to either of the higher levels would be twice forbidden or greater and hence could not possibly compete. If the approximate energy measurement of 1.7 Mev is taken at face value, this 44-minute level would lie about 50 kev higher than the 7.1-hour level. The partial lifetime for an $M4$ transition between the two states would be of the order of 10^4 to 10^5 days, based on the Weisskopf lifetime formulas. The branching would be negligible, and thus the decay schemes would be more or less independent of one another.

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Systematics of Fission Thresholds

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An examination of the experimental data on fission thresholds and ground-state masses of nine nuclei shows that (i) there is experimental evidence for a difference between the masses of even-even, odd-A and odd-odd nuclei in the deformed saddle-point configurations, analogous to the well-known difference for the ground states, (ii) the trend with Z^2/A in the *saddle-point masses* is consistent with simple theoretical estimates based on a liquid drop type of model. This leads to a semiempirical formula for fission thresholds, which is found to be consistent also with estimates of thresholds made with the aid of the observed spontaneous fission half-lives of 28 nuclei. An analogous semiempirical formula for spontaneous fission half-lives is given.

It is well known that the observed fission thresholds do not show the expected decrease with increasing Z^2/A.[1] The threshold energy is the difference between the mass of the nucleus in its ground state and the mass in the saddle-point configuration[2] through which a nucleus must pass when its excitation energy is only just sufficient to surmount the potential barrier opposing division. It is instructive to examine, separately, the available experimental information on the trends with Z^2/A in these two sets of masses.

Of the lower set of points in Fig. 1 those joined together by lines show the experimental ground-state masses[3] of the nine compound nuclei for which the threshold energies have been measured, as seen from a smooth reference surface $M_{\text{ref}}(A,Z)$, where

$$\begin{aligned} M_{\text{ref}}(A,Z)-A=&-8.3557A+19.120A^{\frac{2}{3}} \\ &+0.76278Z^2/A^{\frac{1}{3}}+25.444(N-Z)^2/A \\ &+0.420(N-Z) \text{ millimass Units.} \quad (1) \end{aligned}$$

This is the semiempirical liquid drop formula for nuclear masses,[4] which follows closely the experimental values except for oscillations associated with shell structure. The nine points exhibit the well-known difference between the masses of even-even, odd-A and odd-odd nuclei (representable,[3] on the average, by a term ± 0.77 mMU in this region of A).

The *experimental* masses of the same nuclei in their saddle-point shapes are shown in the upper part of Fig. 1. They are obtained by adding the threshold γ-ray energy to the ground-state mass in the case of photofission, or the neutron energy and mass to the ground-state mass of the target nucleus in the case of neutron fission. (The neutron binding energy need not be known if the experimental target mass is available.)

It will be noted that there is experimental evidence for a difference between *e-e*, *o-A* and *o-o* masses also in the

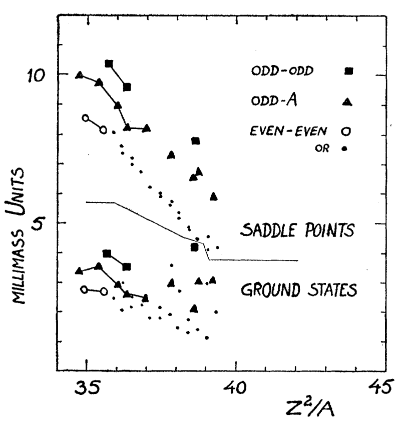

FIG. 1. The masses of nuclei in the ground state and at the saddle point, as seen from a smooth reference surface. All ground-state masses are from reference 3. The nine saddle-point masses shown by points joined together by lines were obtained by adding measured threshold energies (Table I), the remaining ones by adding thresholds estimated from the known spontaneous fission half-lives (reference 8). In the latter case, the threshold of U^{238} was taken as standard for even-even nuclei and Pu^{239} for odd-A nuclei.

[1] See, for example, D. L. Hill and J. A. Wheeler, Phys. Rev. **89**, 1102 (1953).
[2] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).
[3] Glass, Thompson, Seaborg, J. Inorg. Nuc. Chem. **1**, 3 (1955).
[4] A. E. S. Green, Phys. Rev. **95**, 1006 (1954).

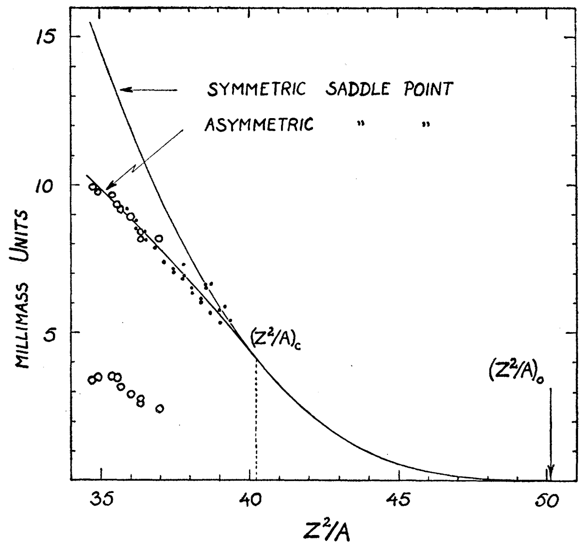

FIG. 2. The points refer to the same set of nuclei as in Fig. 1, but the even-even masses were increased by 0.77 mMU for the ground state and 1.2 mMU for the saddle-point configuration, with similar reductions for odd-odd nuclei. (Also all except nine of the ground state masses shown in Fig. 1 were omitted for the sake of clarity). The curves are the saddle-point masses as given by a theoretical formula with one parameter adjusted arbitrarily.

deformed saddle-point configurations. This is not unexpected since this difference is associated with spin-degeneracy which is not removed by the deformation. The splitting of the three mass surfaces appears to be somewhat larger than for the ground states (about ± 1.2 mMU).

With the object of bringing out more clearly the trends with Z^2/A, we have made a second plot (Fig. 2) in which the *e-e* masses were increased by 0.77 mMU in the ground state and by 1.2 mMU in the saddle-point configuration, with similar reductions for *o-o* nuclei. (All nine points are now shown circled.)

In order to examine the resulting trends in the light of the theory of fission we note that, according to the liquid drop model (in fact more generally[5,6]), the mass of the drop in the symmetric saddle-point configuration is expected to exceed the ground-state mass [Eq. (1)] by an amount $E_{\text{th}}^{\text{sym}}$ given by[7]

$$E_{\text{th}}^{\text{sym}} = c_1[(Z^2/A)_0 - Z^2/A]^3, \qquad (2)$$

where $(Z^2/A)_0 \sim 50$.[4] The energy of the asymmetric saddle point shape, which appears for $Z^2/A < (Z^2/A)_c$ and which is what the experimental results refer to, is lower by an amount[5,6]

$$E_{\text{th}}^{\text{sym}} - E_{\text{th}}^{\text{asym}} = c_3[(Z^2/A)_c - Z^2/A]^2, \qquad (3)$$

where $(Z^2/A)_c = 40.2 \pm 0.7$, as determined empirically from the observed trends in fission asymmetries.[5]

The curves in Fig. 2 show the masses of the symmetric and asymmetric saddle-point configurations calculated with $(Z^2/A)_0 = 50.13$, $c_1 = 0.00424$ [the values given by the liquid drop formula (1)], $(Z^2/A)_c = 40.2$, and only c_3 adjusted arbitrarily. (No reliable theoretical estimates of c_3 are available.) It will be seen that with $c_3 = 0.178$ the nine observed saddle-point masses are fairly well reproduced. This leads to the following semiempirical formula for fission thresholds:

$$E_{\text{th}}\begin{Bmatrix} e\text{-}e \\ o\text{-}A \\ o\text{-}o \end{Bmatrix} = \begin{Bmatrix} -1.2 \\ 0 \\ +1.2 \end{Bmatrix} + 0.00424[50.13 - (Z^2/A)]^3 - 0.178[40.2 - (Z^2/A)]^2 - \delta M \text{ mMU}, \qquad (4)$$

where δM is the deviation of the measured ground state mass from formula (1). Table I compares the observed thresholds with the values calculated by means of (4).

The range of Z^2/A values in the comparison may be extended by using the observed spontaneous fission rates (known for nuclei with Z^2/A values up to 39.4) to estimate thresholds that have not been measured experimentally. An analysis of the empirical material[8] shows that there is a correlation between the spontaneous fission half-lives and the fluctuations in the quantity δM from nucleus to nucleus. This correlation suggests that every mMU of extra stability (as revealed by a small δM) corresponds to a lengthening of the half-life by a factor $\sim 10^5$. (There is evidence for a decrease of this factor from $\sim 10^7$ around Th to $\sim 10^{3.5}$ around Fm). If the hypothesis is made that this reflects primarily the increased height of the barrier against fission as seen from a relatively low ground state, it becomes possible to deduce the difference between the threshold energies of two nuclei from the difference between their spontaneous fission half-lives. This hypothesis is supported by a test in the case of the even-even pair Th^{232} and U^{238} for which both the thresholds and the half-lives are known. (The difference in the values of $\log_{10}$

TABLE I. Fission thresholds.

Compound nucleus[c]	Experimental[a] (in Mev)	Formula (4) (in Mev)	Remarks[b]
Th^{233}	6.11	6.27	*n*
Th^{232}	5.40	5.63	*ph*
U^{239}	5.75	5.51	*n*
U^{238}	5.08	5.03	*ph*
Pa^{232}	5.99	5.97	*n*
U^{235}	5.60	5.49	*n*
U^{235}	5.31	5.49	*ph*
U^{233}	5.18	5.47	*ph*
Np^{238}	5.62	5.73	*n*
Pu^{239}	5.31	4.97	*ph*

[a] These values follow from the data summarized in reference 1, coupled with the use of nuclear masses from reference 3 in the case of neutron fission.
[b] Here *n* stands for neutron fission, *ph* for photofission. The error in the absolute values of the photofission thresholds is of the order of 0.25 Mev (less for relative values).
[c] The nuclei are listed in order of increasing Z^2/A.

[5] W. J. Swiatecki, Phys. Rev. **100**, 936 (1955).
[6] W. J. Swiatecki (to be published).
[7] This formula is derived from an expansion which assumes $[1-(Z^2/A)/(Z^2/A)_0] \ll 1$. More exact calculations with the liquid drop model[6] show that this lowest order formula continues to represent the (liquid drop) threshold energies to about 10% for Z^2/A values down to about 35, even though the expansion ceases to be good already at $Z^2/A \sim 45$.
[8] W. J. Swiatecki, Phys. Rev. **100**, 937 (1955).

(half-life) is 2.25, the difference in the thresholds is 0.3_5 mMU. The ratio is therefore $\sim$6.) It may be noted that a theoretical estimate based on the liquid drop model[9] also suggested that for nuclei around U every mMU in the barrier height should correspond to a factor $\sim 10^7$ in the lifetime.

The thresholds, estimated by assuming that a factor 10^5 in the half-life corresponds to a reduction of the barrier by 1 mMU, have been added to the ground state masses and plotted in Figs. 1 and 2 for the 28 nuclei for which spontaneous fission half-lives are known. The decrease with Z^2/A found for these estimated saddle-point masses joins on with the trend in the nine directly measured masses, and, within the uncertainty of the estimate, is consistent with the semi-empirical formula (4).

According to the present discussion, the reason why the nine observed thresholds do not show a more marked decrease with Z^2/A is to be found in the presence of a systematic trend in the ground-state masses—a decrease of δM with increasing Z^2/A (lower part of Fig. 2). The deviation δM is associated with shell structure and illustrates the limitations of the use of a liquid drop model for nuclear ground states. On the other hand, in analogy with the case of spontaneous fission half-lives,[8] there is evidence that, apart from shell structure in the ground-state configuration (which may be studied with the aid of the empirical quantity δM), the trends in the masses of the distorted saddle-point shapes can be accounted for in terms of a model in which single-particle features are treated in an average way.

In connection with Fig. 2 and formula (4), it is of interest to make an analogous plot of the spontaneous fission half-lives discussed in reference 8. As found there, if the logarithm to base 10 of the experimental half-lives is increased by an empirical correction $k\delta M$ ($k=42.5-Z^2/A$) and then plotted against Z^2/A, three parallel curves are suggested. The result of making the curves coalesce by adding 6.6 units to the even-even points and subtracting 4.9 units from the odd-odd point is shown (by dots) in Fig. 3. These points represent the result of "correcting" the half-lives empirically for shell structure in the ground state and it is of interest to discuss the resulting trend with reference to the liquid drop theory of spontaneous fission.

An argument of a degree of generality similar to that underlying the derivation of formulas (2) and (3)[5,6] suggests that for $1-(Z^2/A)/(Z^2/A)_0 \ll 1$ the half-life for symmetric fission should be proportional to $\exp\{c_4[(Z^2/A)_0-(Z^2/A)]^{5/2}\}$,[10] whereas for $Z^2/A<(Z^2/A)_c$ the exponent should be less by $c_5[(Z^2/A)_c-(Z^2/A)]^2$. Writing $\log_{10}$ (half-life)$=\tau$ we have, approximately,

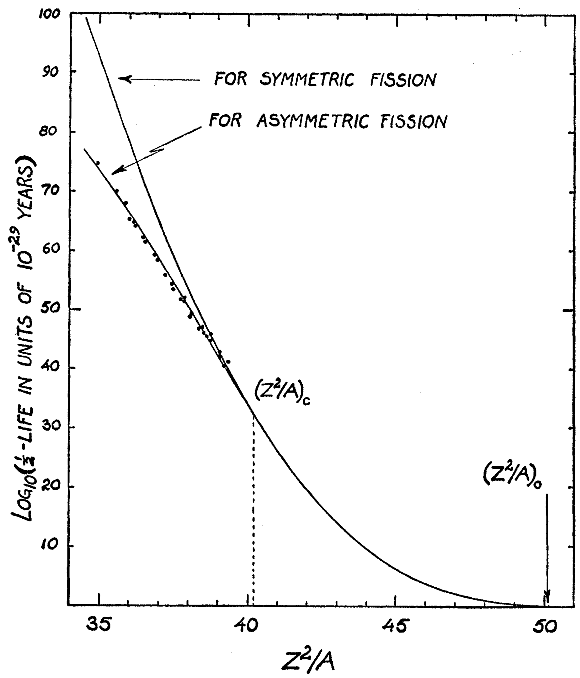

FIG. 3. The points are the experimental spontaneous fission half-lives, empirically corrected for shell structure in the ground state (reference 8). The curves refer to a theoretical formula with two adjusted parameters.

$$(\tau-\tau_0)^> = c_4[(Z^2/A)_0-(Z^2/A)]^{5/2} \quad \text{for } Z^2/A>(Z^2/A)_c,$$

and (5)

$$(\tau-\tau_0)^< = (\tau-\tau_0)^> - c_5[(Z^2/A)_c-(Z^2/A)]^2 \quad \text{for } Z^2/A<(Z^2/A)_c.$$

Here τ_0 corresponds to the half-life in the limit $Z^2/A = (Z^2/A)_0$, of the order of 10^{-29} years, i.e., $\tau_0=-29$.

Figure 3 shows that with the values of $(Z^2/A)_0$ and $(Z^2/A)_c$ as before and c_4 and c_5 adjusted to 0.1041 and 0.691, respectively, a fair representation of the 28 experimental points is possible. The result can be stated in the form of a semiempirical formula for half-lives:

$$\tau\begin{Bmatrix} e\text{-}e \\ o\text{-}A \\ o\text{-}o \end{Bmatrix} = \begin{Bmatrix} -6.6 \\ 0 \\ +4.9 \end{Bmatrix} -29+0.1041[(Z^2/A)_0-(Z^2/A)]^{5/2} -0.691[(Z^2/A)_c-(Z^2/A)]^2-k\delta M, \quad (6)$$

where, in the range $Z^2/A=35$ to 39.5, $k=42.5-Z^2/A$.

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[9] S. Frankel and N. Metropolis, Phys. Rev. **72**, 914 (1947).

[10] The penetration through a potential barrier of height E_{th} and thickness b is taken to be proportional to $\exp\{-E_{th}^{\frac{1}{2}}b\}$. For $1-(Z^2/A)/(Z^2/A)_0 \ll 1$, the height E_{th} is proportional to $[(Z^2/A)_0-Z^2/A]^3$ and b is proportional to $[(Z^2/A)_0-Z^2/A]$; hence the result quoted. See also reference 7.