Nuclear Matrix Elements for Allowed β Transitions*

W. C. GRAYSON, JR.,† ‡ AND L. W. NORDHEIM
Department of Physics, Duke University, Durham, North Carolina

(Received January 13, 1956)

Nuclear matrix elements are evaluated for all allowed β transitions in the strict j-j coupling shell model, i.e., for states of lowest seniority. The wave functions for the j^N configurations are obtained by algebraic means in the formalism of coefficients of fractional parentage. The matrix elements are given for the two assumptions, firstly that isotopic spin is a good quantum number, and secondly that the neutrons and are coupled separately to their lowest seniority states. The latter includes the case where the protons end in a $j=l+\frac{1}{2}$ shell, while the neutrons end in the $j=l-\frac{1}{2}$ shell. A similar explicit form is given for the magnetic moments of the states of the j^N configuration with seniority one.

I. INTRODUCTION

THIS paper is devoted to a derivation of the nuclear matrix elements for all allowed β transitions, using the strict j-j coupling shell model. As shown in the companion paper,[1] hereafter referred to as II, these matrix elements lead to a consistent treatment of the observed transitions which provides some insight into the validity of the model and, in particular, clarifies the role of configuration mixing.

Matrix elements have been calculated for specific transitions in this scheme by Wigner,[2] Feenberg,[3] Kurath,[4] Talmi,[5] and Kofoed-Hansen.[6] In the present treatment all matrix elements of this type are derived in a systematic manner by making use of the powerful techniques developed by Racah[7–10] and Flowers.[11,12]

The complete pairwise coupling of the strict j-j coupling shell model is expressed by choosing the states of lowest seniority,[8,11,13] i.e., of maximum symplectic[14] symmetry. These states have been shown[4,12,15–18] to lead to lowest energy for attractive charge-independent central forces of very short range.

For light nuclei neutrons and protons occupy the same levels and there is considerable evidence that the total isotopic spin, T, is approximately a good quantum number, at least for the ground states. The existence of favored transitions up through the $f_{7/2}$ shell[19] indicates that this should be a valid approximation up to $A\sim 56$.

For higher A the neutron excess becomes large and the last neutron levels are accessible to protons only at high excitation energies. Isotopic spin will then cease to be a good quantum number, and the shell model states will be better represented by coupling the protons and neutrons separately to their configurations of lowest seniority (odd-group coupling model).

The nuclear matrix elements are derived here for both of these extreme cases. However, while the expressions obtained differ in form, they are of comparable magnitude, so that the conclusions to be drawn in II do not depend sensitively on the actual purity of the isotopic spin states.

In Sec. II explicit forms are obtained for the wave functions of states of lowest seniority for the configuration j^N, using the coefficients of fractional parentage (cfp) technique of Racah.

In Sec. III these wave functions (cfp) are used to obtain the nuclear matrix elements for allowed β transitions.

II. WAVE FUNCTIONS FOR THE CONFIGURATION j^N

Assuming charge-independent central forces and introducing the classification according to symplectic symmetry, $\sigma=(s,t)$, of Flowers[11] and Umezawa,[13] the wave functions for the configuration $(nlj)^N\equiv j^N$ of N nucleons in the same shell may be written

$$\Psi=\Psi(j^N\alpha\sigma=(s,t)TT_\zeta JM), \tag{1}$$

where α denotes any additional quantum numbers which may be required to specify the state completely.

Such a state may be interpreted as one in which $N-s$ nucleons couple off in pairs with zero angular

* Work supported by the National Science Foundation and the U. S. Atomic Energy Commission.

† This article is based on a thesis submitted by W. C. Grayson, Jr., to the Graduate School of Arts and Sciences, Duke University, in partial fulfillment of the requirements for the degree of Doctor of Philosophy (1955).

‡ Present address: University of California, Radiation Laboratory, Livermore, California.

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[19] The last observed mirror transition is $Sc^{41}\rightarrow Ca^{41}$, the last triad $Co^{54}\rightarrow Fe^{54}$.

momentum and unit isotopic spin, while the remaining s nucleons couple to angular momentum J and isotopic spin t. The resultant total angular momentum is J, with z-component M, and resultant total isotopic spin T, with "z"-component T_ζ.

States of this type first occur for $N=s$ and $T=t$; s is called the seniority of the state and t its reduced isotopic spin.

In this scheme the ground states predicted by the strict j-j coupling shell model correspond to the states of lowest seniority, i.e., for even-even configurations $\sigma=(0,0)$ and $J=0$; for odd-odd configurations $\sigma=(2,0)$ and $J=1,\ 3,\ \cdots 2j$; for odd-even or even-odd configurations $\sigma=(1,\frac{1}{2})$ and $J=j$.

A. Coefficients of Fractional Parentage

In principle, the wave functions

$$\Psi(j^N\alpha\sigma=(s,t)TT_\zeta JM)$$

can be obtained by vector coupling the single particle wave functions to J, T in such a way that $N-s$ particles couple off in pairs with zero angular momentum and unit isotopic spin and the remaining s couple to J, t, then explicitly antisymmetrizing.[4]

However, for more than three particles this procedure becomes rather lengthy and it is more convenient to use the coefficients of fractional parentage (cfp) technique developed by Racah.[8]

In this scheme the antisymmetric states for N particles are obtained by vector-coupling the last particle to all possible parent states of the $N-1$ particle "ions," the latter being presumed already known from a previous calculation, i.e.,

$$\Psi(j^N\alpha\sigma TT_\zeta JM) = \sum_{\alpha_1\sigma_1T_1J_1}(j^N\alpha\sigma TJ\{\!|j^{N-1}(\alpha_1\sigma_1T_1J_1)j) \times\psi(j^{N-1}(\alpha_1\sigma_1T_1J_1),j_N;\,TT_\zeta JM), \quad (2)$$

where ψ is obtained by vector coupling the wave function of the Nth particle, $\phi_N(jm)$, to

$$\Psi(j^{N-1}\alpha_1\sigma_1T_1T_{1\zeta}J_1M_1),$$

and the coefficients

$$(j^N\alpha\sigma TJ\{\!|j^{N-1}(\alpha_1\sigma_1T_1J_1)j)$$

which give the fractional contribution from each parent state are called *coefficients of fractional parentage* (cfp).

Tables of the cfp in this scheme for the states of $j^{3,4}$ with $j=3/2,\ 5/2$ have been given by Edmonds and Flowers.[12] However, their method is difficult to extend to N particles since it involves a chain calculation and all of the cfp for j^{N-1} are required to obtain the cfp for j^N.

For the purpose of this paper the cfp are needed only for states with $\sigma=(0,0)$ and $\sigma=(1,\frac{1}{2})$. These simple cfp can be obtained for arbitrary N by a treatment closely analogous to that used by Racah[9] for atomic L-S coupling.

B. Factorization of the cfp

The evaluation of the cfp is considerably simplified by using a lemma of Racah[9,20] to factor the cfp into three factors, each depending on a smaller number of variables:

$$(j^{N-1}(\sigma_1\sigma_1T_1J_1)j|\!\}j^N\alpha\sigma TJ) = \langle N-1\ T_1|NT\rangle\langle j^{N-1}(\beta_1\sigma_1T_1)j|\,j^N\beta\sigma T\rangle \times\langle\beta_1\sigma_1\gamma_1J_1\times(1,\tfrac{1}{2})j|\beta\sigma\gamma J\rangle, \quad (3)$$

where the notation indicates the variables upon which each factor depends and $(\beta,\gamma)=\alpha$ denotes any additional quantum numbers which may be required to distinguish states with the same σ, J in this scheme.

In terms of these factors, the requirement that the wave functions be orthogonal and normalized leads to the following orthonormality conditions:

$$\sum_{T_1}|\langle N-1\ T_1|NT\rangle|^2=1, \quad (4a)$$

$$\sum_{\beta_1\sigma_1}\langle j^N\beta\sigma T|\,j^{N-1}(\beta_1\sigma_1T_1)j\rangle \times\langle j^{N-1}(\beta_1\sigma_1T_1)j|\,j^N\beta\sigma T'\rangle=\delta_{T,T'}, \quad (4b)$$

$$\sum_{\gamma_1j_1}\langle\beta\sigma\gamma J|\beta_1\sigma_1\gamma_1J_1\times(1,\tfrac{1}{2})j\rangle\langle\beta_1\sigma_1\gamma_1J_1x(1,\tfrac{1}{2})j|\beta'\sigma'\gamma'J\rangle =\delta_{\beta,\beta'}\delta_{\gamma,\gamma'}\delta_{\sigma,\sigma'}. \quad (4c)$$

C. Additional Relations for Determining the cfp

Three further independent relations between the cfp can be obtained by using Eq. (30) of Edmonds and Flowers[12] to express the matrix elements of various two-particle operators

$$G=\sum_{i<j}^N G_{ij},$$

which are diagonal in this scheme, with known eigenvalues, in terms of the cfp.

Two such operators are

$$G(J)\equiv\sum_{i<k}^N(\mathbf{j}_i\cdot\mathbf{j}_k),$$

with eigenvalues

$$g_N(J)=\tfrac{1}{2}[J(J+1)-Nj(j+1)]$$

and the similar operator for the isotopic spin

$$G(T)\equiv\sum_{i<k}^N(\mathbf{t}_i\cdot\mathbf{t}_k),$$

with eigenvalues

$$g_N(T)=\tfrac{1}{2}[T(T+1)-\tfrac{3}{4}N].$$

[20] The factorization here corresponds to the reduction scheme $U(4j+2)\rightarrow U(2)\times U(2j+1)\rightarrow R(3)\times Sp(2j+1)\rightarrow R(3)\times R(3)$ used in classifying the states; see Flowers[11] and Racah.[9,10]

A third such operator[21] is

$$G(Sp)\equiv 4\sum_{i<j}^{N}\sum_{\substack{K=1\\(K\text{ odd})}}^{2j}(2K+1)(u_i^{(K)}\cdot u_j^{(K)}),$$

where the $u_i^{(K)}$ are unit tensor operators of rank K defined for a particle with angular momentum j by

$$(j\|u^{(K)}\|j')=\delta_{j,j'},\text{ with } K=0,\,1,\,\cdots 2j.$$

From Edmonds and Flowers,[12] the eigenvalues for $G(Sp)$ can be derived:

$$g_N(s,t)=(s-N)(2j+2)-\tfrac{1}{2}s(s-1)+\tfrac{3}{2}s-2t(t+1).$$

When $G(J)$, $G(T)$, and $G(Sp)$ with their eigenvalues are inserted into Eq. (30) of Edmonds and Flowers,[12] three relations are obtained between the cfp themselves:

$$g_N(J)=\frac{N}{N-2}\sum_{\alpha_1\sigma_1T_1J_1}g_{N-1}(J_1)\times|(j^{N-1}(\alpha_1\sigma_1T_1J_1)j]\!]j^N\sigma\alpha TJ)|^2,\quad(5a)$$

$$g_N(T)=\frac{N}{N-2}\sum_{T_1}g_{N-1}(T_1)|\langle N-1\;T_1|NT\rangle|^2,\quad(5b)$$

$$g_N(s,t)=\frac{N}{N-2}\sum_{\beta_1\sigma_1T_1}g_{N-1}(s_1,t_1)|\langle N-1\;T_1|NT\rangle|^2\times|\langle j^{N-1}(\beta_1\sigma_1T_1)j|j^N\beta\sigma T\rangle|^2.\quad(5c)$$

These relations and the orthonormality conditions (4), along with the reciprocity relation (9.6) of Edmonds and Flowers[12] and the j-j coupling analog of Eq. (19) of Racah[8] are sufficient to determine the cfp of interest here.

D. Evaluation of the cfp

It is convenient to consider the three factors separately. The first factor $\langle N-1\;T_1|NT\rangle$ can be evaluated in general.[22] Noting that $T_1=T\pm\frac{1}{2}$, Eqs. (4a) and (5b) give

$$\langle N-1\;T_1=T+\tfrac{1}{2}|NT\rangle=\left[\frac{(N-2T)(T+1)}{N(2T+1)}\right]^{\frac{1}{2}},\quad(6a)$$

$$\langle N-1\;T_1=T-\tfrac{1}{2}|NT\rangle=\left[\frac{(N+2T+2)T}{N(2T+1)}\right]^{\frac{1}{2}}.\quad(6b)$$

No such general result has been obtained for the other two factors and the remainder of the discussion is restricted to those states with seniority one or zero. These states are uniquely determined by N, T, σ, and J so that $\alpha=(\beta,\gamma)$ can be dropped.[23]

Consider now the seniority-zero states, N even, $\sigma=(0,0)$, and $J=0$. For these states the only possible parents are the states with $\sigma_1=(1,\frac{1}{2})$, $J_1=j$ and the orthonormality conditions (4) give

$$\begin{aligned}\langle j^{N-1}((1,\tfrac{1}{2})T_1)j|j^N(0,0)T\rangle&=\langle(1,\tfrac{1}{2})j\times(1,\tfrac{1}{2})j|(0,0)J=0\rangle\\&=1.\end{aligned}\quad(7)$$

The nonzero total cfp for the states of seniority zero then are

$$(j^{N-1}((1,\tfrac{1}{2})T_1J_1=j)j]\!]j^N(0,0)T\;J=0)=\langle N-1\;T_1|NT\rangle.\quad(8)$$

The treatment of the seniority one states with $\sigma=(1,\frac{1}{2})$, $J=j$ is slightly more complicated. Here the possible parent states have $\sigma_1=(0,0)$ and $J_1=0$, $\sigma_1=(2,0)$ and $J_1=1,\,3,\,\cdots 2j$, or $\sigma_1=(2,1)$ and $J_1=2,\,4,\,\cdots 2j-1$.

If one uses the reciprocity relation[24] (9b) of Edmonds and Flowers[12] and the orthonormality condition (4c), the nonvanishing third factors are

$$\langle(0,0)J_1=0\times(1,\tfrac{1}{2})j|(1,\tfrac{1}{2})j\rangle=1,\quad(9a)$$

$$\langle(2,1)J_1=2,4,\cdots 2j-1\times(1,\tfrac{1}{2})j|(1,\tfrac{1}{2})j\rangle=[(2J_1+1)/(j+1)(2j-1)]^{\frac{1}{2}},\quad(9b)$$

$$\langle(2,0)J_1=1,3,\cdots 2j\times(1,\tfrac{1}{2})j|(1,\tfrac{1}{2})j\rangle=-[(2J_1+1)/(j+1)(2j+1)]^{\frac{1}{2}}.\quad(9c)$$

To determine the remaining factors,

$$\langle j^{N-1}(\sigma_1T_1)j|j^N(1,\tfrac{1}{2})T\rangle$$

an additional relation can be obtained by using the j-j coupling analog of Eq. (19) of Racah[8] to relate the total cfp for the configuration with N particles to those for the configuration with $N-1$ holes.

$$\begin{aligned}(j^{N-1}((0,0)T_1J_1=0)j]\!]j^N(1,\tfrac{1}{2})Tj)&=(-1)^{T+T_1+\frac{1}{2}}\left[\frac{(4j+3-N)(2T_1+1)}{N(2j+1)(2T+1)}\right]^{\frac{1}{2}},\\&\quad(j^{4j+2-N}((1,\tfrac{1}{2})Tj)j]\!]j^{4j+3-N}(0,0)T_1J_1=0).\end{aligned}\quad(10)$$

[21] See Edmonds and Flowers.[12] The operators $G(J)$, $G(T)$, and $G(Sp)$ are related to the Casimir operators for the rotation group $R(3)$ and for the symplectic group $Sp(2j+1)$. The reduced "double barred" matrix elements are defined in Racah.[7]

[22] It can also be shown that $\langle N-1\;T_1|NT\rangle=(n[\lambda_1]/n[\lambda])^{\frac{1}{2}}$, where $n[\lambda_1]$, $n[\lambda]$ are the dimensionalities of the representations $[\lambda_1]$ and $[\lambda]$ of $U(2J+1)$ determined by $(N-1,\,T_1)$ and (N,T), considered now as representations of the symmetric group on N letters, S_N. See Racah.[10]

[23] The seniority-one states are uniquely determined by N, T, σ, and J, but for $7\leqslant N\leqslant 2j+1$ there are two sets of parent states with $\sigma_1=(2,1)$, (see Flowers[11]). For these cases the quantity listed in Table I should be interpreted as the rms of the two cfp. In Sec. III these cfp can be expressed in terms of the cfp with $\sigma_1=(0,0)$ using the orthonormality condition (4b), thus avoiding the necessity of determining the two separately. See Appendix I.

[24] Note that the reduction $(\sigma)\rightarrow D_J$ implies that the dimensionality of (σ) is the sum of the dimensionalities of the D_J, i.e.,

$$n(\sigma)=\sum_{J\text{ in }(\sigma)}(2J+1).$$

TABLE I. Nonvanishing total cfp for seniority-one states of the configuration j^N, for N odd, $\sigma=(1,\frac{1}{2})$, $J=j$.[a]

Case (A): $2T=N, N-4, \cdots$

$$(j^{N-1}((0,0)T_1=T-\tfrac{1}{2}\,J_1=0)j\rrbracket j^N(1,\tfrac{1}{2})T\,J=j)=-\left[\frac{(4j+4-N-2T)}{2N(2j+1)}\right]^{\frac{1}{2}}$$

$$(j^{N-1}((2,1)T_1=T-\tfrac{1}{2}\,J_1=2,4,\cdots 2j-1)j\rrbracket f^N(1,\tfrac{1}{2})T\,J=)j=\left[\frac{(2J_1+1)(4j+4)[T(N+2T)-1]+N-2T}{N(2T+1)(2j+2)(2j+1)(2j-1)}\right]^{\frac{1}{2}}$$

$$(j^{N-1}((2,0)T_1=T+\tfrac{1}{2}\,J_1=1,3,2j)j\rrbracket j^N(1,\tfrac{1}{2})T\,J=j)=-\left[\frac{(N-2T)(2J_1+1)}{2N(2j+2)(2j+1)}\right]^{\frac{1}{2}}$$

$$(j^{N-1}((2,1)T_1=T+\tfrac{1}{2}\,J_1=2,4,\cdots 2j-1)j\rrbracket j^N(1,\tfrac{1}{2})T\,J=j)=\left[\frac{(N-2T)(2T+3)(2J_1+1)}{2N(2T+1)(2j+2)(2j-1)}\right]^{\frac{1}{2}}$$

Case (B): $2T=N-2, N-6, \cdots$

$$(j^{N-1}((2,0)T_1=T-\tfrac{1}{2}\,J_1=1,3,\cdots 2j)j\rrbracket j^N(1,\tfrac{1}{2})T\,J=j)=\left[\frac{(N+2T+2)(2J_1+1)}{2N(2j+2)(2j+1)}\right]^{\frac{1}{2}}$$

$$(j^{N-1}((2,1)T_1=T-\tfrac{1}{2}\,J_1=2,4,\cdots 2J-1)j\rrbracket j^N(1,\tfrac{1}{2})T\,J=j)=\left[\frac{(N+2T+2)(2T-1)(2J_1+1)}{2N(2T+1)(2J+2)(2j-1)}\right]^{\frac{1}{2}}$$

$$(j^{N-1}((0,0)T_1=T+\tfrac{1}{2}\,J_1=0)j\rrbracket j^N(1,\tfrac{1}{2})T\,J=j)=\left[\frac{(4j+6-N+2T)}{2N(2j+1)}\right]^{\frac{1}{2}}$$

$$(j^{N-1}((2,1)T_1=T+\tfrac{1}{2}\,J_1=2,4,\cdots 2j-1)j\rrbracket j^N(1,\tfrac{1}{2})T\,J=j)=\left[\frac{(2J_1+1)(4j+4)[(T+1)(N-2T-2)+1]-(N+2T+2)}{N(2T+1)(2j+2)(2j+1)(2j-1)}\right]^{\frac{1}{2}}$$

[a] For the second cfp of case (A) and the fourth cfp of case (B), see reference 23.

Factoring this expression and inserting the factors already determined gives two of the required factors

$$\langle j^{N-1}((0,0)T_1=T+\tfrac{1}{2})j\,|\,j^N(1,\tfrac{1}{2})T\rangle =\left[\frac{(4j+6-N+2T)(2T+1)}{(N-2T)(2T+2)(2j+1)}\right]^{\frac{1}{2}}, \quad (11a)$$

$$\langle j^{N-1}((0,0)T_1=T-\tfrac{1}{2})j\,|\,j^N(1,\tfrac{1}{2})T\rangle =-\left[\frac{(4j+4-N-2T)(2T+1)}{(N+2T+2)(2T)(2j+1)}\right]^{\frac{1}{2}}. \quad (11b)$$

These two factors with Eq. (5c) and the orthonormality condition (4b) determine the remaining cfp. For this purpose, the seniority-one states are for convenience divided into two classes (see the explicit classification of states by Flowers[11]):

Case (A): $2T=N$, $N-4$, $\cdots$. Here if $T_1=T+\frac{1}{2}$, $\sigma_1=(2,0)$, $(2,1)$ and if $T_1=T-\frac{1}{2}$, $\sigma_1=(0,0)$, $(2,1)$.

Case (B): $2T=N-2$, $N-6$, $\cdots$. Here $T_1=T+\frac{1}{2}$ corresponds to $\sigma_1=(0,0)$, $(2,1)$ and $T_1=T-\frac{1}{2}$ to $\sigma_1=(2,0)$, $(2,1)$.

Collecting all three factors together, the nonvanishing total cfp for seniority one states of the configuration j^N are listed in Table I.

E. cfp for Identical Particles

For the special case where j^N involves only one type of particles, i.e., neutrons, protons, or electrons, a calculation similar to that of the last section gives, for arbitrary seniority s,

$$\langle j^{N-1}((s-1,\tfrac{1}{2}s-\tfrac{1}{2})\;T_1=T-\tfrac{1}{2})j\,|\,j^N(s,\tfrac{1}{2}s)\;T=\tfrac{1}{2}N\rangle =[s(2j+3-N-s)/N(2j+3-2s)]^{\frac{1}{2}}, \quad (12a)$$

$$\langle j^{N-1}((s+1,\tfrac{1}{2}s+\tfrac{1}{2})\;T_1=T-\tfrac{1}{2})j\,|\,j^N(s,\tfrac{1}{2}s)\;T=\tfrac{1}{2}N\rangle =[(N-s)(2j+3-s)/N(2j+3-2s)]^{\frac{1}{2}}. \quad (12b)$$

Since $\langle N-1\;T_1=\frac{1}{2}N-\frac{1}{2}\,|\,N\;T=\frac{1}{2}N\rangle=1$ and the third factors $\langle s\pm 1,\frac{1}{2}s\pm\frac{1}{2})J_1\times(1,\frac{1}{2})j\,|\,(s,\frac{1}{2}s)J\rangle$ can be obtained from the tables of Edmonds and Flowers[12] (for $j-3/2$, $5/2$, $7/2$), this result determines the wave functions for states of arbitrary seniority in the simple coupling scheme where neutrons and protons couple separately, or for atomic j-j coupling.[25] Note that here $\sigma=(s,t)$ is uniquely specified by a single number, s.

III. MATRIX ELEMENTS

For allowed beta transitions, there are only two possible types of transition operators and the transition probability can be written

$$1/ft=g_F^2M_F^2+g_G^2M_G^2, \quad (13)$$

[25] C. Schwartz and A. de-Shalit, Phys. Rev. 94, 1257 (1954), previously obtained this result for the seniority-one states only.

where

$$M_F^2=\frac{1}{2J'+1}\sum_{M,M'}|\langle f|\sum_{k=1}^{N}\tau_\eta(k)|i\rangle|^2 \tag{14}$$

is the Fermi matrix element squared, $|\int 1|^2$, and

$$M_G^2=\frac{1}{2J'+1}\sum_{M,M'}|\langle f|\sum_{k=1}^{N}\tau_\eta(k)\boldsymbol{\sigma}(k)|i\rangle|^2 \tag{15}$$

is the Gamow-Teller matrix element squared, $|\int \boldsymbol{\sigma}|^2$.

In these expressions the isotopic-spin-flip operator τ_η and the spin vector $\boldsymbol{\sigma}$ will be assumed normalized so that τ_ζ and σ_z have eigenvalues ± 1. J' is the total angular momentum of the initial state (i) and M' its z-component. In general, primed quantities will refer to the initial state and unprimed quantities to the final states (f).

The wave functions (cfp) obtained in the previous section will now be used to evaluate the nuclear matrix elements M_F^2 and M_G^2 for allowed transitions involving the seniority-zero or -one states of j^N: (A) for T a good quantum number, and (B) for the simple coupling scheme where neutrons and protons couple separately.

The actual computation of these matrix elements is considerably simplified by using Racah's[7] method of irreducible tensor operators to separate out the M-dependence.

Rewriting the transition operators in terms of irreducible tensor operators, in Racah's notation:

$$F^{(10)}=\sum_{k=1}^{N}\tau_\eta(k)=\sum_{k=1}^{N}\sqrt{2}[t_{-1}^{(1)}(k)-t_1^{(1)}(k)], \tag{16}$$

$$G^{(11)}=\sum_{k=1}^{N}\tau_\eta(k)[\boldsymbol{\sigma}(k)]_q=\sum_{k=1}^{N}\sqrt{2}[t_{-1}^{(1)}(k)-t_1^{(1)}(k)]\sigma_q^{(1)}. \tag{17}$$

The superscripts label the tensor character of the operators in, respectively, isotopic spin space and coordinate space. $\mathbf{t}$ is the isotopic spin vector of a single particle $(t_\zeta=\pm\frac{1}{2})$. $q=0,\ \pm 1$ labels the "spherical" components of the spin vector.

A. T a Good Quantum Number

Consider first the case where the total isotopic spin, T, is a good quantum number.

Here the Fermi matrix element, M_F^2, can readily be evaluated for all transitions, independently of the coupling scheme.[26] If the initial state is $\Psi(a'J'M'T'T_\zeta')$ and the final state $\Psi(aJMTT_\zeta)$, where a', a are any additional quantum numbers required to specify the states, then, from Eq. (29) of Racah,[7]

$$\begin{aligned}\langle f|F|i\rangle&=\langle aJMTT_\zeta|\sum_{k=1}^{N}\tau_\eta(k)|a'J'M'T'T_\zeta'\rangle\\&=\delta_{(aJM),(a'J'M')}(-1)^{T+T_\zeta}\sqrt{2}(T\|T^{(1)}\|T')\times V(TT'1;-T_\zeta T_\zeta'(T_\zeta-T_\zeta')),\end{aligned} \tag{18}$$

$$\begin{aligned}M_F^2&=\frac{1}{2J'+1}\sum_{M',M}|\langle f|F|i\rangle|^2\\&=\delta_{(aJ),(a'J')}V^2(TT'1;-T_\zeta T_\zeta'(T_\zeta-T_\zeta'))\times 2|(T\|T^{(1)}\|T')|^2.\end{aligned} \tag{19}$$

Inserting $|(T\|T^{(1)}\|T')|^2=T(T+1)(2T+1)\delta_{T,T'}$,[7,27] and the explicit form of V^2,[7]

$$M_F^2=\delta_{(aJT),(a'J'T')}(T\pm T_\zeta)(T+1\mp T_\zeta)\quad\text{for } T_\zeta=T_\zeta'\pm 1. \tag{20}$$

Evaluation of the Gamow-Teller matrix elements requires a more detailed knowledge of the character of the states involved.

For the simplest case of a single-particle transition $(n'l'j'm't_\zeta')\rightarrow(nljmt_\zeta)$, the matrix elements are readily evaluated using the explicit single-particle wave functions.[2] However, they will be derived here as a simple example of the formalism (see also Talmi[5]).

From Eq. (29) of Racah[7] it follows that

$$\begin{aligned}\langle f|G_q|i\rangle&=\langle nljmt_\zeta|\tau_\eta\sigma_q|n'l'j'm't_\zeta'\rangle\\&=(-1)^{j+m+\frac{1}{2}+t_\zeta}V(jj'1;-mm'q)V(\tfrac{1}{2}\tfrac{1}{2}1;-t_\zeta t_\zeta'(t_\zeta-t_\zeta'))\sqrt{2}(\tfrac{1}{2}\|t^{(1)}\|\tfrac{1}{2})(nlj\|\sigma\|n'l'j'),\end{aligned} \tag{21}$$

and

$$\begin{aligned}M_G^2&=\frac{1}{2j'+1}\sum_{m,m'}\sum_{q=-1}^{1}|\langle f|G_q|i\rangle|^2\\&=\frac{1}{2j'+1}|(nlj\|\sigma\|n'l'j')|^2.\end{aligned} \tag{22}$$

[26] E. P. Wigner and E. Feenberg, Repts. Progr. Phys. 8, 274 (1941).

[27] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, New York, 1951).

The double-barred matrix elements can be obtained from Eq. (30) of Racah[7] and from Condon and Shortley,[28] noting that $\mathbf{j}=\mathbf{l}+\frac{1}{2}\boldsymbol{\sigma}$. These matrix elements vanish for $n\neq n'$, $l\neq l'$ and for $nl=n'l'$ give:

$$\begin{aligned} M_G{}^2&=(j+1)/j && \text{if } j=j'=l+\tfrac{1}{2},\\ M_G{}^2&=j/(j+1) && \text{if } j=j'=l-\tfrac{1}{2},\\ M_G{}^2&=(2j+1)/j && \text{if } j=j'+1=l+\tfrac{1}{2},\\ M_G{}^2&=(2j+1)/(j+1) && \text{if } j=j'-1=l-\tfrac{1}{2}. \end{aligned} \tag{23}$$

For general transitions $j^N\rightarrow j^N$, Eq. (23) of Racah[8] gives

$$\begin{aligned} \langle f|G_q|i\rangle &= \langle f|\sum_{k=1}^{N}\tau_\eta(k)\sigma_q(k)|i\rangle \\ &= N\langle f|\tau_\eta(N)\sigma_q(N)|i\rangle \\ &= N\langle j^N\alpha\sigma TT_\zeta JM|\tau_\eta(N)\sigma_q(N)|j^N\alpha'\sigma'T'T_\zeta'J'M'\rangle \\ &= N\sum_{\alpha_1\sigma_1T_1J_1}(j^N\alpha\sigma TJ\{|j^{N-1}(\alpha_1\sigma_1T_1J_1)j)(j^{N-1}(\alpha_1\sigma_1T_1J_1)j|\}j^N\alpha'\sigma'T'J') \\ &\qquad\times(T_1J_1,\tfrac{1}{2}j(N);TT_\zeta JM|\tau_\eta(N)\sigma_q(N)|T_1J_1,\tfrac{1}{2}j(N);T'T_\zeta'J'M'). \end{aligned} \tag{24}$$

Evaluating the matrix element on the right in terms of the single-particle matrix element from Eq. (44) of Racah,[7] and simplifying, we obtain

$$\begin{aligned} \langle f|G_q|i\rangle=(-1)^{J+M+T_\zeta+T}[3N^2(2T+1)(2T'+1)(2J+1)(2J'+1)]^{\frac{1}{2}} \\ \times V(TT'1;-T_\zeta T_\zeta'(T_\zeta-T_\zeta'))V(JJ'1;-MM'q)(j\|\sigma\|j)``\textstyle\sum", \end{aligned} \tag{25}$$

where[29]

$$\begin{aligned} ``\textstyle\sum"\equiv\sum_{\alpha_1\sigma_1T_1J_1}(-1)^{T+T_1+\frac{1}{2}+J+J'+M}W(\tfrac{1}{2}T\,\tfrac{1}{2}T';T_11)W(jJjJ';J_11) \\ \times(j^N\alpha\sigma TJ\{|j^{N-1}(\alpha_1\sigma_1T_1J_1)j)(j^{N-1}(\alpha_1\sigma_1T_1J_1)j|\}j^N\alpha'\sigma'T'J'). \end{aligned}$$

Defining S.P. as the Gamow-Teller matrix element for the single-particle transition $j\rightarrow j$ [see Eq. (23)], we get

$$\frac{M_G{}^2}{\text{S.P.}}=3N^2(2T+1)(2T'+1)(2j+1)(2J+1)V^2(TT'1;-T_\zeta T_\zeta'(T_\zeta-T_\zeta'))|\textstyle\sum|^2. \tag{26}$$

This expression then gives the Gamow-Teller matrix elements for arbitrary transitions $j^N\rightarrow j^N$ directly in terms of the cfp for the initial and final states.

For $N=2$, all of the cfp are unity for states with $(-1)^{T+J}=-1$ and zero otherwise, i.e.,

$$M_G{}^2/\text{S.P.}=2(2j+1)(2J+1)W^2(jJjJ';j1), \tag{27}$$

as given by Talmi.[5]

For N odd and both initial and final states of seniority one, the cfp of Table I gives, if $T_\zeta=T_\zeta'\pm1$, for $T=T'$:

Case (A): $2T=N,\ N-4,\ \cdots$

$$\frac{M_G{}^2}{\text{S.P.}}=\frac{[(2T+2)(2j+2)-(N-2T)]^2}{(2T)^2(2T+2)^2(2j+2)^2}\times(T\pm T_\zeta)(T+1\mp T_\zeta); \tag{28}$$

Case (B): $2T=N-2,\ N-6,\ \cdots$

$$\frac{M_G{}^2}{\text{S.P.}}=\frac{[(2T)(2j+2)+(N+2T+2)]^2}{(2T)^2(2T+2)^2(2j+2)^2}\times(T\pm T_\zeta)(T+1\mp T_\zeta).$$

For $T=T'+1$, the cfp of Table I give, under the same conditions,

Case (A): $2T=N,\ N-4,\ \cdots$

$$\frac{M_G{}^2}{\text{S.P.}}=\frac{(4j+4-N)^2-(2T)^2}{4(2T)^2(2j+2)^2}(T\pm T_\zeta)(T-1\pm T_\zeta);$$

Case (B): $2T=N-2,\ N-6,\ \cdots$

$$\frac{M_G{}^2}{\text{S.P.}}=\frac{(N+2)^2-(2T)^2}{4(2T)^2(2j+2)^2}(T\pm T_\zeta)(T-1\pm T_\zeta).$$

For $T=T'-1$, the cfp of Table I give, under the same conditions,

[28] Reference 27, p. 64 ff.

[29] For the definition and properties of the Racah functions W, see Racah[8] and L. C. Biedenharn, Oak Ridge National Laboratory Report ORNL-1098, 1952 (unpublished).

Case (A): $2T=N, N-4, \cdots$

$$\frac{M_G^2}{\text{S.P.}}=\frac{(N+2)^2-(2T+2)^2}{4(2T+2)^2(2j+2)^2}(T+1\mp T_\zeta)(T+2\mp T_\zeta);$$

Case (B): $2T=N-2, N-6, \cdots$

$$\frac{M_G^2}{\text{S.P.}}=\frac{(4j+4-N)^2-(2T+2)^2}{4(2T+2)^2(2j+2)^2}(T+1\mp T_\zeta)(T+2\mp T_\zeta).$$

No such general result for states of arbitrary seniority has been obtained, but for $N=3$, 4 and $j=3/2$, 5/2 the cfp of Edmond and Flowers[12] may be inserted in (25) to evaluate M_G^2 for transitions involving states of higher seniority.

Similarly, for transitions of the type $j^{N-x-1}(j')^{x+1}\rightarrow j^{N-x}(j')^x$, Eq. (28) of Racah[8] can be used to obtain M_G^2 in terms of the cfp for the initial and final states. For

$$[j^{N-x-1}(\sigma_1'T_1'J_1')(j')^{x+1}(\sigma_2'T_2'J_2')]_{T'J'}\rightarrow [j^{N-x}(\sigma_1T_1J_1)(j')^x(\sigma_2T_2J_2)]_{TJ},$$

M_G^2 is given by

$$\begin{aligned}M_G^2/\text{S.P.}=3(N-x)(x+1)(2T+1)(2T'+1)(2T_1+1)(2T_2'+1)(2J+1)(2j'+1)(2J_1+1)(2J_2'+1)\\ \times V^2(TT'1;-T_\zeta T_\zeta'(T_\zeta-T_\zeta'))|((j')^x(\sigma_2T_2J_2)j']\!](j')^{x+1}\sigma_2'T_2'J_2')|^2\\ \times|(j^{N-x-1}(\sigma_1'T_1'J_1')j]\!]j^{N-x}\sigma_1T_1J_1)|^2|\textstyle\sum_J|^2|\sum_T|^2,\end{aligned}\tag{29}$$

where

$$\textstyle\sum_J\equiv\sum_{J_3}(2J_3+1)W(J_1'j'J'J_2;J_3J_2')W(J_1JJ_3J';J_21)W(jJ_1j'J_3;J_1'1),$$

$$\textstyle\sum_T\equiv\sum_{T_3}(2T_3+1)W(T_1'\tfrac{1}{2}T'T_2;T_3T_2')W(T_1TT_3T';T_21)W(\tfrac{1}{2}T_1\tfrac{1}{2}T_3;T_1'1)$$

and S.P. $\equiv M_G^2$ for the single-particle transition $j'\rightarrow j$.

This expression cannot be readily evaluated for general transitions of this type, even for those between states of lowest seniority. However, for the special case where $x=0$ (i.e., transitions of the type $j^{N-1}j'\rightarrow j^N$),

$$\begin{aligned}M_G^2/\text{S.P.}=3N(2j'+1)(2J+1)(2T+1)(2T'+1)V^2(TT'1;-T_\zeta T_\zeta'(T_\zeta-T_\zeta'))W^2(\tfrac{1}{2}T\,\tfrac{1}{2}T';T_1'1)W^2(jJj'J';J_1'1)\\ \times|(j^{N-1}(\sigma_1'T_1'J_1')j]\!]j^N\sigma TJ)|^2.\end{aligned}\tag{30}$$

In particular, for $N=2$,

$$M_G^2/\text{S.P.}=(2j'+1)(2J+1)W^2(jJj'J';j1).\tag{31}$$

M_G^2 for specific transitions of interest may be evaluated from (29) by using tables of cfp and of the Racah functions, W.[29]

All other allowed transitions can be reduced to one one of these types $[j^N\rightarrow j^N$ or $j^{N-x-1}(j')^{x+1}\rightarrow j^{N-x}(j')^x]$ by using the method of Condon and Shortley.[30]

B. Odd-Group Coupling Scheme

In cases where the neutron excess is large, and in particular where there are additives (such as doubly occupied orbits with different l for the neutrons alone), the isotopic spin, T, will no longer be a good quantum number.

The logical formulation of the shell model will then be that the neutrons and protons will couple separately to their states of lowest seniority. In odd-A nuclei this means that the even configurations have zero angular momentum, while the nucleons of the odd group couple to $J=j$.

The nuclear matrix elements for allowed β transitions in this scheme are easily obtained from Eq. (11) of Nordheim and Yost[31] and Eq. (28) of Racah.[8]

Consider first transitions of the type

$$[j^p(\sigma_1'J_1');j^n(\sigma_2'J_2')]_{J'}\rightarrow[j^{p+1}(\sigma_1J_1);j^{n-1}(\sigma_2J_2)]_J,$$

where p and n are the proton and neutron numbers in the initial state. The expressions for the matrix elements in terms of the cfp for the initial and final states are then

$$M_F^2=n(p+1)(2j_1+1)(2J_2'+1)|(j^p(\sigma_1'J_1')j]\!]j^{p+1}\sigma_1J_1)|^2|(j^{n-1}(\sigma_2J_2)j]\!]j^n\sigma_2'J_2')|^2W^2(J_1'jJJ_2;J_1J_2')\delta_{J,J'},\tag{32}$$

and

$$M_G^2/\text{S.P.}=n(p+1)(2J+1)(2J_1+1)(2J_2'+1)(2j+1)|(j^{n-1}(\sigma_2J_2)j]\!]j^n\sigma_2'J_2')|^2|(j^p(\sigma_1'J_1')j]\!]j^{p+1}\sigma_1J_1)|^2|\textstyle\sum|^2,\tag{33}$$

where

$$\textstyle\sum\equiv\sum_{J_3}(2J_3+1)W(J_1'J'J_2;J_3J_2')W(J_1JJ_3J';J_21)W(jJ_1jJ_3;J_1'1).$$

The cfp of (12) can be used to evaluate these matrix elements (and the analogous ones for the corresponding β^+ transitions) for all transitions of this type. In particular, for the transitions between the states of lowest seniority

[30] Reference 27, Sec. 1, Chap. 8.

[31] L. W. Nordheim and F. L. Yost, Phys. Rev. **51**, 942 (1937).

which are of interest here, we have, *for even A*, $J'=1$, and $J=0$,

$$\frac{M_G^2}{\text{S.P.}}=\frac{(p+1)(2j+2-n)}{3(2j+1)}, \tag{34}$$

for odd A:

(*a*) if n is odd, p even,

$$M_F^2=\frac{M_G^2}{\text{S.P.}}=\frac{(2j+2-n)(2j+1-p)}{(2j+1)^2};$$

(*b*) if n is even, p odd,

$$M_F^2=\frac{M_G^2}{\text{S.P.}}=\frac{n(p+1)}{(2j+1)^2}. \tag{35}$$

Similarly, for transitions[32]

$$[j^p(\sigma_1'J_1');(j')^n(\sigma_2'J_2')]_{J'}\rightarrow[j^{p+1}(\sigma_1J_1);(j')^{n-1}(\sigma_2J_2)]_J,$$

M_G^2 is given by:

$$M_G^2/\text{S.P.}=n(p+1)(2J+1)(2j'+1)(2J_1+1)(2J_2'+1)\times|(((j')^{n-1}(\sigma_2J_2)j']\!](j')^n\sigma_2'J_2')|^2|(j^p(\sigma_1'J_1')j]\!]j^{p+1}\sigma_1J_1)|^2|\textstyle\sum|^2, \tag{36}$$

where

$$\textstyle\sum\equiv\sum_{J_3}(2J_3+1)W(J_1'j'J'J_2;J_3J_2')\times W(J_1JJ_3J';J_21)W(jJ_1j'J_3;J_1'1),$$

and S.P. is defined as M_G^2 for the single-particle transition $j'\rightarrow j$.

Explicitly, for transitions between states of lowest seniority, we have, *for even-A*, $J'=1$, $J=0$:

$$\frac{M_G^2}{\text{S.P.}}=\frac{(p+1)(2j'+2-n)}{3(2j+1)}; \tag{37}$$

for odd-A:

(a) if n is odd, p even,

$$\frac{M_G^2}{\text{S.P.}}=\frac{(2j'+2-n)(2j+1-p)}{(2j+1)(2j'+1)}, \tag{38}$$

(b) if n is even, p odd,

$$\frac{M_G^2}{\text{S.P.}}=\frac{n(p+1)}{(2j+1)^2}.$$

Matrix elements for allowed transitions between other configurations may be reduced to one of these two types by using the method of Condon and Shortley.[30]

APPENDIX I: EVALUATION OF M_G^2 FOR SENIORITY-ONE TRANSITIONS $j^N\rightarrow j^N$

As a typical example, consider transitions

$$[j^N\,\sigma'=(1,\tfrac{1}{2})\;J'=j\;T']\rightarrow[j^N\,\sigma=(1,\tfrac{1}{2})\;J=j\;T=T'+1],$$

with $2T=N$, $N-4$, $\cdots$ (i.e., case A). Since $T=T'+1$, $T_1=T'+\frac{1}{2}=T-\frac{1}{2}$, then when we insert the $W(T)$ from the ORNL tables,[29] Eq. (26) becomes

$$\frac{M_G^2}{\text{S.P.}}=\frac{(2j+1)^2[(N+2)^2-(2T)^2]^2}{4(2T-1)(2T+1)}(T\pm T_\zeta)(T-1\pm T_\zeta)|\textstyle\sum|^2 \tag{39}$$

for $T_\zeta=T_\zeta'\pm1$, where

$$\sum_{\sigma_1}\equiv\sum_{\sigma_1J_1}(-1)^{J+J_1+j}W(jjjj;J_11)\langle j^N(1,\tfrac{1}{2})T|j^{N-1}(\sigma_1T_1=T-\tfrac{1}{2})j\rangle\times\langle j^{N-1}(\sigma_1\,T_1=T'+\tfrac{1}{2})j|j^N(1,\tfrac{1}{2})\;T'=T-1\rangle|\langle\sigma_1J_1\times(1,\tfrac{1}{2})j|(1,\tfrac{1}{2})j\rangle|^2.$$

Carry out the summation over J_1 first by defining

$$S(\sigma_1)\equiv\sum_{J_1}(-1)^{J+J_1+j}W(jjjj;J_11)|\langle\sigma_1J_1\times(1,\tfrac{1}{2})j|(1,\tfrac{1}{2})j\rangle|^2$$
$$=-\sum_{J_1}\frac{[2j(j+1)-J_1(J_1+1)]}{2j(j+1)(2j+1)}|\langle\sigma_1J_1\times(1,\tfrac{1}{2})j|(1,\tfrac{1}{2})j\rangle|^2. \tag{40}$$

[32] This case includes the important class of transitions in which the protons end in the shell $j=l+\frac{1}{2}$, which is completely filled by neutrons, the latter ending in the states with $j'=l-\frac{1}{2}$. For states of this type T happens still to be a good quantum number. The neutrons in the filled shell can be omitted in the computation of the matrix elements and p and n taken as the numbers in the unfilled shells alone.

The possible parent states have $\sigma_1=(0,0)$, $J_1=0$; $\sigma_1=(2,1)$, $J_1=2, 4, \cdots, 2j-1$; and $\sigma_1=(2,0)$, $J_1=1, 3, \cdots 2j$. Inserting the proper cfp from (9):

$$S(0,0)=-1/(2j+1),\quad S(2,1)=-1/[(2j+2)(2j+1)],\quad S(2,0)=1/[(2j+2)(2j+1)]. \tag{41}$$

For case (A), $T_1=T-\frac{1}{2}$ corresponds to $\sigma_1=(0,0)$, (2,1) and

$$\sum_{\sigma_1}=S(0,0)\langle j^N(1,\tfrac{1}{2})T|\,j^{N-1}((0,0)\ T_1=T-\tfrac{1}{2})j\rangle\langle j^{N-1}((0,0)\ T_1=T'+\tfrac{1}{2})j\,|\,j^N(1,\tfrac{1}{2})\ T'=T-1\rangle$$
$$+S(2,1)\langle j^N(1,\tfrac{1}{2})T|\,j^{N-1}((2,1)\ T_1=T-\tfrac{1}{2})j\rangle\langle j^{N-1}((2,1)\ T_1=T'+\tfrac{1}{2})j\,|\,j^N(1,\tfrac{1}{2})\ T'=T-1\rangle. \tag{42}$$

Using the orthonormality conditions (4), the $\sigma_1=(2,1)$ cfp can be eliminated and

$$|\sum_{\sigma_1}|^2=[S(0,0)-S(2,1)]^2|\langle j^{N-1}((0,0)\ T_1=T-\tfrac{1}{2})j\,|\,j^N(1,\tfrac{1}{2})\ T\rangle|^2|\langle j^{N-1}((0,0)\ T_1=T'+\tfrac{1}{2})j\,|\,j^N(1,\tfrac{1}{2})\ T'=T-1\rangle|^2. \tag{43}$$

Substituting $S(\sigma_1)$ from (41) and the cfp from Table I [using (6) and (9)], we have

$$|\sum_{\sigma_1}|^2=\frac{[(4j+4-N)^2-(2T)^2](2T+1)(2T-1)}{(2T)^2[(N+2)^2-(2T)^2](2j+2)^2(2j+1)^2}. \tag{44}$$

Then, from (39),

$$\frac{M_G^2}{\text{S.P.}}=\frac{[(4j+4-N)^2-(2T)^2]}{4(2T)^2(2j+2)^2}(T\pm T_\zeta)(T-1\pm T_\zeta) \tag{45}$$

as given by (28).

APPENDIX II. MAGNETIC MOMENTS FOR THE SENIORITY-ONE STATES OF j^N, T A GOOD QUANTUM NUMBER

Using the wave functions (cfp) of Table I, the magnetic moments for the states $[j^N\sigma=(1,\frac{1}{2})TJ=j]$ are found to be

Case (A): $2T=N, N-4, \cdots$

$$\mu=\frac{1}{(2T)(2T+2)(2j+2)}\{\mu_n[(2j+2)(2T+2)(T+T_\zeta)-(N-2T)T_\zeta]$$
$$+\mu_p[(2j+2)(2T+2)(T-T_\zeta)+(N-2T)T_\zeta\};$$

Case (B): $2T=N-2, N-6, \cdots$ (46)

$$\mu=\frac{1}{(2T)(2T+2)(2j+2)}\{\mu_n[(2j+2)(2T)(T+1-T_\zeta)-T_\zeta(N+2T+2)]$$
$$+\mu_p[(2j+2)(2T)(T+1+T_\zeta)+T_\zeta(N+2T+2)]\},$$

where $\mu_n=g_nj$ is the magnetic moment of a single neutron with the same l, j and μ_p is the similar quantity for a proton, i.e., the Schmidt values.

Flowers[33] and Umezawa[13] have given discussions of the magnetic moments for such states and for particular cases of states with seniority three.

[33] B. H. Flowers, Phil. Mag. 43, 1380 (1952).