102**, 12 (1956)]. Equations in Part I will be referred to here as Eq. (I-1), etc. Part I itself will be referred to as I, for brevity. The notation is the same as in I: **E** is the electric field, **B** the magnetic field, **C** the particle velocity, etc. The unit vector $\hat{e}_1$ is in the direction of $\mathbf{B}_0$, $\hat{e}_2$ in the direction of the principal radius of curvature of the $\mathbf{B}_0$-lines. $\hat{e}_3$ is $\hat{e}_1\times\hat{e}_2$.