Dynamics of Ionized Media*

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The behavior of an ionized plasma is discussed in an approximation in which an individual particle is assumed to obey a Fokker-Planck equation, and where its interaction with the environment is incorporated in the coefficients of the partial differential equation. It is found that if the interaction of the test particle with the medium is divided into a "nearest neighbor" interaction (which manifests itself in "large-angle collisions") and an interaction with the rest of the medium, then the latter can be adequately treated by a perturbation method. If the nearest neighbor interaction is neglected, the coefficients of successive derivatives form a rapidly decreasing sequence, provided the average kinetic energy greatly exceeds the mean potential energy (which is usually the case). Within the framework of this approximation the coefficients of damping (dynamical friction) and diffusion in velocity space are calculated and the higher (small) coefficients are estimated.

I. INTRODUCTION

THE state of a gas, or any group of particles regarded as a complete dynamical system, is governed by the Liouville equation, which expresses the conservation of extension in phase, as the system proceeds in time according to the equations of motion for the individual particles. As the classical mechanics involved is deterministic, the only stochastic element embodied in the solution of the equation is an uncertainty about the initial conditions. In principle, the distribution function for a single particle is derivable from this equation, together with some assumption concerning the probability distribution for various initial configurations. In the study of actual problems associated with gases, one generally assumes that this exceedingly complicated equation may be replaced by a simple one in which not only the initial conditions, but also the dynamical process itself, as viewed by a single particle, is of a stochastic nature. The Boltzmann equation represents one method of specifying the latter. It proceeds on the assumption that the dynamical history of a molecule may be analyzed in terms of a series of discrete, relatively rare events (collisions), involving only one other member of the system, using the rigorous solution for the motion of two particles which are only interacting with each other, and not with the remainder of the system. This picture, corresponding closely to a stochastic process of the Poisson type, appears to be quite adequate when the gas is of low density and the range of the forces between molecules is quite short, so that such idealized two-particle interactions closely represent the physical system.

In an ionized plasma, however, the latter condition does not obtain and the collision picture is therefore much less applicable. The ionized particle is never quite free either before or after the collision, and is moreover always subject to the long-range force of other ions. It therefore seems worthwhile to explore in this connection the opposite limit of a stochastic process pictured in terms of very frequent (almost continuous) events which individually are insignificant compared to a "collision" but whose cumulative effect may be quite large. The very small-angle collisions would presumably be included here, but the large-angle ones completely left out. It is realized that ideally the two pictures should be combined because larger angle collisions are not always negligible. To avoid complicating the treatment, and to bring out more clearly the features of the method, however, we have entirely neglected this "Poisson aspect" of the problem.

The problem of treating particles which undergo numerous weak interactions has been extensively developed in connection with the Brownian motion of macroscopic particles interacting with microscopic ones. For this treatment the Fokker-Planck (F.P.) equation was developed, and we feel that this equation gives a natural starting point for the present investigation. In choosing the F.P. equation to describe an ionized plasma, the assumption is implicitly made that the time variation of the one-particle distribution function is approximately a Markovian process (one in which only the present and not the past determines the future distribution). The most general form of the F.P. equation may be considered as a differential characterization of such a process, in which there are an infinite number of coefficients dependent only on the instantaneous state of the system. These coefficients are in principle deducible from the solution of the Liouville equation *if* the motion of a single particle is indeed Markovian. However, there is no reason to believe that this is rigorously true, and in any case it is not possible in practice to solve the many-body problem. The F.P. equation must therefore be regarded as a new point of departure for treating the specific problem of an ionized plasma, and the coefficients must be arrived at from physical considerations. This paper is primarily concerned with the determination of the F.P. coefficients, on the basis of a certain physical picture of the stochastic elements that enter into the dynamics of an ionized particle. That the physical picture is adequately expressed by the approximation scheme will be seen from the fact that the higher order coefficients calculated

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according to the scheme are relatively small. From this one must not infer, however, that the method itself, which is limited to the weak, frequent interactions, is entirely adequate, and that the effect of the large-angle collisions is negligible. It is precisely their exclusion that produces the extremely rapid convergence.

In spatially uniform systems, with which we shall be dealing in this paper, the F.P. coefficients take the form of "averages" of successive powers of the change in velocity Δv, in an infinitesimal time interval τ. In Sec. II of this paper, the coefficient of damping, $\langle\Delta v\rangle/\tau(\equiv\alpha^{(1)})$, is calculated. This damping (commonly called "dynamical friction") is considered as coming from two sources. First, as a result of the interaction between the particle under observation ("test particle"), whose distribution function we wish to calculate, and the rest of the particles of the system ("field particles") the average velocity of that particle relative to its environment goes to zero. This is a statistical effect resulting from correlations of the forces on the test particle at different times, even if the average force is zero. Secondly, because of the reaction of the test particle on the field, which modifies the distribution of the field particles so that the average force is not zero, an additional effective damping occurs. This "polarization" effect is calculated in Sec. IIA, while the statistical effect is calculated in Sec. IIB. In Sec. III, the diffusion coefficient $\langle(\Delta v)^2\rangle/\tau(\equiv\alpha^{(2)})$, is calculated, and the higher moments are examined. To order τ, the polarization of the medium does not affect the rate of diffusion or any of the higher terms, so that all coefficients from the second on are of a purely statistical origin. Certain formal divergence difficulties occur in these estimates of the higher coefficients, and in Sec. IV a method for circumventing these difficulties by a slight reformulation of the expressions for the higher moments (based on a closer examination of the physical effects involved) is suggested. The higher F.P. coefficients are then estimated, and are found to be small within the framework of the physical assumptions made.

The polarization calculation, which only enters in the damping coefficient, differs fundamentally from our treatment of the statistical effects in that the former explicitly takes into account the average effects of the many-body forces and thus leads directly to the existence of a long-range cutoff in the two-body force. On the other hand, the statistical effect is treated by a method of successive approximations about rectilinear motion in which effectively only two-body forces contribute. Effects of many-body forces are included in the statistical treatment only insofar as they produce a long range cutoff in the two-body force and provide a natural mechanism for avoiding the formal divergences in the higher moments.

II. DYNAMICAL FRICTION

The Fokker-Planck equation describing the one-particle distribution function $\omega(x,t|x_0,t_0)$ in phase space is[1]

$$\frac{\partial\omega(x,t|x_0,t_0)}{\partial t}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}[\alpha^{(n)}(x,t)\omega(x,t|x_0,t_0)],\quad(1)$$

where the $\alpha^{(n)}$ are quantities of the form $\langle(\Delta x)^n\rangle/\tau$ [where $\Delta x\equiv(\Delta\mathbf{r},\Delta\mathbf{v})$ is the "displacement" in time τ]. Now the position of a particle can change only through its velocity, and therefore for times τ short enough so that higher order terms in τ may be neglected, the coefficients of the derivatives with respect to position are trivially zero (except for $n=1$ where the usual streaming term obtains), so that we may restrict ourselves to the velocity coefficients. In this case, $\alpha^{(1)}$ is generally known as the coefficient of dynamical friction, and $\alpha^{(2)}$ as the coefficient of diffusion. Since the dynamical friction is thus defined to be the average change in velocity, over a short time, of a particle resulting from its interaction with the field particles, it may be expressed by the relation

$$\langle\Delta\mathbf{v}\rangle/\tau=\left\langle\tau^{-1}\int_0^\tau\mathbf{F}[\mathbf{z}(t)]dt\right\rangle\equiv\langle\langle\mathbf{F}[\mathbf{z}(t)]\rangle\rangle,$$

where $\mathbf{F}[\mathbf{z}(t)]$ is the force on the test particle as a functional of its orbit, $\langle\ \rangle$ denotes the ensemble average over the initial conditions, and $\langle\langle\ \rangle\rangle$ represents both time and ensemble average. It is to be noted that the ensemble average must be carried out over the *initial* conditions since when the test particle is singled out by having its velocity specified, the remaining system is no longer in equilibrium.[2] The time interval must be chosen short enough so that the motion of the particle is effectively unchanged, yet long enough to allow the particle to undergo many interactions, so that fluctuations about the average damping force are effectively eliminated. Now

$$\mathbf{F}[\mathbf{z}(t)]=\mathbf{F}\left[\mathbf{z}_0+\mathbf{v}_0t+\int_0^t dt'\int_0^{t'}dt''\mathbf{F}[\mathbf{z}(t'')]\right],$$

and since we are interested in times for which the change in velocity is small, we may expand about the free-particle motion, so that[3]

$$\langle\langle\mathbf{F}[\mathbf{z}(t)]\rangle\rangle\cong\langle\langle\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t]\rangle\rangle+\left\langle\left\langle\int_0^t dt'\int_0^{t'}dt''\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t'']\cdot\nabla\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t]\right\rangle\right\rangle.\quad(2)$$

[1] For a discussion of the assumptions involved in using this equation, see Appendix A.

[2] J. G. Kirkwood, J. Chem. Phys. 14, 180 (1946).

[3] After the expansion of $\mathbf{F}[\mathbf{z}(t)]$, the dependence on the initial values appears explicitly, and one is then able to interchange the time averaging with the ensemble average (integration over the distribution of initial values).

We shall first calculate the term $\langle\langle \mathbf{F}[\mathbf{z}_0+\mathbf{v}_0 t]\rangle\rangle$. This force is not zero because the distribution of the field particles is modified by the test particle.

A. Polarization Effect

As a particle moves through the distribution of particles, the latter becomes polarized because the test particle attracts the field particles, so that they tend to concentrate behind it, and thus the force from particles behind is larger than the force due to those in front, with the result that the test particle is slowed down.[4] Since we are here concerned with the effect of the test particle on the field particles, we need an equation for the distribution of the latter. For this purpose, we use the integrated Liouville equation,

$$\frac{\partial f_1}{\partial t}+\mathbf{v}\cdot\nabla_r f_1+\frac{1}{m}\mathbf{F}^e\cdot\nabla_v f_1 =\frac{1}{m}\int d\mathbf{r}'d\mathbf{v}'\nabla_r U(\mathbf{r},\mathbf{r}')\cdot\nabla_v f_2(\mathbf{r},\mathbf{v};\,\mathbf{r}',\mathbf{v}'), \quad (3)$$

where f_1 is the single-particle distribution function for a member of the field, f_2 is the two-particle distribution function, and U is the interparticle potential. $\mathbf{F}^e$ here is the Coulomb force due to the test particle. Although we deal with a medium of one type of particle only, the tacit assumption is made that a uniform static charge density of opposite sign is present. It is of a magnitude such as to neutralize the whole medium on the average. Results taking both types of particle explicitly into account can be obtained by a simple extension of this work, and are stated later.

To proceed, we now make the approximation of neglecting the correlations between pairs of particles, so that

$$f_2(\mathbf{r},\mathbf{v};\,\mathbf{r}',\mathbf{v}')\cong f_1(\mathbf{r},\mathbf{v})f_1(\mathbf{r}',\mathbf{v}'), \quad (4)$$

and further, that $f_1(\mathbf{r},\mathbf{v})$ departs only slightly from the equilibrium distribution due to the presence of $\mathbf{F}^e$, so that we write

$$f_1(\mathbf{r},\mathbf{v})=f_1^{(0)}(\mathbf{v})+f_1^{(1)}(\mathbf{r},\mathbf{v}), \quad (5)$$

where $f_1^{(0)}(v)$ corresponds to the Maxwellian distribution at temperature T. In addition, instead of "testing" the field by means of a particle with a fixed velocity $\mathbf{v}$, we find it more convenient to allow the field particles to stream by the stationary test particle (located at the origin), with average velocity $-\mathbf{v}$. Substituting Eqs. (4) and (5) into Eq. (3), we get, to first order,

$$(\partial f_1^{(1)}/\partial t)+\mathbf{v}\cdot\nabla_r f_1^{(1)}+(1/m)(\mathbf{F}^e-\nabla\varphi)\cdot\nabla_v f_1^{(0)}=0, \quad (6)$$

where $\varphi(\mathbf{r})$ is the potential energy at the point $\mathbf{r}$ due to the field particles. Thus,[5]

$$\varphi(\mathbf{r})=\int U(\mathbf{r},\mathbf{r}')f_1^{(1)}(\mathbf{r}',\mathbf{v}')d\mathbf{r}'d\mathbf{v}'. \quad (7)$$

Since $U(\mathbf{r},\mathbf{r}')=e^2/|\mathbf{r}-\mathbf{r}'|$, we have

$$\nabla^2\varphi(\mathbf{r})=-4\pi e^2\int f_1^{(1)}(\mathbf{r},\mathbf{v})d\mathbf{v}. \quad (8)$$

We are interested in the steady-state solution of Eq. (6), which would suggest setting $\partial f_1^{(1)}/\partial t=0$ there. However, as has been pointed out by Landau,[6] it is more convenient to perform a Laplace transformation with respect to the time coordinate, which helps to define unambiguously certain contours arising in integrations appearing later on. The velocity $\mathbf{v}$ appears only as a parameter in $f_1^{(1)}$, and it is therefore sufficient to perform a Fourier transform with respect to the coordinate r only. After transformation, Eqs. (6) and (8) become

$$(s+i\mathbf{k}\cdot\mathbf{v})g(\mathbf{k},\mathbf{v},s)=-(1/ms)\int \mathbf{F}^e\cdot\nabla_v f_1^{(0)}e^{-i\mathbf{k}\cdot\mathbf{r}}d\mathbf{r} +(i/m)(\nabla_v f_1^{(0)}\cdot\mathbf{k})\Phi(\mathbf{k},s),$$

and

$$k^2\Phi(\mathbf{k},s)=4\pi e^2\int g(\mathbf{k},\mathbf{v},s)d\mathbf{v}, \quad (10)$$

where

$$g(\mathbf{k},\mathbf{v},s)=\int d\mathbf{r}e^{-i\mathbf{k}\cdot\mathbf{r}}\int_0^\infty dt e^{-st}f_1^{(1)}(\mathbf{r},\mathbf{v})$$

and $\Phi(\mathbf{k},s)$ is the Fourier-Laplace transform of $\varphi(\mathbf{r},t)$. In Eq. (9), the contribution of the initial value of $g(k)$ has been omitted. The justification for this is that the final ("polarized") equilibrium distribution is reached in a time very short compared to a Debye period.[7] The resulting algebraic equation for $g(k)$ may be solved, and the result substituted into Eq. (10). One then obtains

$$\Phi(\mathbf{k},s)=\frac{1}{ms}\left(\frac{4\pi e^2}{k^2}\right)^2 I(\mathbf{k},\mathbf{v})\left[1-\frac{4\pi e^2}{mk^2}I(\mathbf{k},\mathbf{v})\right]^{-1}, \quad (11)$$

where

$$I(\mathbf{k},\mathbf{v})\equiv I(\mu,v)=i\int d\mathbf{v}_0\frac{\mathbf{k}\cdot\nabla_{v_0}f_1^{(0)}(\mathbf{v}_0-\mathbf{v})}{s+i\mathbf{k}\cdot\mathbf{v}_0}, \quad (12)$$

where μ is the cosine of the angle between $\mathbf{k}$ and $\mathbf{v}$, and use has been made of the fact that the Fourier transform of $\mathbf{F}^e$ is just $-4\pi i e^2\mathbf{k}/k^2$. To obtain the steady-state solution, we compute the residue of $\Phi(\mathbf{k},s)$ at $s=0$,

[4] For repulsive potentials, the force from particles in front, which now tends to decelerate the test particle, is larger than the accelerating force due to the particles which have been dispersed behind it.

[5] In this equation only the quantity $f_1^{(1)}$ appears under the integral, since $f_1^{(0)}$ does not contribute in a medium of total charge zero.

[6] L. D. Landau, J. Phys. (U.S.S.R.) 10, 25 (1946).

[7] See reference 6. Although the small $\mathbf{k}$ waves are not rapidly damped, they contribute but little to the force.

making sure, however, that s approaches zero from the positive direction.[6]

Using this steady-state solution, we are able to compute the force on the test particle due to the polarized field. This force is

$$(-\nabla\varphi)_{r=0}=i(2\pi)^{-3}\int \mathbf{k}\Phi(\mathbf{k})d\mathbf{k}. \tag{13}$$

Clearly, the only nonvanishing component of this force lies in the direction of $\mathbf{v}$, and its value is

$$\frac{\mathbf{F}\cdot\mathbf{v}}{|\mathbf{v}|}=\frac{2ie^2}{m}\int_{-1}^{1}d\mu\,\mu I(\mu,v)\int_0^\infty dk\,k\left[k^2-\frac{4\pi e^2}{m}I(\mu,v)\right]^{-1}. \tag{14}$$

The integration over k is divergent for large k, which corresponds to small distances. This divergence reflects our inadequate treatment of dynamical correlations between two particles when they get close together. To secure convergence, we will cut off the integral at $k_{\max}\sim 1/d$, where d is the interparticle distance. This cut-off procedure essentially excludes from consideration the region about the test particle containing one nearest neighbor. In this excluded region, the test particle effectively feels only the force due to its nearest neighbor, and here a two-body collision treatment is indicated. Carrying out the k integration, we get

$$F_{\mathrm{v}}=\frac{2ie^2}{m}\int_{-1}^{1}d\mu\,\mu I(\mu,v)\log\left\{\frac{k_{\max}^2-(4\pi e^2/m)I(\mu,v)}{-(4\pi e^2/m)I(\mu,v)}\right\}. \tag{15}$$

Here the μ integration has to be carried out numerically. It is possible, however, to make an approximate evaluation of this term. It is easily seen that the force is a real quantity here, and it is therefore sufficient to take only the imaginary part of the integrand into account. Writing $I(\mu,v)=R(\mu,v)+iJ(\mu,v)$, we have approximately (since $k_{\max}^2\gg 4\pi e^2\beta n\gtrsim 4\pi e^2|I|/m$ for cases of interest)

$$F_{\mathrm{v}}\cong\frac{2e^2}{m}\int_{-1}^{1}d\mu\,\mu\left\{J(\mu,v)\times\frac{k_{\max}^2}{(4\pi e^2/m)(R^2+J^2)^{\frac{1}{2}}}+R\tan^{-1}J/R\right\}.$$

Now, from Eq. (12), we find that

$$R(\mu,v)=-\beta mn\left[1-v\mu(m/2\pi\kappa T)^{\frac{1}{2}}\times \mathrm{P}\int_{-\infty}^{\infty}du\frac{\exp[-\beta mu^2/2]}{u+v\mu}\right],$$

and

$$J(\mu,v)=\beta mn\pi(m/2\pi\kappa T)^{\frac{1}{2}}v\mu\exp[-\beta mv^2\mu^2/2],$$

where $\beta=(\kappa T)^{-1}$. For large μv, we see that $R\sim(v\mu)^{-2}$, and so for large $v\mu$, $J\ll R$. This also turns out to be true for small $v\mu[v\mu\ll(\beta m)^{-\frac{1}{2}}\equiv v_T]$, so that we assume that we can always neglect J in comparison with R. Then we can write

$$F_{\mathrm{v}}=(2e^2/m)\int_{-1}^{1}d\mu\,\mu J(\mu,v)\log[k_{\max}^2/(4\pi e^2|R|/m)].$$

Since $J(\mu,v)$ provides a sharp cutoff for large μv, we can use the small μv limit of R in the logarithm. Defining

$$\lambda_D^2\equiv(4\pi e^2\beta n)^{-1},$$

we get

$$F_{\mathrm{v}}=\frac{2e^2}{m}\log(k_{\max}^2\lambda_D^2)\int_{-1}^{1}d\mu\,\mu J(\mu,v)$$
$$=-\frac{4\pi e^2}{mv^2}\left(\int_0^v f_1^{(0)}(u)du\bigg/\int_0^\infty f_1^{(0)}(u)du\right)\log(k_{\max}\lambda_D). \tag{16}$$

This friction coefficient must be added to a similar coefficient coming from the statistical effect, which is calculated in the following section. Before going on to this calculation, however, we would like to point out an additional consequence of the polarization, namely the well-known Debye screening of the Coulomb force between two charged particles in the medium.

If for simplicity we consider only the $v=0$ limit, then $I(\mu,0)=-\beta mn$, which gives [Eq. (11)]:

$$\Phi(k,s)=-\frac{1}{ms}\left(\frac{4\pi e^2}{k^2}\right)^2\frac{\beta mn}{1+(4\pi e^2\beta n/k^2)}. \tag{17}$$

This is the Fourier transform of the potential due to the polarization (charge separation) of the plasma, as is clear from Eq. (8). To obtain the total potential due to a charged particle in such a plasma, we must add the unmodified Coulomb potential due to a point charge. Thus

$$\Phi^{\mathrm{tot}}(k,s)=\frac{4\pi e^2}{s}\frac{1}{k^2+4\pi e^2\beta n}=\frac{4\pi e^2}{s}\frac{1}{k^2+\lambda_D^{-2}}, \tag{18}$$

i.e., the potential in this limit has the Yukawa shape $e^{-r/\lambda_D}/r$. If the velocity v does not go to zero, the screening will be modified by velocity-dependent terms.

B. Statistical Effect

In the preceding section, we calculated the average force on a test particle due to the modifications it makes in the motion and, thus, in the distribution of field particles. In the present section, the modification which the random fluctuations in the distribution of field particles makes in the motion of the test particle is investigated. That a damping may arise as a result of the interaction of the test particle with the fluctuations is clear when one notes that the test particle starts out with a definite velocity and as a result of random interactions its direction will be changed so that its average velocity with respect to the medium decreases with time. Although the modification in the distribution function as calculated in Sec. IIA acts to produce a potential, as far as the average density is concerned, it is negligible.[8] We can therefore confine our discussion to the case of a uniform average density ρ_0; deviations of the density from ρ_0 have as their sole consequence the polarization force calculated in the last section. As just indicated, there exists a damping force even in the case of uniform average density. This force arises as a result of correlations between the motion of the test particle and fluctuations in density about the mean [e.g., as taken into account by the second term in Eq. (2)], so that an additional change in velocity is obtained. The leading term of this damping force is just

$$\left\langle\left\langle\int_0^t dt'\int_0^{t'} dt''\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t'']\cdot\nabla_{\mathbf{z}_0}\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t]\right\rangle\right\rangle = \left\langle \tau^{-1}\int_0^\tau dt\int_0^t dt'\int_0^{t'} dt''\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t''] \cdot\nabla_{\mathbf{z}_0}\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t]\right\rangle. \tag{19}$$

To perform the ensemble average, we need a probability distribution function for the forces due to a system of particles distributed in space with a constant average density ρ_0, and having a given distribution in velocity space. Such a distribution has been obtained by Holtsmark[9] on the assumption of equal *a priori* probability for finding a field particle anywhere in the total volume Ω. However, since the effect of the nearest-neighbor interactions is to be treated by collision methods (which are outside the considerations of this paper), we wish to disregard these interactions and thus we need a modified probability-distribution function that excludes nearest neighbor effects. If this separation is not carried out, difficulties appear because the Holtsmark distribution has divergent second and higher moments, reflecting singularities at short distances.

We outline here a simple derivation of this modified Holtsmark distribution for the case of one nearest neighbor.[10] Since the argument is independent of the velocity distribution, we ignore that aspect of the problem. In terms of the probability distribution for a set of particles $W(x_1,x_2,x_3,\cdots x_N)$, the probability distribution for the forces is

$$W[\mathbf{F}(0)=\mathbf{F}]=\int d\mathbf{x}_1\cdots d\mathbf{x}_N W[\mathbf{x}_1\cdots\mathbf{x}_N] \times\delta\Big(\mathbf{F}-\sum_{i=1}^{N}{}' \mathbf{F}_i(\mathbf{x}_i)\Big), \tag{20}$$

where the prime denotes the exclusion of the particle nearest the origin. We assume that

$$W[x_1\cdots x_N]=\prod_{i=1}^N \omega(\mathbf{x}_i),$$

where $\omega(x_i)$ is the probability of finding a particular particle at x_i. In the light of the work of the preceding section, this assumption is equivalent to the statement that except for providing the screening of the Coulomb force between particles, the interparticle correlations are negligible. Since the integrand is symmetric in all its variables, we may arbitrarily label the particles in order of their distance from the origin, so that the nearest neighbor is the one at x_1. The characteristic function, defined by

$$\varphi(\mathbf{k})=\int d\mathbf{F}\exp(-i\mathbf{k}\cdot\mathbf{F})W[\mathbf{F}(0)=\mathbf{F}], \tag{21}$$

therefore is

$$N\int_0^\infty d\mathbf{x}_1\omega(\mathbf{x}_1)\left(\int_{|\mathbf{x}_1|}^\infty d\mathbf{x}'\omega(\mathbf{x}')\exp[-i\mathbf{k}\cdot\mathbf{F}(\mathbf{x}')]\right)^{N-1},$$

where the factor N comes from the number of ways in which the nearest neighbor can be chosen from the N particles, and the lower limit on the integrals for the "external" particles insures that particle No. 1 is nearest to the origin. Now

$$\left(\int_{|\mathbf{x}_1|}^\infty d\mathbf{x}'\omega(\mathbf{x}')\exp[-i\mathbf{k}\cdot\mathbf{F}(x')]\right)^{N-1} =\left\{1-\int_0^{|\mathbf{x}_1|}\omega(\mathbf{x}')d\mathbf{x}'+\int_{|\mathbf{x}_1|}^\infty d\mathbf{x}'\omega(\mathbf{x}') \times(\exp[-i\mathbf{k}\cdot\mathbf{F}(\mathbf{x}')]-1)\right\}^{N-1},$$

since $\int_0^\infty d\mathbf{x}'\omega(\mathbf{x}')=1$. Writing $\omega(\mathbf{x}')=\rho_0/N$, where ρ_0 is the particle density (which for the purpose of this derivation need not be taken as constant), we can let $N\to\infty$, and using $\lim_{N\to\infty}(1+\alpha/N)^N=e^\alpha$ we obtain

$$\varphi(\mathbf{k})=\int_0^\infty \rho_0 d\mathbf{x}\exp\left\{\int_0^{|\mathbf{x}|}\rho_0 d\mathbf{x}' -\int_{|\mathbf{x}|}^\infty \rho_0 d\mathbf{x}'\{\exp[-i\mathbf{k}\cdot\mathbf{F}(\mathbf{x}')]-1\}\right\}. \tag{22}$$

Actually for the calculation of the average in Eq. (19) we need the joint probability $W[\mathbf{F}(\mathbf{x}_1)=\mathbf{F}_1;\mathbf{F}(\mathbf{x}_2)=\mathbf{F}_2]$ for the force to be $\mathbf{F}_1$ at $\mathbf{x}_1$ and $\mathbf{F}_2$ at $\mathbf{x}_2$, the force at $\mathbf{x}_2$ being due to the "external" particles, and that at $\mathbf{x}_1$ being due to *all* particles. By including the nearest neighbors in $\mathbf{F}_1$, we take approximate account of the fact that in Eq. (19) the term $\mathbf{F}[\mathbf{z}_0+\mathbf{v}_0t'']$ represents the acceleration of the test particle that is due to the action of all the

[8] As can be verified using the results of Sec. IIA, the density change is of the order of $\rho_0\beta e^2k_{\max}$, which means that in most plasma problems, $\delta\rho/\rho_0\ll 1$.

[9] See, e.g., S. Chandrasekhar, Revs. Modern Phys. **15**, 1 (1943).

[10] A general discussion of the modified Holtsmark distribution excluding n nearest neighbors appears in Appendix B.

particles. This generalized distribution function has as its characteristic function

$$\varphi(\mathbf{p},\mathbf{q})=\int\rho_0 d\mathbf{x}\exp[-i\mathbf{p}\cdot\mathbf{F}(\mathbf{x}-\mathbf{x}_1)]\times\exp\left(-\int_{S_I}\rho_0(\mathbf{y}-\mathbf{x}_2)d\mathbf{y}\right)\times\exp\left(\int_{S_0}d\mathbf{y}\rho_0(\mathbf{y}-\mathbf{x}_2)[\exp(-i\mathbf{p}\cdot\mathbf{F}(\mathbf{y}-\mathbf{x}_1)-i\mathbf{q}\cdot\mathbf{F}(\mathbf{y}-\mathbf{x}_2))-1]\right),\quad(23)$$

where S_I and S_0 are the regions interior and exterior, respectively, to a sphere of radius x about the point $\mathbf{x}_2$. Since this function is the Fourier transform of $W[\mathbf{F}(\mathbf{x}_1)=\mathbf{F}_1;\mathbf{F}(\mathbf{x}_2)=\mathbf{F}_2]$, the averages are directly obtainable in terms of the derivatives of the characteristic function with respect to $\mathbf{p}$ and $\mathbf{q}$ evaluated at $\mathbf{p}=\mathbf{q}=0$. For example,

$$\langle F_i(\mathbf{x}_1)F_j(\mathbf{x}_2)\rangle=-[(\partial/\partial p_i)(\partial/\partial q_j)\varphi(\mathbf{p},\mathbf{q})]_{\mathbf{p}=\mathbf{q}=0}.$$

The quantity in Eq. (19) to be calculated is

$$\left\langle\left\langle F_i(\mathbf{x}_1)\frac{\partial}{\partial x_{2j}}F_j(\mathbf{x}_2)\right\rangle\right\rangle=\rho_0\int d\mathbf{x}\exp[-\rho_0V(x)]\left[-\rho_0\int_{|\mathbf{x}|}^{\infty}d\mathbf{x}'F_i(\mathbf{x}'-\mathbf{r})\times\frac{\partial}{\partial x_i'}F_j(\mathbf{x}')-\rho_0F_i(\mathbf{x}-\mathbf{r})\int_{|\mathbf{x}|}^{\infty}d\mathbf{x}'\frac{\partial}{\partial x_i'}F_j(\mathbf{x}')-\rho_0^2\int_{|\mathbf{x}|}^{\infty}d\mathbf{x}'F_i(\mathbf{x}-\mathbf{r})\int_{|\mathbf{x}|}^{\infty}d\mathbf{x}''\frac{\partial}{\partial x_i''}F_j(\mathbf{x}'')\right],\quad(24)$$

where $\mathbf{r}=\mathbf{x}_1-\mathbf{x}_2$, $V(x)=4\pi x^3/3$ and a constant ρ_0 is being used. We notice that in the last two terms $\int_{|x|}^{\infty}d\mathbf{x}'(\partial/\partial x_i')F_j(\mathbf{x}')$ occurs. The angular integration yields zero unless $i=j$, in which case the integral becomes $\frac{1}{3}\int_{|x|}^{\infty}\nabla'\cdot\mathbf{F}'d\mathbf{x}'$. When F is replaced by $-\nabla\varphi$, where φ is the screened potential (taken in the static limit for simplicity) $\nabla\cdot F$ becomes $-\nabla^2\varphi=-\lambda_D^{-2}\varphi$, the contact term not contributing because of the finite lower limit on the integral. These terms are eventually to be integrated over time. On interchanging the final $\int d\mathbf{x}$ and $\int dt$, one finds that only the force in the direction perpendicular to the relative velocity does not disappear on integrating over time. The final $\int d\mathbf{x}$, however, then gives zero on grounds of symmetry. Thus we need only calculate

$$-\rho_0\int d\mathbf{x}\exp[-\rho_0V(x)]\int_{|\mathbf{x}|}^{\infty}d\mathbf{x}'F_i(\mathbf{x}'-\mathbf{r})\frac{\partial}{\partial x_i'}F_j(\mathbf{x}'),$$

which upon interchanging of order of integration gives

$$I(\mathbf{x}_1-\mathbf{x}_2)=-\rho_0\int d\mathbf{x}\{1-\exp[-\rho_0V(x)]\}\times F_i(\mathbf{x}-\mathbf{x}_1+\mathbf{x}_2)\frac{\partial}{\partial x_i}F_j(\mathbf{x}).\quad(25)$$

The exclusion of the nearest neighbor results in the appearance of the cutoff function $[1-\exp(-\rho_0V(x))]$. To avoid complicated numerical integrations, we shall approximate its effect by using a more tractable function, which also has the feature of approaching unity for $x\gg\rho_0^{-\frac{1}{3}}$ and vanishing for $x\ll\rho_0^{-\frac{1}{3}}$. Finally, the total expression to be evaluated is

$$\mathcal{J}=\int_0^t dt'\int_0^{t'}dt''I[\mathbf{u}(t''-t)]=\int_0^t dt''(t-t'')I[\mathbf{u}(t''-t)],$$

where $\mathbf{u}$ is clearly the velocity of the test particle relative to the field particle under consideration. To evaluate this integral, we Fourier-transform with respect to x to obtain

$$-\frac{2i}{\pi}\int_0^t(t-t'')dt''\int_0^{k_{\max}}d\mathbf{k}k^2k_i(k^2+\mu^2)^{-2}\times\exp[-i\mathbf{k}\cdot\mathbf{u}(t''-t)],$$

where $\mu=\lambda_D^{-1}$, and where the effect of the nearest neighbor exclusion is taken to introduce a cutoff in the k-integration at $k_{\max}\sim\rho_0^{\frac{1}{3}}$. Performing the time integration, we get

$$\frac{2}{\pi}\int_0^{k_{\max}}dk\frac{k^2k_i}{(k^2+\mu^2)^2}\frac{d}{d\lambda}\left[\pi\delta(\lambda)+iP\frac{1}{\lambda}\right]_{\lambda=\mathbf{k}\cdot\mathbf{u}},\quad(26)$$

where P stands for principal value.[11] The angular integration shows that a nonzero contribution appears only in the case for which k_i is parallel to $\mathbf{u}$, and only the δ function has a nonzero integral. Thus one gets, for the component of the force parallel to $\mathbf{u}$,

$$\mathcal{J}=-(2\pi/u^2)[2\log(\lambda_Dk_{\max})-(k_{\max}\lambda_D)^2/(1+k_{\max}^2\lambda_D^2)].\quad(27)$$

In most cases of interest, the second term may be neglected.

Finally, we recall that the velocity distribution of the field particles was ignored, since all effects leading to the expression for $\mathcal{J}$ were independent of it. However, to obtain the complete statistical effect, the ensemble average must be extended to include this distribution,

[11] In performing the time integration we have taken the limit of $t\rightarrow\infty$. This corresponds to taking τ large enough so as to average over the fluctuations (see Appendix A).

so that we want

$$\langle\langle \mathbf{F}\rangle\rangle = -\frac{2\pi\rho_0 e^4}{m}\log(\lambda_D k_{\max})^2 \times\int \frac{\mathbf{v}-\mathbf{V}}{|\mathbf{v}-\mathbf{V}|}\frac{f(\mathbf{V})}{|\mathbf{v}-\mathbf{V}|^2}d\mathbf{V}, \quad (28)$$

where $\mathbf{v}$ is the velocity of the test particle, $\mathbf{V}$ is the absolute velocity of the field particle, and $(\mathbf{v}-\mathbf{V})/|\mathbf{v}-\mathbf{V}|$ appears because the forces due to the various members of the distribution in velocity add vectorially. If $f(\mathbf{V})$ is spherically symmetric, the angular integrations can be carried out. In this case only the component of $\mathbf{F}$ parallel to $\mathbf{v}$ remains, and yields

$$\langle\langle F_{\mathrm{v}}\rangle\rangle = -\frac{4\pi e^4\rho_0}{mv^2}\log(\lambda_D k_{\max})\frac{\int_0^v f(\mathbf{V})d\mathbf{V}}{\int_0^\infty f(\mathbf{V})d\mathbf{V}}. \quad (29)$$

In the limit of $|\mathbf{v}|\gg\langle V^2\rangle^{\frac{1}{2}}$ the integrals in Eq. (29) cancel, and we are left with the result

$$\langle\langle F_{\mathrm{v}}\rangle\rangle = -\frac{4\pi e^4\rho_0}{mv^2}\log(\lambda_D k_{\max}),$$

independent of the form of the distribution. Assuming a Maxwellian distribution, we can also calculate the low-velocity limit $v\ll v_T$:

$$\langle\langle F_{\mathrm{v}}\rangle\rangle = -(2\pi)^{\frac{1}{2}}\frac{4e^4\rho_0}{3\kappa T}\frac{v}{v_T}\log(\lambda_D k_{\max}).$$

Combining the results of Eq. (29) with those obtained in Sec. IIA [Eq. (16)], we get the total dynamical friction:

$$\langle\langle F_{\mathrm{v}}\rangle\rangle = -\frac{4\pi e^4\rho_0}{v^2}\left(\frac{1}{m_t}+\frac{1}{m_f}\right) \times\log(\lambda_D k_{\max})\frac{\int_0^v f(\mathbf{V})d\mathbf{V}}{\int_0^\infty f(\mathbf{V})d\mathbf{V}}. \quad (30)$$

We have distinguished by labels t and f the masses of the test and field particles, the former appearing in the contribution from the statistical effect, and the latter in the polarization damping.

Before closing this section, we might remark that, strictly speaking, this result is limited to a case in which there is only one type of particle present, so that $m_t\equiv m_f$. However, if two kinds of particles are in the field, the preceding analysis goes through for both the polarization and the statistical effects, so that the total force is given by the sum of two terms, in each of which m_f and $f(V)$ correspond to the different field particles. It is to be noted that in this case $\lambda_D=(4\pi e^2\beta n')^{\frac{1}{2}}$, where n'= electron density + |ionic charge density|.

III. HIGHER MOMENTS

To proceed with the calculation, we must now calculate $\langle(\Delta v)^2\rangle$ and examine the higher F.P. coefficients to see whether the sequence of $\alpha^{(n)}$ converges rapidly enough so that the general F.P. equation may be approximated by a diffusion equation. The diffusion coefficient is given by

$$\alpha_{ij}^{(2)} = \tau^{-1}\langle\Delta v_i\Delta v_j\rangle = (1/m^2\tau) \times\int_0^\tau dt'\int_0^\tau dt''\langle F_i[\mathbf{z}(t')]F_j[\mathbf{z}(t'')]\rangle, \quad (31)$$

where, as before, the limit of $\tau\to\infty$ is to be taken. Using the Holtsmark distribution and the rectilinear motion approximation, we can write $\alpha_{ij}^{(2)}$ as[12]

$$\alpha_{ij}^{(2)} = (\rho_0/m^2\tau)\int_0^\tau dt'\int_0^\tau dt'' \times\int d\mathbf{x}_0 F_i(\mathbf{x}_0-\mathbf{z}_0-\mathbf{u}t')F_j(\mathbf{x}_0-\mathbf{z}_0-\mathbf{u}t''), \quad (31')$$

where $\mathbf{u}$ is the velocity of the test particle relative to the field particle. Taking the Fourier transform, we find that

$$\alpha_{ij}^{(2)} = \frac{2e^4\rho_0}{m^2\pi}\frac{1}{\tau}\int^\tau dt'\int_0^\tau dt'' \times\int d\mathbf{k}\frac{k_ik_j}{(k^2+\mu^2)^2}\exp[i\mathbf{k}\cdot\mathbf{u}(t''-t')]. \quad (32)$$

Introducing the relative time $s=t''-t'$, we notice that $\int_0^\tau dt''$ becomes $\int_{-t'}^{\tau-t'}ds$, which, as $\tau\to\infty$, and for $\tau\gg t'\gg 0$, can be replaced by $\int_{-\infty}^\infty ds$. The remaining t' integration cancels the τ in the denominator, so that we get

$$\alpha_{ij} = \frac{2e^4\rho_0}{m^2\pi}\int d\mathbf{k}k_ik_j(k^2+\mu^2)^{-2}\delta(\mathbf{k}\cdot\mathbf{u}). \quad (33)$$

Clearly only the components of k perpendicular to u contribute. If we choose u in the z direction, only $\alpha_{11}^{(2)}$ and $\alpha_{22}^{(2)}$ do not vanish:

$$\alpha_{11}^{(2)} = \alpha_{22}^{(2)} = \frac{4\pi e^4\rho_0}{m^2u}[\log(\lambda_D k_{\max}) -\tfrac{1}{2}(\lambda_D k_{\max})^2/(1+\lambda_D^2 k_{\max}^2)]. \quad (34)$$

(Again we shall neglect the second term.)

[12] As previously, the effect of nearest-neighbor exclusion is expressed as a short-range cutoff, and again we do not take the velocity distribution of the field particles into account until the end of the calculation.

To perform the velocity distribution average, we must transform the tensor $\alpha_{ij}^{(2)}$ to the coordinate system of the test particle. We find that the coefficient obtained above, $\delta_{ij}(1-\delta_{i3})$, becomes $\delta_{ij}-u_iu_j/u^2$. Taking v, the absolute velocity of the test particle, to be in the z direction, we find that if the velocity distribution $f(\mathbf{V})$ is spherically symmetric, the azimuthal integration removes all off-diagonal terms, and leaves

$$\alpha_{ij}^{(2)}(v)=(4\pi e^4\rho_0/m^2)\Phi_{ij}(v)\log(\lambda_D k_{\max}), \qquad (35)$$

where

$$\Phi_{ij}(v)=\delta_{ij}\left(\frac{1}{v}\int_0^v f(\mathbf{V})d\mathbf{V}-\frac{1}{3v^3}\int_0^v f(\mathbf{V})V^2d\mathbf{V}+\tfrac{2}{3}\int_v^\infty (f(\mathbf{V})/V)d\mathbf{V}\right),\quad i=1,2$$

$$=\tfrac{2}{3}\delta_{ij}\left(\frac{1}{v^3}\int_0^v f(\mathbf{V})V^2d\mathbf{V}+\int_v^\infty (f(\mathbf{V})/V)d\mathbf{V}\right),\quad i=3 \qquad (35')$$

and $\int_0^\infty f(\mathbf{V})d\mathbf{V}=1$. For large v, this reduces to

$$\alpha_{ij}^{(2)}=\frac{4\pi e^4\rho_0}{m^2v}\delta_{ij}(\delta_{i1}+\delta_{i2})\log(\lambda_D k_{\max}), \qquad (35a)$$

while for a Maxwellian distribution we can also compute the low-velocity limit

$$\alpha_{ij}^{(2)}=\frac{16(\pi)^{\frac{1}{2}}e^4\rho_0}{3m^2v_T}\delta_{ij}[\log(\lambda_D k_{\max})]. \qquad (35b)$$

As in the treatment of the dynamical friction, these relations are strictly valid only for one type of field particle. However, it is clear from the derivation that if several types of field particle are present we simply add their effects.

The next coefficient in the F.P. equation, $\alpha_{ijk}^{(3)}$, is of the form $\tau^{-1}\langle\Delta v_i\Delta v_j\Delta v_k\rangle$. In the exact expression, analogous to Eq. (31), we again make the rectilinear motion approximation to obtain

$$\alpha_{ijk}^{(3)}=(\rho_0/m^3\tau)\int_0^\tau dt_1\cdots\int_0^\tau dt_3\int d\mathbf{x}_0F_i(\mathbf{x}_0-\mathbf{z}_0-\mathbf{u}t_1)\times F_j(\mathbf{x}_0-\mathbf{z}_0-\mathbf{u}t_2)F_k(\mathbf{x}_0-\mathbf{z}_0-\mathbf{u}t_3)$$

$$\simeq(\rho_0/m^3)\int_{-\infty}^\infty dt\int_{-\infty}^\infty dt'\int d\mathbf{x}_0F_i(\mathbf{x}_0')\times F_j(\mathbf{x}_0'-\mathbf{u}t)F_k(\mathbf{x}_0'-\mathbf{u}t'),$$

where we have used relative times as before, and one of the time integrations has been cancelled against τ^{-1}. Taking the Fourier transform as before, and performing the remaining time integrations, we get

$$\alpha_{ijk}^{(3)}=\frac{4\rho_0e^6}{\pi^2m^3}\int d\mathbf{k}_1d\mathbf{k}_2d\mathbf{k}_3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\delta(\mathbf{k}_1\cdot\mathbf{u})\delta(\mathbf{k}_2\cdot\mathbf{u})\times k_{1i}k_{2j}k_{3k}(k_1^2+\mu^2)^{-1}(k_2^2+\mu^2)^{-1}(k_3^2+\mu^2)^{-1}. \qquad (36)$$

Since $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ are constrained to lie in a plane perpendicular to $\mathbf{u}$, if we choose $\mathbf{u}$ as z axis, $\alpha_{ijk}^{(3)}=0$ unless $i, j, k\neq 3$. Also, since $\mathbf{k}_1$, $\mathbf{k}_2$, and $\mathbf{k}_3$ form a triangle, only two azimuthal angular integrations are involved. If these are carried out, one readily sees that $\alpha_{ijk}^{(3)}=0$ in all cases. When corrections to the rectilinear motion approximation analogous to those in Eq. (2) are made, $\alpha_{ijk}^{(3)}$ no longer vanishes, but involves the correlation between forces at four different times, and thus is of the same order as $\langle(\Delta v)^4\rangle$.[13] Thus, instead of calculating the modified $\alpha_{ijk}^{(3)}$, we shall examine the next coefficient. This can be set up as before, and if the same manipulations are carried out, we find that for a particular relative velocity u,

$$\alpha_{ijkl}^{(4)}=\frac{4\rho_0e^8}{\pi^2u^3m^4}\int d\mathbf{k}_1\cdots d\mathbf{k}_4\delta(\mathbf{k}_1+\cdots+\mathbf{k}_4)\times\delta(\cos\theta_2)\delta(\cos\theta_3)\delta(\cos\theta_4)\times k_{1i}\cdots k_{4l}(k_1^2+\mu^2)^{-1}\cdots(k_4^2+\mu^2)^{-1}. \qquad (37)$$

The integral is just a function of $\mu=\lambda_D^{-1}$ and $k_{\max}$, and therefore $\alpha_{ijkl}^{(4)}$, when averaged over the velocity distribution of the field particles, diverges logarithmically for small relative velocities. Physically, the reason for this divergence is that, in a rectilinear-motion approximation, particles of small relative velocity interact with each other over long periods of time, thus producing a very large effect. In the complete system, however, two such particles do not stay together indefinitely but rather diffuse away from each other, as a result of interactions with other members of the system. To obtain a more realistic result this effect has to be taken into account. This is done in the following section, in which it is shown that the effect of the spatial diffusion is to introduce an effective cutoff at low relative velocities. Using this result, one can make an estimate of the higher coefficients, and one finds that, apart from numerical and logarithmic factors (which cannot amount to more than one order of magnitude), the coefficients decrease in the ratio $(e^2\rho_0^{\frac{1}{3}}/\kappa T)^2$, i.e., mean potential energy/mean kinetic energy,[2] which for most physically interesting conditions leads to extremely rapid convergence.

IV. HIGHER APPROXIMATIONS

Generally speaking, the determination of the F.P. coefficients hinges on the determination of the values of

[13] This is equivalent to the result that the statistical part of the dynamical friction is of the same order as the diffusion coefficient.

expressions of the type $\langle\langle \mathbf{F}[\mathbf{z}_1(t_1)]\cdots\mathbf{F}[\mathbf{z}_n(t_n)]\rangle\rangle$. The work of the last section indicates that the rectilinear motion approximation fails for $n>2$ (and even for $n=2$ in the case of a gravitational potential, for which there is no Debye shielding). It is thus necessary to depart from the rectilinear motion approximation in the direction of taking into consideration the correlations of the test particle with more than one field particle. Clearly, as long as one considers correlations with only one field particle, the motion is determined completely by the initial conditions, whose indefiniteness introduces the stochastic element into the theory. This determinacy is present whether the motion is expanded about some unperturbed path, or whether an exact solution of the two-body problem is obtained. However, as soon as one considers these two particles (both of which are now considered to be "test particles") to be in the fluctuating force-field due to the remainder of the field particles, their motions no longer are determined, and the "path" of a test particle is now to be considered as a stochastic variable. Since the forces in $\langle\langle \mathbf{F}[\mathbf{z}_1(t_1)]\cdots\mathbf{F}[\mathbf{z}_n(t_n)]\rangle\rangle$ are obtained additively from forces between pairs of particles, the distribution required to calculate such averages is the joint probability distribution for the paths of a pair of interacting particles. This distribution is assumed to be governed by a generalized F.P. equation, in which the coefficients serve to eliminate the interaction with the remaining field particles. Since these coefficients are calculated from expressions of the type $\langle\langle \mathbf{F}[\mathbf{z}_1(t_1)]\cdots\mathbf{F}[\mathbf{z}_n(t_n)]\rangle\rangle$, ideally this presents a complicated set of equations for the coefficients, which must be solved in a self-consistent manner. Such a generalized two-body F.P. equation represents a rather complicated picture of the diffusion of two particles interacting with each other in a common random field. If, as is the case in most problems of interest, the "mean" kinetic energy greatly exceeds the "mean" potential energy, one expects that the mutual interactions will play a small role in the diffusion (the particles behave essentially as if they were free), and under these circumstances it will be a good approximation to decouple the "paths" of the two test particles by neglecting the effect of their mutual interaction on their motion. Thus, the probability distribution for the "path" of each particle is given by the solution of the one-body F.P. equation. In this case, the nth F.P. coefficient obeys an equation of the form

$$\alpha^{(n)}=f_n(\alpha^{(1)}\cdots\alpha^{(n)}\cdots),\quad n=1,2,\cdots.$$

As can readily be seen by cutting off the divergent integrals in the previous section, $\alpha^{(n)}$ decreases rapidly with increasing n, so that such an equation can be solved by successive approximations.

Since $\alpha_{ij}^{(2)}$ as calculated in Sec. III is small, and suffers from no divergence difficulties, higher corrections to it are small. On the other hand, all higher coefficients are divergent if $\alpha^{(2)}$ is set equal to zero, and to prevent such difficulties it is necessary to include $\alpha^{(2)}\neq 0$, which (as will be indicated) removes the divergences. Replacement of the rectilinear motion by the "diffusing" motion, calculated in Appendix C, in the expression for $\alpha_{ijkl}^{(4)}$ results in a very untransparent expression. Since, however, the same convergence-producing modifications arise in a second-order calculation of $\alpha_{ij}^{(2)}$, for the sake of clarity we shall illustrate these features by a calculation of the latter.

If the motions of the particles are now to be governed by a probability distribution, Eq. (31′) is generalized to

$$\begin{aligned}\langle F_i[\mathbf{z}(t_1)]F_j[\mathbf{z}(t_2)]\rangle &= \int d\mathbf{x}_0 d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{z}_1 d\mathbf{z}_2 \\ &\quad\times F_i(\mathbf{z}_1-\mathbf{x}_1)F_j(\mathbf{z}_2-\mathbf{x}_2) \\ &\quad\times P[\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2;\,\mathbf{z}_0,\mathbf{z}_1,\mathbf{z}_2;\,t_1,t_2],\end{aligned}\qquad(38)$$

where $P[\mathbf{x}_0,\mathbf{x}_1,\mathbf{x}_2;\,\mathbf{z}_0,\mathbf{z}_1,\mathbf{z}_2;\,t_1,t_2]$ is the joint probability that the field particle (which is also a "test" particle from this point of view) initially is at $\mathbf{x}_0$, and at $\mathbf{x}_1$ and $\mathbf{x}_2$ at times t_1 and t_2 respectively, and similarly for the test particle. Decoupling the "paths" corresponds to writing

$$\begin{aligned}P[\mathbf{x}_0,\cdots;\,\mathbf{z}_0,\cdots;\,t_1,t_2] &= P[\mathbf{x}_1|\mathbf{x}_0;\,t_1]P[\mathbf{x}_2|\mathbf{x}_1;\,t_2-t_1] \\ &\quad\times P[\mathbf{z}_1|\mathbf{z}_0;\,t_1]P[\mathbf{z}_2|\mathbf{z}_1;\,t_2-t_1]P[\mathbf{x}_0],\end{aligned}\qquad(39)$$

for $t_2>t_1$, and a similar expression if $t_2<t_1$.[14] Here $P(\mathbf{x}_1|\mathbf{x}_0;\,t_1)$ is the conditional probability that a particle arrives at $\mathbf{x}_1$ after a time t_1, given that it started at $\mathbf{x}_0$. Strictly speaking, all these probabilities should involve the velocities, but since the dynamical friction and the diffusion in velocity space cause extremely small changes in the velocity over the time intervals of interest for diffusion in coordinate space, we can neglect such changes, except insofar as they cause spatial diffusion. As before, one time integration in the expression for $\alpha_{ij}^{(2)}$ can be removed, and if relative coordinates are introduced, two spatial integrations can be carried out immediately to yield

$$\alpha_{ij}^{(2)}=\frac{2\rho_0}{m^2}\int_0^\infty dt\int d\mathbf{z}\int d\mathbf{x}_1 d\mathbf{x}_2 P[\mathbf{z}|0;\,t] \times P[\mathbf{x}_2-\mathbf{x}_1|0;\,t]F_i[-\mathbf{x}_1]F_j[\mathbf{z}-\mathbf{x}_2].$$

Taking the Fourier transform, and introducing the solution of the diffusion equation (Appendix C) we get, after carrying out the x integrations,

$$\alpha_{ij}^{(2)}=\frac{4e^4\rho_0}{m^2\pi}\int_0^\infty dt\int d\mathbf{k}k_ik_j(k^2+\mu^2)^{-2} \times\exp\{i\mathbf{k}\cdot(\mathbf{v}-\mathbf{V})t-\tfrac{2}{3}\bar{\alpha}k^2t^3\},$$

[14] If no diffusion is taken into account, the probabilities are replaced by delta functions; in particular, for the rectilinear motion approximation,

$$P[\mathbf{x}_1|\mathbf{x}_0;\,t_1]=\delta(\mathbf{x}_1-\mathbf{x}_0-\mathbf{u}t).$$

where $\bar{\alpha}$ is taken to be the low-velocity limit of the differential coefficient in Eq. (35b). Performing the angular integration, we find

$$\alpha_{ij}^{(2)}=\frac{4\rho_0 e^4}{m^2}\delta_{ij}\int_0^\infty dt\int_0^{k_{\max}} dk k^4(k^2+\mu^2)^{-2} \times\int_{-1}^{1} d(\cos\theta)\, \exp[ikt|\mathbf{v}-\mathbf{V}|\cos\theta-\tfrac{2}{3}\bar{\alpha}k^2t^3] \times\begin{cases}\sin^2\theta & i=1,2\\ 2\cos^2\theta. & i=3\end{cases} \quad (40)$$

If $\bar{\alpha}$ were zero, the integral with $\cos^2\theta$ would vanish, and for small $\bar{\alpha}$ it will be small. Therefore, for the purpose of discussing the integral, we replace $\sin^2\theta$ by $1-\cos^2\theta$ and neglect the integral of the second term. The angular integration then yields

$$\frac{8\rho_0 e^4}{m^2}\int_0^\infty dt\int_0^{k_{\max}} dk\frac{k^4}{(k^2+\mu^2)^2}\frac{\sin kt|\mathbf{v}-\mathbf{V}|}{kt|\mathbf{v}-\mathbf{V}|}\exp(-\tfrac{2}{3}\bar{\alpha}k^2t^3)$$
$$=\frac{8\rho_0 e^4}{m^2|\mathbf{v}-\mathbf{V}|}\int_0^\infty d\varphi\int_0^{k_{\max}} dk\frac{k^3}{(k^2+\mu^2)^2}\frac{\sin\varphi}{\varphi} \times\exp\left[-\tfrac{2}{3}\frac{\bar{\alpha}\varphi^3}{k|\mathbf{v}-\mathbf{V}|^3}\right]. \quad (41)$$

An examination of the integral over φ shows that for $\bar{\alpha}\gg k|\mathbf{v}-\mathbf{V}|^3$ the integral goes to zero, whereas for $\bar{\alpha}\ll k|\mathbf{v}-\mathbf{V}|^3$ it approaches $\pi/2$.[15] This provides a cutoff in the k-integration, i.e. $k\geqslant\bar{\alpha}|\mathbf{v}-\mathbf{V}|^{-3}$, in order that the integrand in the k-integration not vanish. To estimate this integral, we replace the φ integral by a step function, and obtain approximately

$$\frac{4\pi\rho_0 e^4}{m^2|\mathbf{v}-\mathbf{V}|}\log\frac{k_{\max}^2+\mu^2}{\bar{\alpha}^2|\mathbf{v}-\mathbf{V}|^{-6}+\mu^2}\text{ for }\bar{\alpha}|\mathbf{v}-\mathbf{V}|^{-3}<k_{\max},$$
$$0\text{ for }\bar{\alpha}|\mathbf{v}-\mathbf{V}|^{-3}>k_{\max}. \quad (42)$$

Thus, the spatial diffusion provides a natural cutoff in the velocity integrals. This feature persists for higher order F.P. coefficients. It is to be noted that in the above expression, μ, the large-distance (Debye) cutoff, may actually be set equal to zero without destroying convergence. This is of interest in astronomical problems, where all forces are attractive and no natural "screening" distance exists. In such a case, one would assume the existence of an unknown diffusion coefficient α, and solve the implicit equations obtained above for this unknown α.

V. CONCLUSION

In this investigation a form of perturbation theory (p.t.) was used in a treatment of the dynamics of an ionized medium. Although for the purpose of calculating higher order F.P. coefficients a departure from the original p.t. was necessary to avoid formal divergences, this departure merely indicated a natural minimum relative velocity cutoff, which could then be used to make the p.t. convergent. That the p.t. is well suited to the examination of effects on a particle due to the medium, excluding the nearest neighbor, is evident from (i) the smallness of the corrections to a particular F.P. coefficient, and (ii) the rapid convergence of the sequence of successive F.P. coefficients which for practical purposes reduces the general F.P. equation to an ordinary diffusion equation. Of course, to obtain the correct expressions for the F.P. coefficients, the effect of the nearest neighbors must also be considered by a collision treatment, which contains the exact dynamical path of the two (nearest) particles in interaction. One can however extrapolate the p.t. so as to include the nearest neighbors, which corresponds to working with the unmodified Holtsmark distribution (cutoff at the distance of "nearest approach"). It turns out that this extrapolation corresponds to replacing $k_{\max}\sim\rho_0^{\frac{1}{3}}$ by $\sim\kappa T/e^2$, the minimum impact parameter. When this is done, the results can be compared with the calculation of these coefficients using the exact solutions of the two-body equations in a Boltzmann type treatment which was carried out by Judd, MacDonald, and Rosenbluth.[16] The "extrapolated" p.t. coefficients agree with the results of this calculation, from which one can conclude that the perturbation approach is valid over wide ranges of relevant parameters, and its ease of handling may make it particularly useful in more complicated problems, such as that of an ionized medium in a strong magnetic field, where the smallness of the Larmor radius relative to the other "lengths" would make a collision treatment very difficult or even meaningless.

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APPENDIX A

To clarify some aspects of the use of the Fokker-Planck equation in this paper, we now give a formal derivation of that equation. Let $W[x,t|x^{(0)},t_0]$ be the conditional probability that the system be in a state $x\{q_1,\cdots,q_N;p_1,\cdots,p_N\}$ at time t, if it is known to have been in the state $x^{(0)}$ at time t_0. Since Hamiltonian dynamics represents a Markovian process, $W[x,t|x^{(0)},t_0]$ satisfies the Chapman-Kolmogoroff equation,

$$W[x,t+\tau|x^{(0)},t_0]=\int W[x,t+\tau|x',t]dx'W[x',t|x^{(0)},t_0]. \quad \text{(A-1)}$$

[15] The φ integral can actually be done exactly in terms of Bessel functions of order $\frac{1}{3}$ and related functions.

[16] Judd, MacDonald, and Rosenbluth (to be published). See also S. Chandrasekhar, Astrophys. J. 97, 255 (1943).

If we write

$$W[x,t+\tau|x',t]=\delta(x-x')+K_\tau(x,x';t), \quad \text{(A-2)}$$

where by definition

$$\int K_\tau(x,x';t)dx=0, \quad \text{(A-3)}$$

this becomes

$$\Delta_\tau W[x,t|x^{(0)},t_0]\equiv W[x,t+\tau|x^{(0)},t_0]-W[x,t|x^{(0)},t_0]$$
$$=\int K_\tau(x,x';t)W[x',t|x^{(0)},t_0]dx'. \quad \text{(A-4)}$$

It is convenient to separate $K_\tau(x,x';t)$ into a diagonal and a nondiagonal part,

$$K_\tau(x,x';t)=-\delta(x-x')v_\tau^{(0)}(x',t)+T_\tau(x,x';t),$$

where, from (A-3),

$$\int T_\tau(x,x';t)dx=v_\tau^{(0)}(x',t). \quad \text{(A-3')}$$

If $v_\tau^{(0)}(x',t)=0$, then

$$\int T_\tau(x,x';t)dx=0, \quad \text{(A-3'')}$$

and we can proceed by expanding $T_\tau(x,x';t)$ in terms of the elements of the set of improper functions

$$\psi_n(x)=\frac{(-1)^n}{n!}\delta^{(n)}(x), \quad (n=1,2,\cdots) \quad \text{(A-5)}$$

that satisfy the requirement (A-3''), and which together with the set

$$\bar{\psi}_n(x)=x^n, \quad (n=1,2,\cdots) \quad \text{(A-6)}$$

form a biorthonormal base,

$$\int \bar{\psi}_n(x)\psi_m(x)dx=\delta_{mn}. \quad \text{(A-7)}$$

Thus, writing

$$T_\tau(x,x';t)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\delta^{(n)}(x-x')v_\tau^{(n)}(x',t), \quad \text{(A-8)}$$

where by (A-7)

$$v_\tau^{(n)}(x',t)=\int dx(x-x')^nT_\tau(x,x';t),$$

we find that

$$\Delta_\tau W[x,t|x^{(0)},t_0]=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}$$
$$\times[v_\tau^{(n)}(x;t)W[x,t|x^{(0)},t_0]]. \quad \text{(A-9)}$$

We may remark that if $\tau\to 0$ the difference equation becomes a differential equation, and furthermore, as long as we are dealing with the complete system subject to the laws of Hamiltonian dynamics, $v_\tau^{(n)}\sim\tau^n$, so that only the first term in the summation remains and we obtain the well-known Liouville equation. To obtain a corresponding equation for $\omega(x_1,t|x_1^{(0)},t_0)$, the conditional probability for one particle to be found at $x_1\{q_1,p_1\}$ at time t, given that it was to be found at $x_1^{(0)}\{q_1^{(0)},p_1^{(0)}\}$ at time t_0, we integrate the whole equation over $d\xi d\xi^{(0)}(\equiv dx_2\cdots dx_N dx_2^{(0)}\cdots dx_N^{(0)})$ after multiplying both sides by $W[\xi^{(0)}|x_1^{(0)};t_0]$, the conditional probability distribution for the configuration $\xi^{(0)}$, given a certain value $x_1^{(0)}$ at time t_0. Then

$$\Delta_\tau\omega(x_1t|x_1^{(0)},t_0)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\int d\xi d\xi^{(0)}v_\tau^{(n)}(x_1,\xi,t)$$
$$\times W[x_1,\xi,t|x^{(0)},\xi^{(0)},t_0]W[\xi^{(0)}|x_1^{(0)};t_0]$$
$$=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx^n}\int d\xi v_\tau^{(n)}(x_1,\xi,t)$$
$$\times W[x_1,\xi,t|x_1^{(0)},\xi_0,t_0].$$

For small τ, $v_\tau^{(n)}(x_1,\xi;t)$ depends very strongly on ξ, and will exhibit large fluctuations. It is expected, however, that as τ is increased (but still kept small enough so that no large changes occur in the system in that time interval), a secular component in $v_\tau^{(n)}$ tends to become dominant compared with the fluctuations, and over that range (the "plateau region"[2]) $v_\tau^{(n)}(x_1,\xi,t)\simeq\tau\alpha^{(n)}(x,t)$ +smaller terms of $O(\tau^2)$ depending on ξ for the physically interesting distributions in ξ. Equation (A-9) then becomes

$$\tau^{-1}\Delta_\tau\omega(x_1,t|x_1^{(0)},t_0)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{d^n}{dx_1^n}$$
$$\times[\alpha^{(n)}(x_1,t)\omega(x_1,t|x_1^{(0)},t_0)]. \quad \text{(A-10)}$$

Since we have specified τ small enough so as to preclude significant changes in the distribution, the left-hand side can be replaced by the time derivative. It is this equation that forms the starting point for the approach to the problem of ionized media used in this paper.

Note that if in Eq. (A-3'), $v_\tau^{(0)}(x',t)$ does not vanish, condition (A-3'') is no longer satisfied, and (A-5) no longer forms an appropriate base for expanding $T_\tau(x,x';t)$. A differential representation of form (A-9) is then no longer possible. Since $v_\tau^{(0)}$ (at least as $\tau\to 0$) may be interpreted as the rate of depletion of the initial state, a nonvanishing $v_\tau^{(0)}$ suggests finite discontinuous changes in the system, which may better be described by a collision treatment. Thus, if one writes

$$K_\tau(x,x';t)=-\delta(x-x')v_\tau^{(0)}(x',t)+\sum_{n=1}^{N-1}\frac{(-1)^n}{n!}$$
$$\times\delta^{(n)}(x-x')v_\tau^{(n)}(x';t)+R_N(x,x';t),$$

the first and last terms of this expansion correspond to direct and inverse collisions, respectively.

APPENDIX B

We shall now derive the probability distribution function for the force $\mathbf{F}$ on a particle located at the origin of the coordinate system whose action is due to its distant neighbors. The ν particles located closest to the origin are regarded as close neighbors. The positional probability of all particles (both close and distant neighbors) is governed by a Poisson distribution function. This choice of a function is intended to reflect the fact that

(1) the gas is rare. As the volume element $dV(\mathbf{r})$ shrinks to zero the number of particles $dN(\mathbf{r})$ contained in it also approaches zero.

(2) the total number of particles contained in $dV(\mathbf{r}_i)$ is statistically independent of that in $dV(\mathbf{r}_j)(i\neq j)$.

The distribution function for the force $W(\mathbf{F},\nu)$ is readily expressed in terms of the joint distribution function

$$W_\nu(dN(\mathbf{r}_{11})=n_{11},\cdots dN(\mathbf{r}_{1\mu})=n_{1\mu};\; dN(\mathbf{r}_{\mu1})=n_{\mu1},\cdots dN(\mathbf{r}_{\mu\mu})=n_{\mu\mu}),$$

for the position of the distant neighbors by means of the equality

$$W[\mathbf{F},\nu]=\sum_{(n_{i\alpha})}\delta(\mathbf{F}-e^2\sum_{i\alpha}n_{i\alpha}\mathbf{r}_{i\alpha}/r_{i\alpha}{}^3)\times W_\nu[n_{11},\cdots,n_{1\mu};n_{21},\cdots]. \quad \text{(B-1)}$$

The index i in this expression is intended to designate the radial distance of various volume elements from the origin; α denotes the angular parameters of the volume element in some chosen coordinate system. The letter $n_{i\alpha}$ is a possible value of the stochastic variable $dN(\mathbf{r}_{i\alpha})$ restricted to positive integers or zero. To simplify the notation, we employ the value of the stochastic variable $n_{i\alpha}$ to designate the distribution. We also define:

$$n_i=\sum_\alpha n_{i\alpha},\quad d\bar{N}(\mathbf{r}_i)=\sum d\bar{N}(\mathbf{r}_{i\alpha}),$$

where the bar in the last expression designates the mean of the variable $dN(\mathbf{r})$. For an isotropic distribution—which will here be assumed—we also have

$$d\bar{N}(\mathbf{r}_{i\alpha})=(4\pi)^{-1}d\Omega_\alpha d\bar{N}(\mathbf{r}_i).$$

It follows from this assumption together with the fact that the underlying distribution is of a Poisson type that the conditional probability $W[\{n_{i\alpha}\}|\{n_i\}]$, defined by

$$W_\nu[\{n_{i\alpha}\}]=W[\{n_{i\alpha}\}|\{n_i\}]W_\nu[\{n_i\}], \quad \text{(B-2)}$$

may be expressed as

$$W[n_{11},\cdots,n_{1\mu};n_{21},\cdots,n_{2\mu};\cdots|n_1,n_2]=\prod_i n_i!\prod_\alpha\frac{(d\Omega_\alpha)^{n_{i\alpha}}}{n_{i\alpha}!}=\prod_{i,\alpha}\frac{(d\bar{N}(r_{i\alpha}))^{n_{i\alpha}}}{n_{i\alpha}!}\prod_i\frac{n_i!}{[d\bar{N}(\mathbf{r}_i)]^{n_i}}. \quad \text{(B-3)}$$

The statistical dependence arising from the neglect of close neighbors is then reflected in the structure of $W_\nu(n_1n_2\cdots)$ alone.

Combining (B-2) with (B-1) and employing a Fourier representation for the delta function, we readily obtain

$$W[\mathbf{F},\nu]=(2\pi)^{-3}\int d\mathbf{k}\exp(-i\mathbf{k}\cdot\mathbf{F})\varphi(\mathbf{k}), \quad \text{(B-4)}$$

where

$$\varphi(\mathbf{k})=\sum_{(n_i)}e(n_1)e(n_2)\cdots W^\nu(n_1,n_2,\cdots) \quad \text{(B-5)}$$

and

$$e(n_i)=\left[\frac{\sum_\alpha dN(\mathbf{r}_{i\alpha})\exp(ik_{i\alpha})}{d\bar{N}(\mathbf{r}_i)}\right]^{n_i}. \quad \text{(B-6)}$$

The unfamiliar symbol in (B-6) is defined by

$$k_{i\alpha}=\frac{e^2}{4\pi}\frac{\mathbf{k}\cdot\mathbf{r}_{i\alpha}}{|r_i|^3}. \quad \text{(B-7)}$$

We now decompose the sum in (B-5) into summands:

$$\varphi(k)=\sum_{i=1}^\infty s_i,$$

where

$$S_1=\sum_{n_1>0}^\infty\sum_{n_2\geq0}\sum_{n_3>0}\cdots e(n_1)e(n_2)\cdots W_{1\nu}(n_1,n_2,\cdots),$$

$$S_2=\sum_{n_2>0}^\infty\sum_{n_3\geq0}\sum_{n_4\geq0}\cdots e(n_2)e(n_3)\cdots\times W_{2\nu}(0,n_2,n_3,\cdots), \quad \text{(B-8)}$$

$$S_3=\sum_{n_3>0}^\infty\sum_{n_4\geq0}\sum_{n_5\geq0}\cdots e(n_3)e(n_4)\cdots W_{3\nu}(0,0,n_3,\cdots).$$

Because of the restriction on the summations of the right member of (B-8), the quantities $W_{i\nu}(0,0,\cdots0,n_i,n_{i+1},\cdots)$ may be factored as

$$W_{1\nu}=W_1(n_1)P_{n_2}(d\bar{N}(\mathbf{r}_2))P_{n_3}(d\bar{N}(\mathbf{r}_3))\cdots,$$
$$W_{2\nu}=W_2(0,n_2)P_{n_3}(d\bar{N}(\mathbf{r}_3))P_{n_4}(d\bar{N}(\mathbf{r}_4))\cdots. \quad \text{(B-9)}$$

Equations (B-9) are expressions of the obvious fact that once a distant neighbor has made its appearance, in $dV(\mathbf{r}_i)$, the distributions of distant neighbors in $dV(\mathbf{r}_{i+1+S})$ are statistically independent of the particles in the remaining elements of volume. In accord with our assumptions,

$$P_{n_i}(d\bar{N}(\mathbf{r}_i))=\frac{[d\bar{N}(\mathbf{r}_i)]^{n_i}}{n_i!}\exp[-d\bar{N}(\mathbf{r}_i)], \quad \text{(B-10)}$$

we obtain

$$W_1(n_1)=P_{\nu+n_1}(d\bar{N}(\mathbf{r}_1)),$$

$$W_2(0,n_2)=\sum_{s=1}^{\nu} P_s(d\bar{N}(\mathbf{r}_1))P_{\nu+n_2-s}(d\bar{N}(\mathbf{r}_2)),\tag{B-11}$$

$$W_3(0,0,n_3)=\sum_{s=0}^{\nu} P_s(d\bar{N}(\mathbf{r}_1)+d\bar{N}(\mathbf{r}_2))\times P_{\nu+n_3-s}(d\bar{N}(\mathbf{r}_3)),\text{ etc.}$$

The elementary but somewhat tedious summations may now be carried out, yielding the result

$$\varphi(k)=\int d\bar{N}(\mathbf{r})\frac{[\bar{N}(\mathbf{r})]^{\nu-1}}{(\nu-1)!}\exp[-\bar{N}(\mathbf{r})]\times\exp\left[\int_r^\infty d\bar{N}(\mathbf{r}')\times\left\{\exp\left(\frac{ie^2}{4\pi}\mathbf{k}\cdot\mathbf{r}'/|\mathbf{r}'|^3\right)-1\right\}\right].\tag{B-12}$$

APPENDIX C

In this Appendix, we outline a method of solving the constant-coefficient diffusion equation which is used in Sec. IV. The present method yields the solutions in a form more amenable to further integrations than those obtained by Chandrasekhar. The equation to be solved is

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla_{\mathbf{r}}f-\beta\nabla_{\mathbf{v}}\cdot(\mathbf{v}f)-\alpha\nabla_{\mathbf{v}}^2f=0.\tag{C-1}$$

Writing

$$f=\int g(\mathbf{k},\mathbf{l})\exp(i\mathbf{k}\cdot\mathbf{r}+i\mathbf{l}\cdot\mathbf{v})d\mathbf{k}d\mathbf{l},$$

we obtain

$$\frac{\partial g(\mathbf{k},\mathbf{l})}{\partial t}-\mathbf{k}\cdot\nabla_{\mathbf{l}}g(\mathbf{k},\mathbf{l})+\beta\mathbf{l}\cdot\nabla_{\mathbf{l}}g(\mathbf{k},\mathbf{l})+\alpha l^2g(\mathbf{k},\mathbf{l})=0.$$

This equation is easily solved by the method of characteristics[17] for which

$$dt/dS=1,\quad d\mathbf{l}/dS=\beta\mathbf{l}-\mathbf{k},\quad dg/dS=-\alpha l^2g.$$

On integration, these give

$$\ln(g/g_0)=-\alpha\{(\mathbf{l}_0-\mathbf{k}\beta^{-1})^2(e^{2\beta t}-1)/2\beta+2(\mathbf{l}_0-\mathbf{k}\beta^{-1})\cdot\mathbf{k}(e^{\beta-1}-1)/\beta+k^2\beta^{-2}t\}$$

and

$$\mathbf{l}_0=\mathbf{k}\beta^{-1}+(\mathbf{l}-\mathbf{k}\beta^{-1})e^{-\beta t}.$$

Since we require

$$f(\mathbf{x},\mathbf{v},0)=\delta(\mathbf{x}-\mathbf{x}_0)\delta(\mathbf{v}-\mathbf{v}_0),$$
$$g_0=(2\pi)^{-6}\exp(-i\mathbf{k}\cdot\mathbf{x}_0-i\mathbf{l}\cdot\mathbf{v}_0).$$

From these equations, a complete solution is obtained. If $t\beta\ll1$, i.e., little damping has taken place,

$$g\approx(2\pi)^{-6}\exp[-i\mathbf{k}\cdot(\mathbf{x}_0+\mathbf{v}_0t)-i\mathbf{l}\cdot\mathbf{v}_0]\times\exp[-\alpha(l^2t+lkt^2+\tfrac{1}{3}k^2t^3)].$$

In this case,

$$\rho(\mathbf{x},t)=\int d\mathbf{v}f(\mathbf{x},\mathbf{v},t)=(2\pi)^3\int g(\mathbf{k},0,t)e^{i\mathbf{k}\cdot\mathbf{x}}d^3k$$

has the Fourier transform

$$(2\pi)^{-3}\exp[-i\mathbf{k}\cdot(\mathbf{x}_0+\mathbf{v}_0t)]\exp[-\tfrac{1}{3}\alpha k^2t^3],$$

which is the expression used in Sec. IV.

[17] See, for instance, R. Courant and D. Hilbert, *Methods of Mathematical Physics* (Interscience Publishers, Inc., New York, 1953), Vol. II.