Optical-Model Potential for Nucleons Scattered by Nuclei*

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The Serber model of high-energy nuclear interactions permits one to express the "optical-model" potential for nucleons in terms of nucleon-nucleon scattering amplitudes. The potential is written in terms of four well-depth parameters in the form

$$\mathcal{V}_C=-[V_{CR}+iV_{CI}]\rho(r)+[V_{SR}+iV_{SI}]\left(\frac{\hbar}{\mu c}\right)^2\frac{1}{r}\frac{d\rho}{dr}\boldsymbol{\sigma}\cdot\mathbf{l},$$

and experiments necessary to determine the four parameters V_{CR}, V_{CI}, V_{SR}, and V_{SI} are discussed. The explicit relations between the parameters and nucleon-nucleon scattering amplitudes are given and existing experimental data are reviewed.

The phase shifts of Feshbach and Lomon have been used to calculate V_{CR}, V_{SR}, and V_{SI}. These values seem to reproduce the qualitative features of the "measured" values of these quantities. (A quantitative comparison is not possible, since analyses of the experimental data are not sufficiently complete.)

I. INTRODUCTION

THE optical model[1] of nuclear scattering provides a means of studying nuclear structure by comparing nucleon-nucleus with nucleon-nucleon scattering and pion-nucleus[2] with pion-nucleon scattering. In the present note we wish to review briefly the relevant relations between nucleon-nucleus and nucleon-nucleon scattering.[3] The general theory was given in reference 3. However, the results were not put into a convenient form for use.

II. RELATION OF THE OPTICAL-MODEL POTENTIAL TO NUCLEON-NUCLEON SCATTERING AMPLITUDES

To describe the nucleon-nucleon scattering amplitude M, we shall use the notation of Wright.[4] He writes for the matrix M in the spin space of the nucleons:

$$\begin{aligned}M=&BS+C(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\hat{n}+N(\boldsymbol{\sigma}_1\cdot\hat{n})(\boldsymbol{\sigma}_2\cdot\hat{n})T\\&+\tfrac{1}{2}G[(\boldsymbol{\sigma}_1\cdot\hat{n}_-)(\boldsymbol{\sigma}_2\cdot\hat{n}_-)+(\boldsymbol{\sigma}_1\cdot\hat{n}_+)(\boldsymbol{\sigma}_2\cdot\hat{n}_+)]T\\&+\tfrac{1}{2}H[(\boldsymbol{\sigma}_1\cdot\hat{n}_-)(\boldsymbol{\sigma}_2\cdot\hat{n}_-)-(\boldsymbol{\sigma}_1\cdot\hat{n}_+)(\boldsymbol{\sigma}_2\cdot\hat{n}_+)]T.\end{aligned}\qquad(1)$$

Here S and T are the respective singlet and triplet spin projection operators. $\hat{n}$, $\hat{n}_-$, $\hat{n}_+$ are unit vectors in the directions of $\mathbf{p}\times\mathbf{p}'$, $\mathbf{p}'-\mathbf{p}$, and $\mathbf{p}'+\mathbf{p}$, respectively, where $\mathbf{p}'$ and $\mathbf{p}$ are the final and initial center-of-mass (abbreviated c.m.) momenta of the particles. The coefficients B, C, N, G, H depend upon energy and the c.m. scattering angle θ. To distinguish between proton-proton and proton-neutron scattering, we shall use subscripts PP and NP on the coefficients B, C, etc.

To describe the *elastic* scattering of a nucleon by a nucleus in the laboratory system,[5] we suppose the initial and final nucleon momenta to be $\mathbf{P}$ and $\mathbf{P}'$, respectively. The scattering occurs primarily at small angles, so we may consider the nucleus to be at rest both before and after the scattering, and suppose the nucleon energy, E_{lab} (which is the sum of kinetic and rest energy), to be unchanged by the scattering. The "center-of-mass" energy is

$$E_{\text{c.m.}}=c[p^2+M^2c^2]^{\frac{1}{2}}.\qquad(2)$$

The nuclear radius, R_A, volume, V_A, and density, $\rho(x)$, are related by

$$\begin{aligned}\int\rho(x)d^3x&=V_A,\\V_A&=(4/3)\pi R_A^3,\\R_A&=(\hbar/\mu c)\lambda A^{\frac{1}{3}},\end{aligned}\qquad(3)$$

where μ is the pion rest mass and λ is a dimensionless scale parameter. Consequently $\lambda\simeq 1$.

In momentum space the optical-model potential is given by[3]

$$(\mathbf{P}'|\mathcal{V}_C|\mathbf{P})=-\frac{3}{\lambda^3}(\mu c^2)\left[\frac{E_{\text{c.m.}}\mu^2c}{E_{\text{lab}}m\hbar}\right]\langle\bar{M}\rangle\frac{1}{(2\pi\hbar)^3}\times\int\exp\left[-\frac{i}{\hbar}(\mathbf{P}'-\mathbf{P})\cdot\mathbf{x}\right]\rho(x)d^3x,\qquad(4)$$

where $\langle\bar{M}\rangle$ is the M-matrix of Eq. (1) averaged over PP and NP interactions and over the spins of the nuclear particles, and m is the rest mass of the nucleon. The last factor above implies that small scattering angles will be most important, so we expand to *first order* in the scattering angle θ. If we take $Z=\frac{1}{2}A$ for

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[1] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).

[2] For the scattering of pions, the dispersion relation of M. L. Goldberger [Phys. Rev. **99**, 979 (1955)] permit one to establish some general relations between the scattering from nucleons and the scattering from nuclei, if one uses the simple optical model. These relations are discussed by Frank, Gammel, and Watson (submitted to *The Physical Review*).

[3] G. Takeda and K. Watson, Phys. Rev. **94**, 1087 (1954) and Phys. Rev. **97**, 1336 (1955).

[4] S. C. Wright, Phys. Rev. **99**, 996 (1955).

[5] We are treating the nucleus as being infinitely heavy. For nonrelativistic scattering it is more nearly accurate to suppose that our formulas apply to the nucleon-nucleus center-of-mass coordinate system.

the nucleus, so that $\bar{M}=\frac{1}{2}[M_{PP}+M_{NP}]$ we have

$$\langle \bar{M}\rangle=\bar{M}_0+i\boldsymbol{\sigma}\cdot[(\mathbf{p}'\times\mathbf{p})/p^2]\bar{M}_1 \tag{5}$$

with [see Eq. (1)]

$$\begin{aligned}\bar{M}_0&=\tfrac{1}{8}\{[B+N+G]_{PP}+[B+N+G]_{NP}\}\,|_{\theta=0},\\ \bar{M}_1&=-\frac{1}{2i}\left[\frac{1}{\sin\theta}(C_{PP}+C_{NP})\right]\Bigg|_{\theta=0}.\end{aligned} \tag{6}$$

It is convenient to eliminate $\mathbf{p}'\times\mathbf{p}/p^2$ from Eq. (5), using

$$\frac{\mathbf{p}'\times\mathbf{p}}{p^2}=\frac{2E_{\text{c.m.}}}{mc^2}\frac{\mathbf{P}'\times\mathbf{P}}{P^2}, \tag{7}$$

which is valid to first order in θ. We also introduce the following definitions:

$$\begin{aligned}\Gamma&\equiv\frac{3}{\lambda^3}[\mu c^2]\left[\frac{E_{\text{c.m.}}\mu}{E_{\text{lab}}m}\left(\frac{\mu c}{\hbar}\right)\right],\\ Q&\equiv\left[\frac{2E_{\text{c.m.}}}{mc^2}\right]\left[\frac{\mu c}{P}\right]^2,\\ V_{CR}&\equiv+\Gamma\,\mathrm{Re}\{\bar{M}_0\},\\ V_{CI}&\equiv+\Gamma\,\mathrm{Im}\{\bar{M}_0\},\\ V_{SR}&\equiv-Q\Gamma\,\mathrm{Re}\{\bar{M}_1\},\\ V_{SI}&\equiv-Q\Gamma\,\mathrm{Im}\{\bar{M}_1\}.\end{aligned} \tag{8}$$

Using Eqs. (5), (7), and (8), we may write Eq. (4) in the form

$$\begin{aligned}(\mathbf{P}'|\mathcal{V}_C|\mathbf{P})=&\left\{-(V_{CR}+iV_{CI})+\left(\frac{1}{\mu c}\right)^2 i\boldsymbol{\sigma}\cdot\mathbf{P}'\times\mathbf{P}\right.\\ &\left.\times[V_{SR}+iV_{SI}]\right\}\left\{\frac{1}{(2\pi\hbar)^3}\int\rho(x)\right.\\ &\left.\times\exp\left[-\frac{i}{\hbar}(\mathbf{P}'-\mathbf{P})\cdot\mathbf{x}\right]d^3x\right\}.\end{aligned} \tag{9}$$

Together with Eqs. (6) and (8), this expresses the nucleon-nucleus "optical-model" potential explicitly in terms of the nucleon-nucleon scattering amplitudes. As we shall discuss below, the four quantities V_{CR}, V_{CI}, V_{SR}, and V_{SI}, are directly measurable, as are also the coefficients in Eq. (1). Thus a detailed experimental study of the validity of the optical model is possible.

A somewhat more useful form for Eq. (9) is obtained on transforming $\mathcal{V}_C$ to a coordinate representation. To an approximation consistent with our assumption of small θ, we have[6]

$$\begin{aligned}\mathcal{V}_C(\mathbf{x})=&-[V_{CR}+iV_{CI}]\rho(x)\\ &+[V_{SR}+iV_{SI}]\left(\frac{\hbar}{\mu c}\right)^2\frac{1}{x}\frac{d\rho}{dx}\boldsymbol{\sigma}\cdot\mathbf{l},\end{aligned} \tag{10}$$

where

$$\mathbf{l}=\mathbf{x}\times\left(\frac{1}{i}\nabla\right) \tag{11}$$

is the nucleon angular momentum operator. The spin-orbit term above differs from that usually considered in that it contains an imaginary term, V_{SI}, in the well depth.

III. EXPANSION OF OPTICAL MODEL POTENTIALS IN TERMS OF NUCLEON-NUCLEON SCATTERING PHASE SHIFTS

We may also easily express $\mathcal{V}_C$ in terms of the nucleon-nucleon scattering phase shifts. We again use Wright's notation[4]:

ϵ_J = mixing parameter for angular momentum J,
δ_J^1, δ_J^2 = two eigenphase shifts for triplet states of parity $=-(-1)^J$,
$\delta_{J,1}$ = phase shifts for triplet states with parity $=(-1)^J$,
$\delta_{J,0}$ = phase shift for singlet states,

and in addition we write explicitly

$$\delta(^3P_0)=\text{phase shift for scattering in the } {}^3P_0 \text{ state.} \tag{12}$$

Then

$$\begin{aligned}V_{CR}=&+\left[\frac{\hbar}{16p}\Gamma\right]\{[\sum_{l(\text{odd})}+3\sum_{l(\text{even})}](2l+1)\sin2\delta_{l,0}\\ &+[\sum_{l(\text{even})}+3\sum_{l(\text{odd})}][(2l+1)\sin2\delta_{l,1}\\ &+(2l+3)(\sin2\delta_{l+1}{}^1+\sin2\delta_{l+1}{}^2)]\\ &+3\sin2\delta(^3P_0)\},\\ V_{SR}=&\left[\frac{\hbar}{32p}Q\Gamma\right]\Big\{[\sum_{l(\text{even})}+3\sum_{l(\text{odd})}]\{-(2l+1)\sin2\delta_{l,1}\\ &+(2l+3)[\sin2\delta_{l+1}{}^1(l\cos^2\epsilon_{l+1}-(l+3)\sin^2\epsilon_{l+1})\\ &+\sin2\delta_{l+1}{}^2(l\sin^2\epsilon_{l+1}-(l+3)\cos^2\epsilon_{l+1})]\}\\ &-6\sin2\delta(^3P_0)\Big\},\\ V_{SI}=&\left[\frac{\hbar}{16p}Q\Gamma\right]\Big\{[\sum_{l(\text{even})}+3\sum_{l(\text{odd})}]\{-(2l+1)(\sin\delta_{l,1})^2\\ &+(2l+3)[(\sin\delta_{l+1}{}^1)^2(l\cos^2\epsilon_{l+1}-(l+3)\sin^2\epsilon_{l+1})\\ &+(\sin\delta_{l+1}{}^2)^2(l\sin^2\epsilon_{l+1}-(l+3)\cos^2\epsilon_{l+1})]\}\\ &-6(\sin^2\delta(^3P_0))^2\Big\}.\end{aligned} \tag{13}$$

The form for V_{CI} is most easily given in terms of total nucleon-nucleon cross sections by means of the relation

$$V_{CI}=\hbar v_{\text{lab}}/2\lambda_S, \tag{14}$$

[6] See the appendix for a derivation.

where $v_{\rm lab}$ is the nucleon velocity in the laboratory frame of reference and

$$\frac{1}{\lambda_S}=\frac{3}{4\pi}\left[\frac{1}{\lambda^3}\right]\left[\frac{\mu c}{\hbar}\right]\left[\bar{\sigma}\left(\frac{\mu c}{\hbar}\right)^2\right]. \tag{15}$$

In this expression $\bar{\sigma}$ is the average nucleon-nucleon cross section in the nucleus[7,8]:

$$\bar{\sigma}=\tfrac{1}{2}\gamma[\sigma_{NP}+\sigma_{PP}], \tag{16}$$

where σ_{NP} and σ_{PP} are the total scattering cross sections for nucleons on free nucleons and γ is Goldberger's correction factor for binding effects. For energies greater than 100 Mev, $\gamma\simeq 1$.

IV. THE PARAMETERS OF THE M MATRIX AND POLARIZATION EXPERIMENTS IN NUCLEON-NUCLEUS SCATTERING

The effective nucleon-nucleus scattering matrix assumes the form (we are neglecting the nuclear spin)

$$M_N=A+L\boldsymbol{\sigma}\cdot\hat{n}, \tag{17}$$

where $\hat{n}$ is a unit normal to the plane of scattering, and A and L are functions of the scattering angle, readily expressible in terms of the nucleon-nucleon scattering parameters B, G, C, and N, or in terms of the phase shifts and mixing parameters as defined by Wright.[4] Hence the matrix M_N, and consequently the four optical-model potentials V_{CR}, V_{CI}, V_{SR}, V_{SI}, are directly measurable quantities. At each scattering angle only three independent quantities, such as the magnitudes and relative phase of A and L, need be determined. This can be done by means of angular distribution and polarization measurements in double and triple scattering experiments.[9,10]

Letting ρ_0 be the von Neumann density matrix in the spin space of the incoming nucleon beam, we may write the differential cross section $I_1(\theta)$ after the first scattering in the form

$$I_1(\theta)=\mathrm{Tr}[M_1\rho_0M_1^\dagger]/\mathrm{Tr}[\rho_0], \tag{18}$$

where

$$M_1=A+L\boldsymbol{\sigma}\cdot\hat{n}_1.$$

is the M-matrix describing the first scattering. The polarization $\mathbf{P}_1$ after the first scattering is given by

$$\mathbf{P}_1=\mathrm{Tr}[\boldsymbol{\sigma}M_1\rho_0M_1^\dagger]/\mathrm{Tr}[M_1\rho_0M_1^\dagger]. \tag{19}$$

If the incident beam is unpolarized, we may write

$$\rho_0=\tfrac{1}{2}\mathfrak{I} \tag{20}$$

and obtain

$$\begin{aligned}I_1&=\tfrac{1}{2}\,\mathrm{Tr}[M_1M_1^\dagger]\\&=|A|^2+|L|^2,\end{aligned} \tag{21}$$

$$\begin{aligned}\mathbf{P}_1&=\mathrm{Tr}[\boldsymbol{\sigma}M_1M_1^\dagger]/2I_1\\&=\hat{n}_1(J_1/I_1),\end{aligned} \tag{22}$$

where

$$J_1=A^*L+AL^*. \tag{23}$$

The magnitude of the polarization $\mathbf{P}_1$ may be measured by letting the scattered beam undergo a second scattering in the same plane through the same angle, and observing the azimuthal right-left asymmetry in the scattered intensity. If the normal $\hat{n}_2$ to the plane of the second scattering is defined to be parallel to $\hat{n}_1$ for "right" scattering and antiparallel for "left" scattering, then we may write the M-matrix describing the second scattering in the form

$$M_{2R}=A+L\boldsymbol{\sigma}\cdot\hat{n}_1,\quad M_{2L}=A-L\boldsymbol{\sigma}\cdot\hat{n}_1. \tag{24}$$

The differential cross section I_2 has the general form

$$I_2=\mathrm{Tr}[M_2M_1\rho_0M_1^\dagger M_2^\dagger]/\mathrm{Tr}[M_1\rho_0M_1^\dagger]. \tag{25}$$

Using (20), (21), (23), and (24), the right-left asymmetry ratio becomes

$$\begin{aligned}\frac{I_{2R}-I_{2L}}{I_{2R}+I_{2L}}&=\frac{\mathrm{Tr}[(I_1+J_1\boldsymbol{\sigma}\cdot\hat{n}_1)J_1\boldsymbol{\sigma}\cdot\hat{n}_1]}{\mathrm{Tr}[(I_1+J_1\boldsymbol{\sigma}\cdot\hat{n}_1)I_1]}\\&=(J_1/I_1)^2=(\mathbf{P}_1\cdot\hat{n}_1)^2,\end{aligned} \tag{26}$$

which is a positive definite quantity.

Thus the measurements of cross section and polarization in a double scattering experiment determine two parameters, I_1 and $|J_1|$ for example. The third independent parameter may be found by letting the second scattering take place in a plane perpendicular to the plane of the first scattering, and measuring the polarization $\mathbf{P}_2$ of the scattered beam by means of a right-left asymmetry determination in a third scattering. The polarization after the second scattering is given by

$$\mathbf{P}_2=\mathrm{Tr}[\boldsymbol{\sigma}M_2M_1\rho_0M_1^\dagger M_2^\dagger]/\mathrm{Tr}[M_2M_1\rho_0M_1^\dagger M_2^\dagger] \tag{27}$$

which for the special case under consideration becomes

$$\mathbf{P}_2=\frac{1}{I_1^2}[\hat{n}_1J_1G_1+\hat{n}_2I_1J_1+\hat{n}_1\times\hat{n}_2J_1K_1], \tag{28}$$

where

$$\begin{aligned}G_1&=|A|^2-|L|^2,\\K_1&=i[AL^*-A^*L],\\\hat{n}_1\cdot\hat{n}_2&=0.\end{aligned} \tag{29}$$

Since I_1, J_1, and G_1 are independent they determine the M matrix up to a trivial phase factor, and K_1, being similarly fixed, is a dependent parameter. Thus the triple scattering experiments furnish an overdetermination of the parameters and consistency checks are

[7] M. L. Goldberger, Phys. Rev. **74**, 1269 (1948).
[8] N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953).
[9] L. Wolfenstein, Phys. Rev. **96**, 1654 (1954); L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).
[10] R. H. Dalitz, Proc. Phys. Soc. (London) **A65**, 175 (1952).

possible. If the third scattering takes place in a plane parallel to the first scattering, then the sequence of scatterings is described by the following set of matrices:

$$\begin{aligned} M_1 &= A+L\boldsymbol{\sigma}\cdot\hat{n}_1,\\ M_2 &= A+L\boldsymbol{\sigma}\cdot\hat{n}_2,\\ M_{3R} &= A+L\boldsymbol{\sigma}\cdot\hat{n}_1,\\ M_{3L} &= A-L\boldsymbol{\sigma}\cdot\hat{n}_1,\\ \hat{n}_1\cdot\hat{n}_2 &= 0. \end{aligned} \tag{30}$$

The scattered intensity after the third scattering is given by

$$I_3=\mathrm{Tr}[M_3M_2M_1M_1^\dagger M_2^\dagger M_3^\dagger]/\mathrm{Tr}[M_2M_1M_1^\dagger M_2^\dagger]. \tag{31}$$

With the use of (22), (28), and (29) a brief calculation yields the following for the asymmetry ratio:

$$\frac{I_{3R}-I_{3L}}{I_{3R}+I_{3L}}=\frac{J_1^2G_1}{I_1^3}=(\mathbf{P}_1\cdot\hat{n}_1)(\mathbf{P}_2\cdot\hat{n}_1). \tag{32}$$

This type of experiment measures the component of polarization along $\hat{n}_1$ of the beam after the second scattering, and thus provides a direct determination of the parameters I_1, J_1, and G_1 (except for the sign of J_1) without overdetermination.

A more probable experimental situation is one in which the beam of nucleons entering the experimental area has a known polarization $\mathbf{P}_1$. This is the case when the first scattering, which occurs at a target inside the accelerator, is used to extract the beam. A subsequent external scattering with $\hat{n}$ parallel to $\mathbf{P}_1$ leads to the scattered intensities:

$$\left.\begin{matrix} I_{2R}\\ I_{2L}\end{matrix}\right\}=\frac{\mathrm{Tr}[I_1\rho_1\pm J_1\rho_1\boldsymbol{\sigma}\cdot\hat{n}]}{\mathrm{Tr}[\rho_1]}, \tag{33}$$

where ρ_1 is the density matrix for the incoming beam corresponding to polarization $\mathbf{P}_1$; i.e., $\mathbf{P}_1=\mathrm{Tr}[\rho_1\boldsymbol{\sigma}]/\mathrm{Tr}[\rho_1]$. The asymmetry ratio becomes

$$\frac{I_{2R}-I_{2L}}{I_{2R}+I_{2L}}=\frac{J_1\,\mathrm{Tr}[\rho_1\boldsymbol{\sigma}\cdot\hat{n}]}{I_1\,\mathrm{Tr}[\rho_1]}=(J_1/I_1)P_1. \tag{34}$$

Thus J_1 is determined.

Similarly, a double scattering experiment for which the normal $\hat{n}_1$ to the plane of the first external scattering is perpendicular to $\mathbf{P}_1$, and for which the normal $\hat{n}_2$ to the plane of the second external scattering is parallel to $\mathbf{P}_1$, leads to the asymmetry ratio

$$\frac{I_{3R}-I_{3L}}{I_{3R}+I_{3L}}=\frac{J_1G_1}{I_1^2}P_1. \tag{35}$$

This suffices to determine G_1.

As a simple and instructive example, let us consider scattering nearly in the forward direction from a light nucleus. In this case we may use the Born approximation to calculate M_N:

$$M_N\cong-(2\pi)^2(\hbar/c^2)E_{\mathrm{lab}}(\mathbf{P}'|\mathcal{V}_C|\mathbf{P}). \tag{36}$$

By comparing Eqs. (17) and (36), and using (9) to substitute for $\mathcal{V}_C$, we obtain

$$\begin{aligned} A&=+(2\pi)^2\frac{\hbar}{c^2}E_{\mathrm{lab}}[V_{CR}+iV_{CI}]\frac{1}{(2\pi\hbar)^3}\times\int d^3x\rho(x)\exp\left[-\frac{i}{\hbar}(\mathbf{P}'-\mathbf{P})\cdot\mathbf{x}\right],\\ L&=-(2\pi)^2\frac{\hbar}{c^2}E_{\mathrm{lab}}[V_{SR}+iV_{SI}]\left[i\left(\frac{P}{\mu c}\right)^2\sin\theta_{\mathrm{lab}}\right]\frac{1}{(2\pi\hbar)^3}\times\int d^3x\rho(x)\exp\left[-\frac{i}{\hbar}(\mathbf{P}'-\mathbf{P})\cdot\mathbf{x}\right]. \end{aligned} \tag{37}$$

As an approximation we may assume that V_{CR} and V_{CI} are known from simple scattering experiments. Then knowledge of G_1 and J_1 determine *both* V_{SR} and V_{SI}, as can easily be seen by substituting the solution (37) into (34) and (35). There results

$$\frac{I_{2R}-I_{2L}}{I_{2R}+I_{2L}}=-2P_1\frac{[-V_{CR}V_{SI}+V_{CI}V_{SR}](P/\mu c)^2\sin\theta_{\mathrm{lab}}}{V_{CR}^2+V_{CI}^2+[(P/\mu c)^2\sin\theta_{\mathrm{lab}}]^2[V_{SR}^2+V_{SI}^2]}\equiv P_1\frac{J_1}{I_1}, \tag{38}$$

$$\frac{I_{3R}-I_{3L}}{I_{3R}+I_{3L}}=\left[\frac{J_1}{I_1}P_1\right]\times\frac{\{V_{CR}^2+V_{CI}^2-[(P/\mu c)^2\sin\theta_{\mathrm{lab}}][V_{SR}^2+V_{SI}^2]\}}{\{V_{CR}^2+V_{CI}^2+[(P/\mu c)^2\sin\theta_{\mathrm{lab}}]^2[V_{SR}^2+V_{SI}^2]\}}.$$

Most previous investigations have ignored the term V_{SI}.

A similar analysis may be carried through for the nucleon-nucleon scattering problem.[9,11] The M-matrix then will have the form given by Eq. (1) which is the most general expression for the scattering of two spin-$\frac{1}{2}$ particles, assuming symmetry of the interaction under spin exchange. In this case there are nine independent parameters, and again the measurement of angular distribution and asymmetry ratios in triple scattering processes serves to determine these parameters (although much more complex experiments are needed[9]).

[11] Henry P. Stapp, thesis, University of California Radiation Laboratory Report 2825 (unpublished).

V. EXPERIMENTAL VALUES FOR THE OPTICAL-MODEL POTENTIALS

A number of authors[12–21] have employed the optical model to fit the nucleon-nucleus scattering data, and several have interpreted the polarization data in terms of an effective spin-orbit potential as suggested by Fermi.[22] With various assumptions concerning the nuclear radius and shape of the function $\rho(x)$, they estimate numerical values for the optical-model parameter as a function of energy.

Taylor[12] uses the cross-section data for neutrons on Cd, Cu, Al, and C at 85 Mev to evaluate well depths, nuclear radius parameters, and absorption mean free paths as a function of neutron energy ranging from 30 to 400 Mev. Woods and Saxon, and Le Levier and Saxon[13] fit values for V_{CR} and V_{CI} to agree with the elastic cross sections of protons at energies in the neighborhood of 20 Mev on Al, Ni, Pt, Pd, and W, using a diffuse surface optical model. The values quoted by Melkanoff, Moszkowski, Nodvik, and Saxon[14] at 17 and 31.5 Mev were obtained from an analysis of elastic scattering of protons on elements ranging from Fe to Pb, and the 5.25-Mev values from protons on Ni. Clementel and Villi[15] calculate V_{CI} as a function of energy up to 200 Mev, using a Fermi-gas model for the nucleus and relating the energy dependence of V_{CI} to that of the neutron-proton total cross section. Kind and Villi[16] derive a perturbation theoretic expression for the optical-model potential V_{CR} on the basis of the independent particle model, starting from a nucleon-nucleon interaction of Yukawa type. They fit neutron scattering data at energies from 50 to 300 Mev.

It is to be noted that the above-cited investigations[12–16] do not take account of the spin-orbit potential, and that values for the spin-independent potentials will in general depend on the magnitude of the spin-orbit interaction.

Fernbach, Heckrotte, and Lepore[17] and Tamor[18] include a discussion of the spin-orbit interaction in their analyses, and quote values for V_{CR}, V_{CI}, and V_{SR} which give agreement with the 290-Mev scattering data. Snow, Sternheimer, and Yang[19] investigate the effect of the spin-orbit term on polarization, and fit the polarization data[23] for 316-Mev nucleons on Be,

TABLE I. Compilation of numerical values for the optical-model potentials from scattering and polarization data.

Energy Mev	V_{CR} Mev	Reference	Energy Mev	V_{CI} Mev	Reference
<3	42	21	0	1.6	15
5.25	52.5	14	<3	2	21
17	45.5	13, 14	5	3.45	15
20	38	13	5.25	0.9	14
20	30	20	10	5.38	15
31.5	35	13, 14	17	8.5	13, 14
50	30	12	20	9	13
50	28	16	20	20	20
100	20	16	20	8.73	15
100	19.5	12	31.5	15	13, 14
150	15	12	40	11.5	15
150	17	16	50	8.6	12
200	14	12	80	12.5	15
200	12	16	100	12.1	15
250	13	12	100	7.7	12
250	10	16	150	7.4	12
300	9	16	150	10.9	15
400	12	12	200	8.0	12
Energy Mev	V_{SR} Mev	Reference	200	9.85	15
290	2.5	17	290	18	17
316	0.58[a]	19	400	14	12
316	1.23[b]	19			

[a] Square well for V_C.
[b] Harmonic oscillator potential for V_C.

assuming for $\mathcal{V}_C$ both square-well and harmonic-oscillator potentials and for V_{SR} various shapes including the Fermi-Thomas precession form given by Eq. (10).

These results, together with those of Chase and Rohrlich[20] for 20-Mev protons on Al, Cu, and Ag, and those of Feshbach, Porter, and Weisskopf[21] for low-energy neutrons are summarized in Table I and by the graphs of Figs. 1 and 2. For comparison, values of V_{CI} obtained from total nucleon-nucleon cross sections [see Eq. (14)] are included in Fig. 1.

VI. EVALUATION OF POTENTIALS FROM FESHBACH-LOMON PHASE SHIFTS

Recently, Feshbach and Lomon[24] have given a phase-shift analysis of the nucleon-nucleon scattering cross

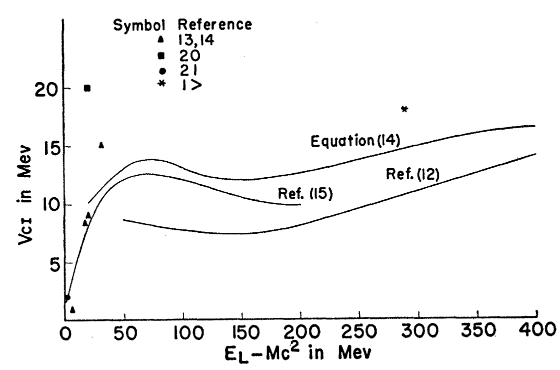

FIG. 1. Several calculated and measured values of V_{CI}.

[12] T. B. Taylor, Phys. Rev. **92**, 831 (1953).
[13] R. D. Woods and D. S. Saxon, Phys. Rev. **95**, 577 (1954); R. E. Le Levier and D. S. Saxon, Phys. Rev. **87**, 40 (1952).
[14] Melkanoff, Moszkowski, Nodvik, and Saxon (to be published).
[15] E. Clementel and C. Villi, Nuovo cimento **2**, 176 (1955); see also A. M. Lane and C. F. Wandel, Phys. Rev. **98**, 1524 (1955).
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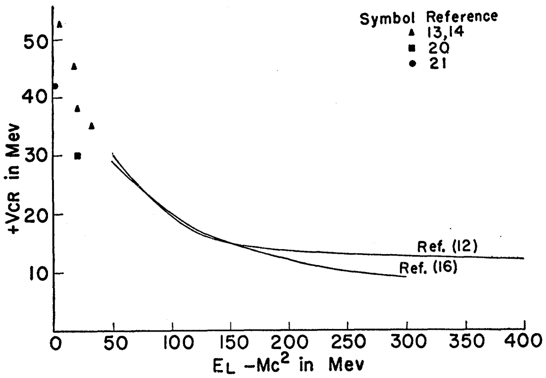

FIG. 2. Experimental values of V_{CR}. The curve from reference 16 represents calculated values.

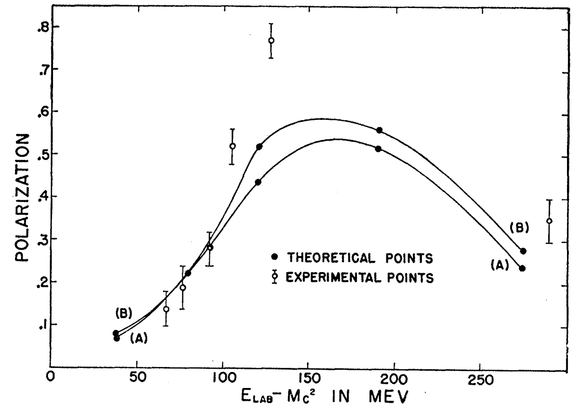

FIG. 4. Calculated polarization of nucleons scattered from carbon at $\theta_{\text{lab}}=20°$. The values of V_{CR}, V_{SR}, and V_{SI} were obtained from the Feshbach-Lomon phase shifts. The experimental points are those of Dickson, Rose, and Salter [Proc. Phys. Soc. (London) **68**, 361 (1955)]. The point at 290 Mev is due to Chamberlain, Segrè, Wiegand, Tripp, and Ypsilantis (quoted in reference 17). The error indicated has been estimated from the scatter of the experimental points. The curve is drawn from points calculated only at the five energies shown, so the extent of the discrepancy between calculated and experimental value at 130 Mev is not clear. Also, for quantitative comparison more accurate evaluation than ours from Eq. (39) are required.

sections. Using their phase shifts, we have evaluated V_{CR}, V_{SR} and V_{SI} from Eqs. (13). The results are shown in Table II and Fig. 3 for their two sets of phase shifts A and B. In making a comparison with the "experimental" values for these quantities, we must recall that: (1) except for the work of Fernbach, Heckrotte, and Lepore[17] (who obtain $V_{CR}\simeq 0$ at 290 Mev), most determinations of V_{CR} have neglected V_{SR} and V_{SI}; (2) determinations of V_{SR} have neglected V_{SI}; (3) values quoted for these parameters do not agree sufficiently well to permit a quantitative comparison with our calculated values. It seems, however, that the qualitative features of the V's are correctly given.

These values of the V's, along with V_{CI} as given by Eq. (14), have been used to calculate the polarization of nucleons scattered from carbon at $\theta_{\text{lab}}=20°$. For this calculation the Coulomb force is neglected (for the case when the incident particle is a proton) and the spin-orbit term is treated in Born approximation. That

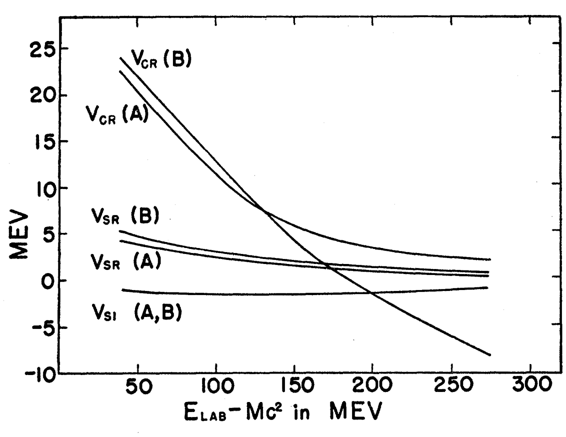

FIG. 3. Values of V_{CR}, V_{SR}, and V_{SI} calculated from the Feshbach-Lomon phase shifts. The labels "A" and "B" refer to the two sets of phase shifts of these authors.

is, we take

$$M_N = M_N(0) - (2\pi)^2 E_{\text{lab}} (\psi_0^{(-)}, \mathcal{V}_{S0}\psi_1^{(+)}), \tag{39}$$

where $M_N(0)$, $\psi_0^{(+)}$, $\psi_0^{(-)}$ are the scattering amplitudes and wave functions for the central potential only and have been evaluated by a simple eikonal approximation. V_{S0} is the spin-orbit term in Eq. (10). The polarization, calculated at five energies, is shown in Fig. 4.

VII. CONCLUSIONS

Use of Eqs. (6) and (8) enables one to express the parameters of the optical-model potential directly in terms of nucleon-nucleon scattering amplitudes. The implication of this is that the potential may be specifically obtained from the results of appropriate nucleon-nucleon scattering experiments. It may also be "meas-

TABLE II. Optical model potentials calculated from Eq. (13) using the two sets (A) and (B) of Feshbach-Lomon phase shifts.[a] All quantities are in Mev.

Kin. energy (lab)	V_{CR} A	V_{CR} B	V_{SR} A	V_{SR} B	V_{SI} A	V_{SI} B
38.5	22.53	23.88	4.23	5.32	−1.13	−1.06
80	14.88	16.86	2.81	3.45	−1.49	−1.40
120	8.65	9.01	2.16	2.76	−1.49	−1.44
190	4.05	−0.57	1.19	1.65	−1.35	−1.39
274	2.23	−8.19	0.33	0.78	−0.83	−0.94

[a] See reference 24.

ured" by studying the elastic scattering of nucleons by nuclei. Comparison of these two sets of values should give an indication of the validity of the Serber model of high-energy nuclear reactions, in which it is assumed that "two-body" processes determine the scattering. This, in turn, has a bearing on a number of related questions such as the relative importance of two-body *versus* many-body forces,[25–27] the relative importance of "compound nucleus" effects *versus* "knock-on" processes,[28] and the contribution from "nonlinear" meson effects.

APPENDIX

We shall derive here the coordinate-space expression[17,29] for the optical model potential.

The optical-model potential in momentum space has the form

$$(\mathbf{P}'|\mathcal{V}_C|\mathbf{P}) = -\Gamma\left(\bar{M}_0 + 2i\frac{E_{\mathrm{c.m.}}}{mc^2}\boldsymbol{\sigma}\cdot\frac{\mathbf{P}'\times\mathbf{P}}{P^2}\bar{M}_1\right)(2\pi\hbar)^{-3}$$

$$\times\int d^3y\rho(y)\exp\left[-\frac{i}{\hbar}(\mathbf{P}'-\mathbf{P})\cdot\mathbf{y}\right]$$

$$=\left\{V_C+\frac{1}{(\mu c)^2}i\boldsymbol{\sigma}\cdot\mathbf{P}'\times\mathbf{P}V_S\right\}(2\pi\hbar)^{-3}$$

$$\times\int d^3y\rho(y)\exp\left[-\frac{i}{\hbar}(\mathbf{P}'-\mathbf{P})\cdot\mathbf{y}\right].$$

using the definitions and notation established in the text, with

$$V_C = -(V_{CR}+iV_{CI}),\quad V_S = V_{SR}+iV_{SI}.$$

The corresponding expression in the coordinate space of the incoming nucleon is obtained by taking the Fourier transform of the foregoing. Thus

$$(\mathbf{x}'|\mathcal{V}_C|\mathbf{x}) = \int\frac{d^3P'd^3P}{(2\pi\hbar)^6}\exp\left(\frac{i}{\hbar}\mathbf{x}'\cdot\mathbf{P}'\right)$$

$$\times\left\{V_C+\frac{1}{(\mu c)^2}V_S i\boldsymbol{\sigma}\cdot\mathbf{P}'\times\mathbf{P}\right\}\rho(y)d^3y$$

$$\times\exp\left[-\frac{i}{\hbar}(\mathbf{P}'-\mathbf{P})\cdot\mathbf{y}\right]\exp\left(-\frac{i}{\hbar}\mathbf{x}\cdot\mathbf{P}\right)$$

$$=\int\frac{d^3yd^3P'd^3P}{(2\pi\hbar)^6}\left\{V_C+\left(\frac{\hbar}{\mu c}\right)^2V_S i\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}'\times\boldsymbol{\nabla}\right\}$$

$$\times\rho(y)\exp\left[\frac{i}{\hbar}(\mathbf{y}-\mathbf{x})\cdot\mathbf{P}\right]\exp\left[\frac{i}{\hbar}(\mathbf{x}'-\mathbf{y})\cdot\mathbf{P}'\right].$$

In view of the small-scattering-angle approximation, the integrations over $\mathbf{P}$ and $\mathbf{P}'$ may be performed by moving the term within the brackets outside the integral sign. The symbols $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}'$ denote gradient operators with respect to the $\mathbf{x}$ and $\mathbf{x}'$ coordinates, respectively. The resultant delta functions render the integration over $\mathbf{y}$ trivial, and there results

$$(\mathbf{x}'|\mathcal{V}_C|\mathbf{x})\cong\left\{V_C+\left(\frac{\hbar}{\mu c}\right)^2V_S i\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}'\times\boldsymbol{\nabla})\right\}[\rho(\mathbf{x})\delta(\mathbf{x}'-\mathbf{x})].$$

If furthermore it is assumed that $\rho(\mathbf{x})$ depends only on the magnitude of $\mathbf{x}$, then it follows that

$$(\nabla'\times\nabla)[\rho(x)\delta(\mathbf{x}'-\mathbf{x})] = -\frac{1}{x}\frac{d\rho}{dx}(\mathbf{x}\times\nabla')\delta(\mathbf{x}'-\mathbf{x})$$

$$=\frac{1}{i}\frac{1}{x}\frac{d\rho}{dx}\mathbf{l}\delta(\mathbf{x}'-\mathbf{x}),$$

where $\mathbf{l}$ is the nucleon orbital angular momentum operator. There results finally

$$(\mathbf{x}'|\mathcal{V}_C|\mathbf{x}) = V_C\rho(x)\delta(\mathbf{x}'-\mathbf{x})+\left(\frac{\hbar}{\mu c}\right)^2V_S\frac{1}{x}\frac{d\rho}{dx}\boldsymbol{\sigma}\cdot\mathbf{l}\delta(\mathbf{x}'-\mathbf{x}).$$

The space-dependent potential operator $\mathcal{V}_C(\mathbf{x})$ is given by the relationship

$$(\mathbf{x}'|\mathcal{V}_C|\mathbf{x}) = \mathcal{V}_C(x)\delta(\mathbf{x}'-\mathbf{x}),$$

so that

$$\mathcal{V}_C(\mathbf{x}) = V_C\rho(x)+V_S\left(\frac{\hbar}{\mu c}\right)^2\frac{1}{x}\frac{d\rho}{dx}\boldsymbol{\sigma}\cdot\mathbf{l}.$$

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