Optical Model Potential for Pion-Nucleus Scattering

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(Received September 12, 1955)

The assumption that two-body forces are not appreciably modified within nuclei offers the possibility of calculating the optical-model potential from two-body scattering phenomena. For instance, the dispersion relations recently proposed by Goldberger permit one to make a calculation of the pion-nucleus well depth directly from observed pion-nucleon scattering cross sections. The results predict a rather striking variation with energy for this potential and appear to be in fair agreement with the available experimental data. The physical basis for the potential is investigated. Numerical calculations using meson theory (in the form proposed by Chew) are in disagreement with the results from the dispersion relations (as well as the experimental values). This involves solving an equation closely related to that used by Brueckner *et al.* in their study of nuclear saturation. The relations of the potential to various nuclear properties are discussed—for instance, it is shown in detail that the resolution of the nuclear radius is limited to the range of the two-body interaction.

I. INTRODUCTION

STUDIES of the elastic scattering of fast particles by atomic nuclei have frequently been made within the framework of the "optical model." According to this model one replaces the interaction between particle and nucleus by a complex potential well having the form

$$V_c(x) \equiv V_0(\omega_0)\rho(x), \tag{1}$$

where $\rho(x)$ is the nuclear density at point x normalized to

$$\int \rho(x) d^3x = V_A,$$

the nuclear volume. The complex quantity $V_0(\omega_0)$ is called the "well-depth" and is a function of ω_0, the energy of the incident particle.[1]

The applications of the optical model to nuclear scattering have been twofold. The most frequent application has involved obtaining the well-depth $V_0(\omega_0)$ [for some $\rho(x)$] which best fits the scattering data at energy ω_0.[2–5] The second application supplements this by attempting to deducethe values of $V_0(\omega_0)$ from the interaction between a pair of elementary particles.[6–8]

It is this latter aspect of the optical model to which this paper is devoted. We shall be specifically concerned with π-meson-nucleus scattering. According to the most elementary theory, $V_0(\omega_0)$ should have the form [excluding the term involving meson absorption—this term is correctly included in Eq. (6)]:

$$V_0(\omega_0) = -\frac{1}{\lambda^3}\frac{3}{2}(\mu c^2)\left[\frac{\mu c^2}{E_\pi}\right]\left[\frac{W_T(\text{c.m.})}{Mc^2}\right]\frac{\mu}{\hbar} f_0(\omega_0), \tag{2}$$

where $f_0(\omega_0)$ is the forward scattering amplitude in the center-of-mass coordinate system (to be abbreviated c.m.) for pion-nucleon scattering averaged over the neutrons and protons of the nucleus.[9] μ is the pion rest mass, M the nucleon rest mass. E is the kinetic plus rest energy of the pion in the laboratory frame of reference. W_T(c.m.) is the kinetic plus rest energy of the system in the c.m. system. Finally λ is determined from the nuclear radius $R[V_A = (4\pi/3)R^3]$ by

$$R = A^{\frac{1}{3}}\left(\frac{\hbar}{\mu c}\right)\lambda. \tag{3}$$

The expression (2) is readily obtained from the form for $V_0(\omega_0)$ given in Sec. II, Eq. (32), on observing that $f_0(\omega_0)$ is related to the forward scattering amplitude in the laboratory system, $f_{0\,\text{lab}}$ by

$$f_{0\,\text{lab}}/p = f_0(\omega_0)/k, \tag{4}$$

where k and p are respectively the momenta in the center-of-mass and laboratory systems for pion-nucleon scattering.

It is convenient to treat separately the real and imaginary parts of V;

$$V_0(\omega_0) = V_R(\omega_0) + iV_I(\omega_0). \tag{5}$$

We briefly review the expected[7,8] form for $V_I(\omega_0)$ which arises as a result of inelastic scattering within

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[1] We shall, for simplicity, treat the nucleus as being sufficiently heavy that its recoil at the time of collision can be neglected.

[2] Fernbach, Serber, and Taylor, Phys. Rev. **75**, 1352 (1949).

[3] T. Taylor, thesis, 1954, Cornell University (unpublished).

[4] Byfield, Kessler, and Lederman, Phys. Rev. **86**, 17 (1952).

[5] J. O. Kessler and L. M. Lederman, Phys. Rev. **94**, 689 (1954).

[6] R. Jastrow, Phys. Rev. **82**, 261 (1951).

[7] K. Watson, Phys. Rev. **89**, 575 (1953).

[8] N. Francis and K. Watson, Phys. Rev. **92**, 291 (1953).

[9] We shall consider only positive or negative pion scattering. For π^+-mesons for instance,

$$f_0 = \tfrac{2}{3}\left[\frac{Z}{A}+\tfrac{1}{2}\right]f_{\frac{3}{2}} + \tfrac{1}{3}\left[1-\frac{Z}{A}\right]f_{\frac{1}{2}},$$

where $f_{\frac{3}{2}}$ and $f_{\frac{1}{2}}$ are forward scattering amplitudes for isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$, respectively and Z and A are the respective atomic number and mass number of the nucleus. We shall assume that $Z \simeq A/2$ in this paper, so π^+- and π^--mesons will have identical f_0's. [For π^--mesons, we replace Z by $(A-Z)$ in the above expression.]

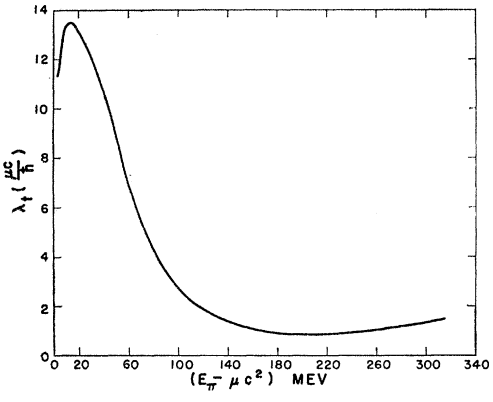

FIG. 1. The mean free path in units of $(\hbar/\mu c)$ for a pion in nuclear matter, as calculated from Eqs. (6), (7), and (9), is given as a function of the pion energy within the nucleus in the laboratory frame of reference.

the nuclear medium. We have

$$V_I(\omega_0)=\hbar\frac{v_L}{2\lambda_t}, \quad \frac{1}{\lambda_t}=\frac{1}{\lambda_s}+\frac{1}{\lambda_a}. \tag{6}$$

Here v_L is the pion velocity in the laboratory frame of reference, λ_t is the mean free path for either an inelastic scattering or absorption of the pion by the nucleus, and λ_s is the mean free path for an inelastic scattering. The latter is given by

$$\frac{1}{\lambda_s}=\frac{3}{4\pi\lambda^3}\left[\bar{\sigma}\left(\frac{\mu c}{\hbar}\right)\right]^2\frac{\mu c}{\hbar}, \tag{7}$$

where $\bar{\sigma}$ is the average of the π^+-proton and π^--neutron total scattering cross sections (see reference 9). It is

$$\bar{\sigma}=\tfrac{1}{2}\gamma[\sigma(\pi^+, \text{prot.})+\sigma(\pi^+, \text{neut.})]. \tag{8}$$

γ represents a correction to the free-nucleon cross sections due to nuclear binding. It has been calculated from the model of Goldberger.[10,11]

For λ_a we use the model[12] which considers $1/\lambda_a$ to be proportional to the capture cross-section σ_d of pions by deuterons.[13,14] We write

$$\frac{1}{\lambda_a}=\left[\frac{\mu c}{\hbar}\right][0.107]\left[\frac{1}{\eta}(0.14+\eta^2)\right], \tag{9}$$

where

$$\eta=k/\mu c \tag{10}$$

is the pion momentum in units of μc.

Values for λ_t are given in Fig. 1. For quantitative reference these are also tabulated in Table I.

We shall be primarily concerned with $V_R(\omega_0)$, the real part of $V_0(\omega_0)$. This depends upon the real part of f_0, which is (by reference 9)

$$\begin{aligned}\mathrm{Re}(f_0)&=(1/3)[2\,\mathrm{Re}(f_{\frac{3}{2}})+\mathrm{Re}(f_{\frac{1}{2}})],\\ &=(1/3k)[J_{\frac{3}{2}}+\tfrac{1}{2}J_{\frac{1}{2}}].\end{aligned} \tag{11}$$

Here

$$\begin{aligned}J_{\frac{3}{2}}&=\sin2\alpha_3+\sin2\alpha_{31}+2\sin2\alpha_{33},\\ J_{\frac{1}{2}}&=\sin2\alpha_1+\sin2\alpha_{11}+2\sin2\alpha_{13},\end{aligned} \tag{12}$$

where the α's are the pion-nucleon phase shifts in the notation of Anderson, Fermi, Martin, and Nagle.[15] The indices $\frac{3}{2}$ and $\frac{1}{2}$ refer to the isotopic spin of the pion-nucleon system.

Given the phase shifts, it is now possible to calculate Eqs. (11) and (12) and thus the real part of the potential depth $V_R(\omega_0)$. A considerably more elegant method has recently become available with the publication by Goldberger[16] and by Goldberger, Mijazawa, and Oehme[17] of dispersion relations obtained from the "principle of causality," by which it is possible to calculate $J_{\frac{3}{2}}$ and $J_{\frac{1}{2}}$ directly from observed total cross sections. Indeed, $J_{\frac{3}{2}}$ and $J_{\frac{1}{2}}$ have been so determined by Anderson, Davidon, and Kruse.[18] In our work we shall take the numerical values of $J_{\frac{3}{2}}$ and $J_{\frac{1}{2}}$ as given by these authors.

TABLE I. Values of $[\lambda_t(\mu c/\hbar)]$ and V_I (in Mev), as calculated from Eqs. (6)–(9) are given as a function of the pion kinetic energy in the laboratory frame of reference.

$(E_\pi-\mu c^2)$ (Mev)	V_I (Mev)	$\lambda_t[\mu c/\hbar]$
4	1.50	11.2
14	2.39	13.5
31	3.44	12.2
54	5.90	8.00
82	12.81	4.20
112	26.30	2.22
185	74.50	0.85
270	59.30	1.11
315	44.60	1.49

We can thus calculate the well-depth $V_0(\omega_0)$ entirely from observed pion-nucleon cross sections and compare the result with experiments concerning *pion-nucleus* scattering. The results should have an important bearing on models of nuclear structure, as has been frequently discussed.[19] In Fig. 2, we give $V_R(\omega_0)$ as

[10] M. L. Goldberger, Phys. Rev. **74**, 1269 (1948). A formal derivation was given in reference 8.
[11] M. H. Johnson, Phys. Rev. **83**, 510 (1951).
[12] Brueckner, Serber, and Watson, Phys. Rev. **84**, 258 (1951).
[13] We take $\sigma_d=(4.45/\eta)[0.14+\eta^2]_{\text{c.m.}}^2$ from the semiempirical formulas of M. Gell-Mann and K. Watson, Ann. Rev. Nuc. Sci., 1954. The proportionality constant Γ is taken to be $\Gamma=4$ (see reference 12).
[14] The model is summarized and some numerical values are given in N. Francis and K. Watson, Am. J. Phys. **21**, 659 (1953).
[15] Anderson, Fermi, Martin, and Nagle, Phys. Rev. **91**, 155 (1953).
[16] M. L. Goldberger, Phys. Rev. **99**, 970 (1955).
[17] Goldberger, Mijazawa, and Oehme, Phys. Rev. **99**, 986 (1955).
[18] Anderson, Davidon, and Kruse, Phys. Rev. **100**, 358 (1955).
[19] See for instance, G. Takeda and K. Watson, Phys. Rev. **97**, 1336 (1955).

calculated from

$$V_R(\omega_0)=-\frac{\mu c^2}{2\lambda^3}\left[\frac{\mu c^2}{E_\pi}\frac{W_T(\text{c.m.})}{Mc^2}\right]_\eta^1[J_{\frac{3}{2}}+\tfrac{1}{2}J_{\frac{1}{2}}], \quad (13)$$

using $J_{\frac{3}{2}}$ and $J_{\frac{1}{2}}$ as given in reference 18. $V_R(\omega_0)$ is also tabulated in Table II.

As has already been mentioned, Eqs. (1) and (2) [and thus Eq. (13)] are not exact, according to the general theory of the optical model. Instead, these equations are approximations which are expected to be valid under certain conditions. These conditions will be discussed in the remainder of this paper, the major points being:

(1) The validity of the form (1), which will be investigated in the next section.

(2) The value of the energy, ω at which $V_0(\omega_0)$ is to be evaluated. This is determined from the equation

$$\omega=\omega_0-V_0(\omega), \quad (14)$$

where ω_0 is the actual energy of the meson.

TABLE II. The value of the real part, V_R, of the optical model potential as calculated from Eq. (13) and the dispersion relations of Goldberger and Anderson *et al.* ($E_\pi-\mu c^2$) is the kinetic energy of the pion "inside the nucleus." $\lambda=1.0$.

$(E_\pi-\mu c^2)$ Mev	V_R Mev
58	−23
78	−30
101	−41
124	−46
148	−39
173	−13
198	+25
223	+38
249	+41
276	+47
302	+49

(3) That it is not the scattering amplitude f_0 for mesons scattered by free nucleons, but the f_0 for pion-nucleon scattering in the presence of the potential $V_c(x)$, which is to be put into Eq. (2). This correction will be considered in Secs. III and IV.

(4) Nuclear binding corrections. These will be discussed in Secs. II and V.

(5) General conditions of validity.[7,8,20] The most important of these is the assumption that two-body forces within the nucleus are very nearly the same as between free particles and that many-body forces are negligible. This is presumably to be tested by comparison with experiment of the results given in Fig. 2 and Table II.

Another likely condition on the simple theory given in this section is that the mean free path λ_t be long enough for the energy of the meson to be fairly well defined between scatterings. This permits the use of the amplitude f_0 only on the energy shell. The uncertainty in the pion energy, ΔE_π, between scatterings is given by $\Delta E_\pi=\hbar/\Delta t$, where $\Delta t=\lambda_t/v_t$. From this $\Delta E_\pi/E_\pi$ can be calculated and is shown in Fig. 3. The condition $\Delta E_\pi/E_\pi\ll 1$ seems to be reasonably well satisfied except near $E_\pi-\mu c^2=200$ Mev.

[20] G. F. Chew and M. L. Goldberger, Phys. Rev. **87**, 778 (1952).

II. GENERAL FORM OF V_c

The potential V_c, in the approximation that detailed structure of the nucleus is neglected,[7,8] is

$$V_c=\sum_{\alpha=1}^{A}\langle t_\alpha\rangle, \quad (15)$$

where t_α is the two-body scattering operator for scattering the meson by the αth nucleon in the nucleus. The symbol $\langle\cdots\rangle$ means the average of t_α over the ground state of the nucleus. If we make the impulse approximation,[20] t_α is given as the solution to (η is the usual infinitesimal positive parameter)

$$t_\alpha=V_\alpha+V_\alpha\frac{1}{E_\pi+i\eta-K_\alpha-V_c}t_\alpha, \quad (16)$$

where E_π is the energy of the meson in the laboratory system (to be abbreviated as "lab") and K_α is sum of the kinetic energies of the pion and nucleon. V_α is the interaction energy of the pion with the nucleon. If the V_c is neglected in Eq. (16), t_α becomes just the usual two-body scattering operator.

Equations (15) and (16) represent a pair of equations to determine V_c. In the next section this determination will be discussed in detail. In the present section we shall assume that t_α is known and discuss only Eq. (15).

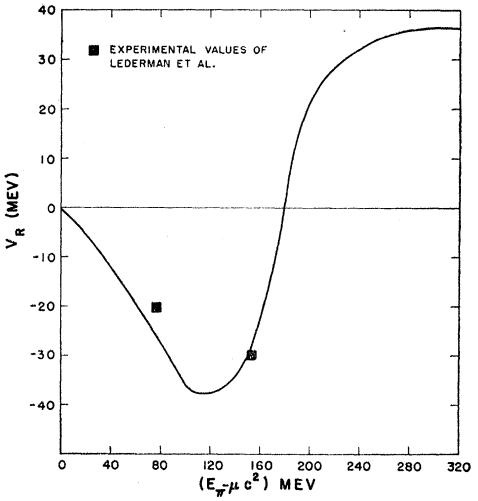

FIG. 2. The real part of the optical model potential is shown as a function of the energy of the pion in the nucleus. This is obtained from the dispersion relations of Goldberger (reference 16) and the evaluation of these by Anderson *et al.* (reference 18). The "experimental points" shown are those of Lederman *et al.* (references 4 and 5).

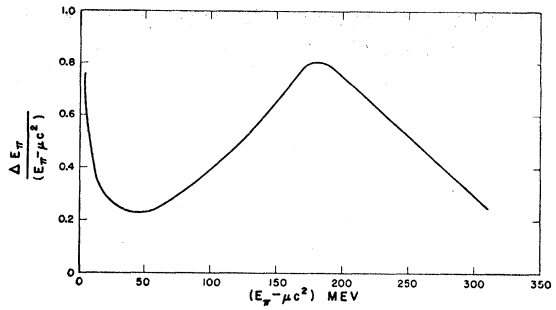

FIG. 3. The relative uncertainty in the pion energy between successive scattering.

We shall also assume that t_α is at least very similar to the actual free pion-nucleon scattering operator.

Since the nucleons are equivalent (in the isotopic spin formalism) we can drop the index α in Eq. (15) and write

$$V_c = A\langle t \rangle. \tag{15}$$

We desire to evaluate V_c in the laboratory frame of reference. In this coordinate system we write explicitly t_L for the t-operator, using the symbol t for this quantity in the center-of-mass coordinate system.

The relation between these is[21]

$$t_L = (\mathbf{k}'|t|\mathbf{k})\left(\frac{\omega^2 W_N^2(\text{c.m.})}{E_{\pi 0} E_{\pi f} W_{N0(\text{lab})} W_{Nf(\text{lab})}}\right)^{\frac{1}{2}}, \tag{17}$$

where ω and W_N(c.m.) are the c.m. pion and nucleon energies and E_π and W_N(lab) are the corresponding lab values. The subscripts 0 and f refer to values before and after the scattering, respectively. Each of these energies is the sum of kinetic and rest-energies. $\mathbf{k}$ and $\mathbf{k}'$ represent the initial and final pion momenta in the c.m.

We shall be primarily concerned with pion energies not much in excess of 200 Mev, so we may treat the nucleon nonrelativistically. This means that the W_N terms may be dropped in Eq. (17). (The approximations of this section actually become better at higher energies, except for the minor modification of treating nucleon energies more carefully.)

In the laboratory system the initial and final pion momenta will be designated as $\mathbf{p}$ and $\mathbf{p}'$, respectively. Those for the nucleon are written as $\mathbf{P}$ and $\mathbf{P}'$. ($\mathbf{P} \neq 0$ since the nucleon has kinetic energy in the nucleus.) We may approximate the Lorentz transformation equations to give

$$\mathbf{k} = \frac{Mc^2\mathbf{p} - \mathbf{P}\omega}{Mc^2 + \omega}, \quad \mathbf{k}' = \frac{Mc^2\mathbf{p}' - \omega\mathbf{P}'}{Mc^2 + \omega}. \tag{18}$$

For a discussion of the effect of the spread in $\mathbf{P}$ due to nuclear binding, we may consider the square-root factor in Eq. (17) to be approximately constant and write

$$t_L = \left(\frac{1}{Mc^2+\omega}[\mathbf{p}'Mc^2 - \mathbf{P}'\omega]|t_L| \times \frac{1}{Mc^2+\omega}[\mathbf{p}Mc^2 - \mathbf{p}\omega]\right) \tag{19}$$

using Eqs. (17) and (18).

To evaluate V_c, according to Eq. (15′), we must average t_L, as given by Eq. (19) over the ground-state nuclear wave function $g_N(\mathbf{P}_1, \cdots \mathbf{P}_A)$. We may suppose spin and isotopic spin terms to nearly average to zero,[19] so t_L in Eq. (19) need not be considered as a function of these operators. Then

$$(\mathbf{p}'|V_c|\mathbf{p}) = A\int t_L(\mathbf{p},\mathbf{p}';\mathbf{P},\mathbf{P}')g_N^*(\mathbf{P}')g_N(\mathbf{P}) \times d^3P'd^3P\delta(\mathbf{p}+\mathbf{P}-\mathbf{p}'-\mathbf{P}')[\textstyle\prod_j' d^3P_j], \tag{20}$$

if we omit writing explicitly the momentum variables of nucleons other than that one whose momentum variable appears in t_L. The last factor in Eq. (20) represents the volume element associated with these other momenta. Henceforth we shall not write this factor, its presence being understood.

Substituting

$$g_N(\mathbf{P}) = (2\pi)^{-\frac{3}{2}}\int d^3z \exp(-i\mathbf{P}\cdot\mathbf{z})g_N(z),$$

etc., into Eq. (20), we have also

$$(\mathbf{p}'|V_c|\mathbf{p}) = [A/(2\pi)^3]\int d^3Pd^3z'd^3z \exp[-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{z}'] \times \exp[-i\mathbf{P}\cdot(\mathbf{z}'-\mathbf{z})]t_L(\mathbf{p},\mathbf{p}',\mathbf{P})g_N^*(z')g_N(z). \tag{21}$$

In a qualitative way, we see that Eq. (19) depends only weakly on $\mathbf{P}$ when $Mc^2/\omega \gg 1$, so the integral over $\mathbf{P}$ in Eq. (21) can be approximated by a δ function of $(\mathbf{z}'-\mathbf{z})$. The integral over $\mathbf{z}$ then shows that $|\mathbf{p}'-\mathbf{p}| \simeq \hbar/R_A$, where R_A is the nuclear radius. This is much smaller than $p \simeq p'$ for energies of interest to us, which suggests that we can set $\mathbf{p}'=\mathbf{p}$ in t_L and take it out of the integral in Eq. (21). This forms the basis of Eqs. (1) and (2).

To do this a little more carefully, we rewrite Eq. (20) as

$$(\mathbf{p}'|V_c|\mathbf{p}) = A\int d^3Pt(\mathbf{p},\mathbf{p}',\mathbf{P})g_N(\mathbf{p}-\mathbf{p}'+\mathbf{P})g_N(\mathbf{P}) \simeq A\int d^3Pt_L(\mathbf{p},\mathbf{p}',p)|g_N(P)|^2. \tag{22}$$

The last step follows from setting $\mathbf{p}=\mathbf{p}'$ in g_N^*, which is a reasonable approximation in view of the arguments of the previous paragraph. Equation (22) tells us that

[21] C. Møller, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 23, 1 (1945).

t_L should be averaged over the momentum spectrum of the nucleons in the nucleus. This procedure is, of course, more or less self-evident intuitively.

We shall suppose that this has been done and insert the averaged t_L into Eq. (21). The new t_L is now no longer a function of $\mathbf{P}$ so it can be taken out of the integral. Equation (21) may then be integrated over $\mathbf{P}$ and $\mathbf{z}'$ to give

$$(\mathbf{p}'|V_c|\mathbf{p})=A(\mathbf{p}'|t_L|\mathbf{p}) \times\int d^3z\exp[-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{z}]|g_N(z)|^2. \quad (23)$$

Here $(\mathbf{p}'|t_L|\mathbf{p})$ is understood to have been averaged over the P values in the nucleus. This represents the first-order correction for nuclear motion. Numerical values will be given later. We now write

$$|g_N(z)|^2=(1/V_A)\rho(z), \quad (24)$$

where $\rho(z)$ is the nuclear density normalized according to Eq. (1).

Equation (23) is now transformed to coordinate space:

$$\begin{aligned}(\mathbf{x}'|V_c|\mathbf{x})&=(2\pi)^{-3}\int d^3p'd^3p\exp(i\mathbf{p}'\cdot\mathbf{x}')(\mathbf{p}'|V_c|\mathbf{p})\exp(-i\mathbf{p}\cdot\mathbf{x})\\&=(A/V_A)[1/(2\pi)^3]\int d^3p'd^3p(\mathbf{p}'|t_L|\mathbf{p})\exp(i\mathbf{p}'\cdot\mathbf{x}')\\&\quad\times\exp(-i\mathbf{p}\cdot\mathbf{x})\int d^3z\exp[-i(\mathbf{p}'-\mathbf{p})\cdot\mathbf{z}]\rho(z). \quad (25)\end{aligned}$$

In accordance with our previous arguments, we shall set $\mathbf{p}'=\mathbf{p}$ in t_L above. Now define

$$\mathbf{n}=\mathbf{p}'-\mathbf{p},\quad \mathbf{l}=\tfrac{1}{2}(\mathbf{p}'+\mathbf{p}), \quad (26)$$

so t_L is a function of $|l|$ only. This definition gives

$$(\mathbf{x}'|V_c|\mathbf{x})=(A/V_A)\int d^3l t_L(l) \times\exp[-\mathbf{l}\cdot(\mathbf{x}'-\mathbf{x})]\rho\left(\frac{\mathbf{x}'+\mathbf{x}}{2}\right). \quad (27)$$

Let us next consider V_c operating on a wave function describing the scattering of a meson by the nucleus:

$$\phi(x)=\int d^3p'a(\rho_0,\rho')\exp(i\varrho'\cdot\mathbf{x}). \quad (28)$$

Since we are supposing nuclear dimensions to be large compared to the wavelength of the meson, we may suppose a to be sharply peaked about $\rho'=\rho_0$, where ρ_0 may be considered to be the *momentum of the meson.*

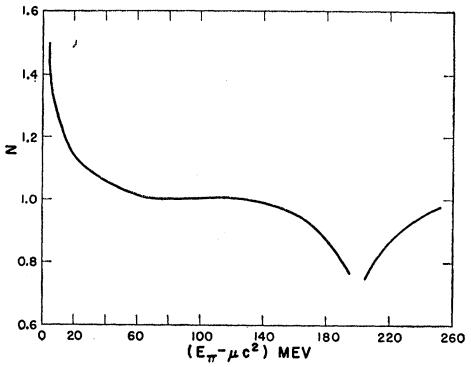

FIG. 4. The modification of V_R due to nucleon motion. The V_R of Eq. (2) should be multiplied by the factor N [Eq. (22)] to correct for the nucleon velocities, assuming a Fermi momentum distribution. (The curve is indeterminate near $V_R=0$.)

Under this condition, we find

$$\int(\mathbf{x}'|V_c|\mathbf{x})\phi(x)d^3x\simeq(2\pi)^3(A/V_A)t_L(\rho_0)\rho(\mathbf{x}')\phi(x'). \quad (29)$$

Thus, we may finally set

$$(\mathbf{x}'|V_c|\mathbf{x})=\delta(\mathbf{x}'-\mathbf{x})V_c(x), \quad (30)$$

where

$$V_c(x)=(2\pi)^3(A/V_A)(\mathbf{p}|t_L|\mathbf{p})\rho(x). \quad (31)$$

$\mathbf{p}$ is the momentum of the meson inside the nucleus. This has the form of Eq. (1), with

$$V_0(\omega_0)=(2\pi)^3(A/V_A)(\mathbf{p}|t_L|\mathbf{p}), \quad (32)$$

which leads immediately to Eq. (2) on expressing t_L in terms of $f_0(\omega_0)$. The requisite formulas for expressing t_L in terms of $f_0(\omega_0)$ will be reviewed in the next section.

Let us close this section by recalling the approximations made:

(1) We must first consider the t_L in Eq. (23) to have been averaged over the nuclear momentum distribution. This means averaging the V of Eq. (32) in this manner. In Fig. 4, we see how this modifies the V of Fig. 2.

(2) Equation (23) could be used as it stands as the optical model potential. It is more complex than Eq. (1) and involves more than just the forward scattering amplitudes, so it is evidently desirable to use Eq. (1) when possible. We see that

$$|\mathbf{p}'-\mathbf{p}|\simeq\hbar/R_A,$$

so setting $\mathbf{p}'=\mathbf{p}$ in t_L can be justified when this quantity is negligible. The next approximation (30) is evidently justified under similar conditions. It would appear that Eqs. (31) and (32) are adequate, although perhaps marginal, for scattering well into the forward hemisphere for elements as light as carbon with pions of $(\omega_0-Mc^2)>50$ Mev. Explicit calculations by Kisslinger[22] using V_c in the form (23) seem to be in agree-

[22] L. Kisslinger, Phys. Rev. **98**, 764 (1955).

ment with this conclusion. An explicit calculation to check these arguments is made in Sec. IV.

In connection with Eq. (31) we remarked that $\mathbf{p}$ is the momentum inside the nucleus. This follows, since in Eq. (28) it is the value of the wave function inside the nucleus which is important. Indeed the Schrödinger equation for elastic scattering is

$$[E_\pi(p)+V_c]\phi=E_{\pi 0}\phi. \tag{33}$$

In the nuclear interior we can approximately rewrite Eq. (23) as

$$(\mathbf{p}'|V_c|\mathbf{p})=(2\pi)^3(A/V_A)(\mathbf{p}|t_L|\mathbf{p})\delta(\mathbf{p}'-\mathbf{p}), \tag{34}$$

so Eq. (33) becomes an algebraic equation:

$$E_\pi(p)+V_0(p)=E_{\pi 0}. \tag{35}$$

Solving this equation for p, we find the value of p at which V_c is to be evaluated. This means that in Fig. 2 we must add ordinate and abscissa to find the energy of the meson when it is outside the nucleus. We also note that, in Eq. (16) defining t_α, it is again the combination $E_{\pi 0}-V_c$ which occurs. Thus it is the two-body scattering in the dispersive medium which is important. In the next section we investigate this point in more detail.

Finally, we note that in Eq. (27) the nuclear density is a function of $\frac{1}{2}(\mathbf{x}'+\mathbf{x})$ rather than of $\mathbf{x}'$ only, as implied by the approximate Eq. (29). Physically, this means that the nuclear boundary is defined only to within the range of the two-body interaction by elastic scattering experiments [when interpreted according to Eq. (1)].

III. SELF-CONSISTENT EVALUATION OF t

We now consider Eq. (16) for the evaluation of t_α. A derivation of this integral equation was given in reference 7 (Appendix A). In the course of the derivation terms of relative order $(1/A)$ were discarded.[23]

For the discussion of Eq. (16) we shall use the expression (34) for V_c; that is, we shall use V_c in momentum space and in this representation consider it to be a diagonal operator and thus just a complex number. We shall furthermore [for the evaluation of Eq. (16)] neglect the imaginary part V_I of V_c, writing $V_0=V_R$. Comparison of Fig. 2 with Table I indicates that this is a reasonable approximation over the energy range for which our theory is expected to be applicable. [In the Appendix the effect of including V_I in Eq. (16) is considered. No essential complication appears to be added to the integral equations; however, for a first study of the formidable Eq. (16) some simplification seems in order.] Finally, to solve Eq. (16), we shall work in the c.m. system for the pion-nucleon pair and so re-express Eq. (34) in terms of t in the c.m. system:

$$V_R(k)=(2\pi)^3(A/V_A)\,\mathrm{Re}[(\mathbf{k}|t|\mathbf{k})]. \tag{36}$$

It is convenient now to effect decomposition into eigenstates of the isotopic spin I. If $t^{\frac{3}{2}}$ and $t^{\frac{1}{2}}$ are the t-matrices for $I=\frac{3}{2}$ and $\frac{1}{2}$, respectively, Eq. (36) may be written as

$$V_R(\omega_0,k)=(2\pi)^3(1/2)(A/V_A)\,\mathrm{Re}\{2(\mathbf{k}|t^{\frac{3}{2}}|\mathbf{k})+(\mathbf{k}|t^{\frac{1}{2}}|\mathbf{k})\}. \tag{37}$$

Equation (16) becomes for each isotopic spin state (explicitly in the c.m. system)

$$t=V+V\frac{1}{\omega_0+i\eta-h}t, \tag{38}$$

where

$$h(k)=\omega(k)+V_R(\omega_0,k). \tag{39}$$

(We shall frequently omit writing the isotopic spin index.) Here ω_0 is the c.m. energy of the meson (outside the nuclear medium) and $\omega(k)=c(k^2+\mu^2c^2)^{\frac{1}{2}}$.

Equation (38) is next decomposed into equations for angular and orbital angular momentum labeled by (j,l). We write (for either isotopic spin state)

$$\begin{aligned}(\mathbf{k}'|t|\mathbf{k})&=\sum_{j,l}\sum_M(k'|t_{j,l}|k)Y_{j,l}{}^M(\theta',\phi')Y_{j,l}{}^{M*}(\theta,M),\\ (\mathbf{k}'|V|\mathbf{k})&=\sum_{j,l}\sum_M(k'|V_{j,l}|k)Y_{j,l}{}^M(\theta',\phi')Y_{j,l}{}^{M*}(\theta,M),\end{aligned} \tag{40}$$

where

$$Y_{j,l}{}^M(\theta,\phi)=\sum_{\nu=\pm\frac{1}{2}}(\tfrac{1}{2},l;\nu,M-\nu|j,M)Y_l{}^{M-\nu}(\theta,\phi)\xi^\nu. \tag{41}$$

Here ξ^ν is the nucleon spin wave function. The angles (θ,ϕ) represent the direction of $\mathbf{k}$, etc. Substituting Eqs. (40) into (38) we obtain a set of equations for $t_{j,l}$:

$$(k'|t_{j,l}|k)=(k'|V_{j,l}|k)+\int\frac{(k'|V_{j,l}|k'')(k''|t_{j,l}|k)}{\omega_0+i\eta-h(k'')}(k'')^2dk''. \tag{42}$$

The forward scattering amplitude is most easily evaluated from Eq. (40) by choosing the axis of quantization as the polar axis, so that $\theta'=\theta=0$. Recalling that $Y_l{}^{M-\nu}(0,\phi)=\delta_{M,\nu}[(2l+1)/4\pi]^{\frac{1}{2}}$, we obtain

$$(\mathbf{k}|t|\mathbf{k})=[1/(4\pi)]\textstyle\sum_l\{(l+1)t_{j=l+\frac{1}{2},l}+lt_{j=l-\frac{1}{2},l}\}\mathfrak{I}, \tag{43}$$

where $\mathfrak{I}$ is the unit matrix in spin space. [We have continued to omit the isotopic spin index from Eq. (43).]

Equation (42) is most easily evaluated in terms of

[23] In the meantime an improved derivation using

$$t_\alpha=V_\alpha+V_\alpha\frac{1}{E_{\pi 0}+\eta-K_\alpha-V_c}I_\alpha,$$

in the notation of reference 7, has been found by one of us (K.W.), which indicates more clearly the nature of the approximation. (This will be published separately.) In a related context, K. A. Brueckner [Phys. Rev. **96**, 508 (1954)] has made a detailed examination of the $1/A$ terms.

the K-matrix, which satisfies

$$(k_B|K_{j,l}|k_A)=(k_B|V_{j,l}|k_A)+\int\frac{(k_B|V_{j,l}|k'')(k''|K_{j,l}|k_A)}{\omega_0-h(k'')}k''^2dk'', \quad (44)$$

and

$$t_{j,l}=K_{j,l}-i\pi K_{j,l}\delta(\omega_0-h)t_{j,l}. \quad (45)$$

That the $(k'(t_{j,l}|k)$ given by Eq. (45) satisfies Eq. (42) is easily shown. As a matter of fact, the only new aspect of our Eqs. (42), (44), and (45) is that there is no restriction to the energy shell implied for k, k'.

We define k_0 as the root of

$$\omega_0-h(k_0)=0. \quad (46)$$

Then, doing the integral over the δ function in Eq. (45), we are led to

$$(k'|t_{j,l}|k)=(k'|K_{j,l}|k)-ib(k'|K_{j,l}|k_0)(k_0|t_{j,l}|k), \quad (47)$$

where

$$b(k_0)=\frac{\pi k^2}{dh(k)/dk}\bigg|_{k=k_0}. \quad (48)$$

Equation (47) can be solved for $(k|t_{j,l}|k)$ to give

$$(k|t_{j,l}|k)=(k|K_{j,l}|k)-ib\frac{(k|K_{j,l}|k_0)(k_0|K_{j,l}|k)}{1+ib(k_0|K_{j,l}|k_0)}. \quad (49)$$

[It is to be remembered that k_0 is defined by Eq. (46).]

The real part of Eq. (49) is ($K_{j,l}$ is real and symmetric)

$$\mathrm{Re}[(k|t_{j,l}|k)]=(k|K_{j,l}|k)-\frac{b^2(k_0|K_{j,l}|k_0)[(k_0|K_{j,l}|k)]^2}{1+b^2[(k_0|K_{j,l}|k_0)]^2}. \quad (50)$$

To actually calculate $V_R(\omega_0,k)$ from the $K_{j,l}$ we evaluate the $t_{j,l}$ from Eq. (50), and in terms of these $(\mathbf{k}|t|\mathbf{k})$ for each isotopic spin state is obtained using Eq. (43). Next Eq. (37) is used to give $V_R(\omega_0,k)$. This value of $V_R(\omega_0,k)$ must then be inserted into Eq. (44) to find the $K_{j,l}$. This is evidently a complex mathematical problem. In the next section two specific examples of this problem will be solved in detail. The first is a simple one which can be treated analytically. The second involves choosing an interaction V as obtained by the Tamm-Dancoff method in meson field theory. In this case the problem was solved using a high speed digital computer (an IBM-701).

In the approximation of Sec. II, it is only $V_R(\omega_0,k_0)$, or the potential on the energy shell, which we desire when we are finished. [The off-the-energy-shell values are needed, of course, to solve Eq. (44).] Setting

$$(k_0|K_{j,l}|k_0)=-(1/b)\tan\delta(j,l), \quad (51)$$

where $\delta(j,l)$ is the scattering phase shift in the (j,l) state, we solve Eqs. (49) and (50) to give the usual expressions:

$$(k_0|t_{j,l}|k_0)=-(1/b)\exp[i\delta(j,l)]\sin\delta(j,l),$$
$$\mathrm{Re}[(k_0|t_{j,l}|k_0)]=-(1/b)[\tfrac{1}{2}\sin2\delta(j,l)]. \quad (52)$$

Using the usual notation[15] for the phase shifts $\delta(j,l)$, we evaluate Eq. (43) for the $I=\frac{3}{2}$ state:

$$\mathrm{Re}[(\mathbf{k}|t^{\frac{3}{2}}|\mathbf{k})]=-\frac{1}{(2\pi)^2\omega(k_0)k_0}\left[1+\frac{\omega(k_0)}{k_0}\frac{dV_R(k_0)}{dk_0}\right]\times\tfrac{1}{2}\{\sin2\alpha_3+\sin2\alpha_{31}+2\sin2\alpha_{33}\}. \quad (53)$$

The quantity in $\{\cdots\}$ is recognized as being $J_{\frac{3}{2}}$ of Eq. (12). $\mathrm{Re}[t^{\frac{1}{2}}]$ is the same, but with $J_{\frac{3}{2}}$ replaced by $J_{\frac{1}{2}}$. Then Eq. (37) gives [if we set $k=k_0$ and replace t by t_{lab} according to Eq. (17)]

$$V_R(\omega_0,k_0)_{\mathrm{lab}}=-\left(\frac{2\pi}{3}\right)(A/V_A)\frac{W_N(\mathrm{c.m.})}{E_\pi Mc^2}\frac{1}{k_0}\times\left[1+\frac{\omega(k_0)}{k_0}\frac{dV_R(k_0)}{dk_0}\right][J_{\frac{3}{2}}+\tfrac{1}{2}J_{\frac{1}{2}}]. \quad (54)$$

If we neglect $dV_R(k_0)/dk_0$ this reduces exactly to Eq. (13).[24] [E_π is the lab energy which corresponds to the c.m. energy $\omega(k_0)$.]

IV. SPECIFIC SELF-CONSISTENT CALCULATIONS OF V_R

In the last section, we discussed the manner in which the "two-body scattering amplitude" f_0 of Eq. (2) is to be modified by solving the two-body problem "in the dispersive medium." Following the notation used by Brueckner[25] and his collaborators, we shall refer to the latter as the "self-consistent" V_R. The value of V_R obtained from Eq. (2) we shall call the "free particle" V_R. It is evidently desirable, where possible, to use the approximate "free particle" V_R, since this is much easier to obtain than is the "self-consistent" V_R. In the present section, we cmpare the values of V_R obtained by these two methods.

We might remark in this connection that Brueckner *et al.*[25] found it necessary to do the "self-consistent" calculation in their work on nuclear saturation.

A. Simple Model

We suppose the two-body interaction V to have the form

$$(k'|V_{\frac{3}{2},1}|k)=-G\phi(k')\phi(k), \quad (55)$$

where G is a constant and the functional form of ϕ will be specified later. The $V_{\frac{3}{2},1}$ of Eq. (55) is supposed to

[24] *Note added in proof.*—It should be noted that ω_T occurs in Eqs. (2) and (13) and not ω_N. In the no-recoil approximation, $\omega_T=\omega_N$. We wish to thank Dr. W. Riesenfeld for calling our attention to the fact that we inadvertently used ω_N in place of ω_T in preparing Fig. 2. Table II is correct, however.

[25] K. A. Brueckner, Phys. Rev. **96**, 508 (1954); Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 217 (1954).

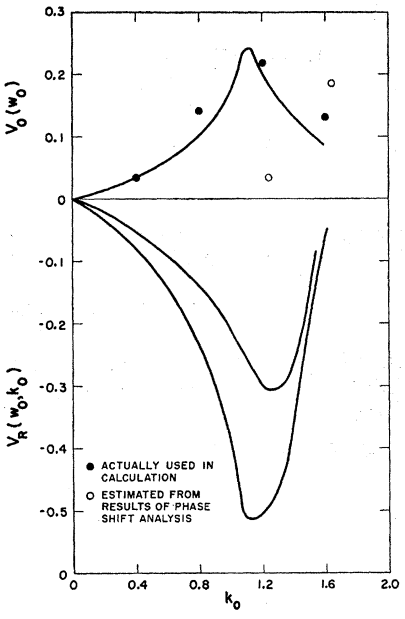

FIG. 5. The free-particle values of $V_R(\omega_0,k)$ in the center-of-mass system calculated from the "principle of causality" (1) and from the $T=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$ phase shifts calculated by Chew (2). The difference between the two curves is $V_0(\omega_0)$, Eq. (61).

be the submatrix $V_{j,l}$ ($I=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$) of Eq. (40). Then Eq. (44) for the K-matrix has the exact solution (as may be verified by substitution)

$$(k'|K_{\frac{3}{2},1}|k)=-G\phi(k')\phi(k)/(1+GI), \tag{56}$$

where

$$I=\int k'^2\frac{\phi^2(k')}{\omega_0-h(k')}dk'. \tag{57}$$

Again, the $K_{\frac{3}{2},1}$ of Eq. (56) is the submatrix for the ($I=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$) state.

Using Eqs. (48) and (49) to relate K to t and Eq. (36) to relate t to V_R, we obtain [assuming for simplicity that V acts only in the ($I=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$) state]

$$V_R=-\phi^2(k)\left[\frac{G}{1+GI}\frac{2\pi}{1+b^2[(k_0|K_{\frac{3}{2},1}|k_0)]^2}\right]. \tag{58}$$

This is the "self-consistent" V_R. The "free-particle" V_R is obtained by replacing $h(k')$ in Eq. (57) by $\omega(k')$ [see Eq. (39)].

For numerical evaluation two models were chosen:

$$\omega(k)=\mu c^2+(k^2/2\mu),\quad G=0.08028 \tag{59}$$

(in both models), and

Case I

$$\begin{aligned}\phi(k)&=k/(\mu^2c^2+k^2)^{\frac{1}{2}} &&\text{for}\quad k<6.65\mu c\\ &=0 &&\text{for}\quad k>6.65\mu c.\end{aligned}$$

Case II

$$\begin{aligned}\phi(k)&=k/(\mu^2c^2+2k^2)^{\frac{1}{2}} &&\text{for}\quad k<6.65\mu c\\ &=0 &&\text{for}\quad k>6.65\mu c.\end{aligned} \tag{60}$$

On evaluating I, Eq. (58) may be solved algebraically for V_R. The results are shown in Table II, where $V_R(\omega_0,k_0)$ is given as a function of k_0 [$\omega_0-h(k_0)=0$ determines k_0, we recall] for both the "free-particle" and "self-consistent" problems.

From Table II we observe that in Case I the "free particle" V_R is about ten times larger than the "self-consistent" V_R when $(E_\pi-\mu c^2)$ is much less than 200 Mev. At higher energies the two values are nearly equal. For Case II, on the other hand, the two values are about equal at all energies.

The similarity of "free-particle" and "self-consistent" V_R's at high energies is quite understandable. Reference to Eqs. (38) and (39) suggests that for $\omega_0\gg|V_R|$, the latter is negligible, so Eq. (38) reduces to a free-particle scattering equation.

At low energies the difference between the two V_R's for Case I is due to the considerable strength of the potential V at high momenta. In Case II, the V is less strong at high k and the "self-consistent" corrections are less important. Similar conclusions are drawn in Part B, where a much more complex problem is considered.

IV. B

A pseudoscalar symmetric theory of the pion nucleon interaction which neglects nucleon recoil (that is, the cutoff theory proposed by Chew) predicts that there is no interaction except in those states for which the orbital angular momentum l is equal to 1. It predicts that the phase shifts even for these states are small unless in addition the isotopic spin T is $\frac{3}{2}$ and the total angular momentum j is $\frac{3}{2}$. Thus it predicts that the main contribution to $V_R(\omega_0,k)$ in Eq. (37) results from the term with $(\mathbf{k}|t^{\frac{3}{2}}|\mathbf{k})$, and the main contribution to this in Eq. (43) results from the term with $l=1$, $j=\frac{3}{2}$.

In this section we apply the cut-off theory for the $T=\frac{3}{2}$, $l=1$, $j=\frac{3}{2}$ state, and treat the other states only in the following rough approximation. We replace Eq. (37) by

$$V_R(\omega_0,k)=(2\pi/\lambda^3)(k|t(T=\tfrac{3}{2},\ j=\tfrac{3}{2},\ l=1)|k)+V_0(\omega_0), \tag{61}$$

where $V_0(\omega_0)$ represents the contribution of the other states. We have taken it to be independent of k as it would be (roughly) if S states made the main contribution to it.

The magnitude of $V(\omega_0)$ was estimated in the following way. We may also write

$$V_R(\omega_0,k_0)=\frac{1}{\lambda^3\omega_0k_0}(\tfrac{1}{2}J_{\frac{3}{2}}+\tfrac{1}{4}J_{\frac{1}{2}}),$$

where $J_{\frac{3}{2}}$ and $J_{\frac{1}{2}}$ are defined in Eq. (12). This is, of course, the free-particle $V_R(\omega_0,k_0)$, and not the self-consistent one. This expression can be evaluated from the "principle of causality",[17,18] and the result is shown in Fig. 5. The first term on the right of Eq. (61) can be

evaluated in the free-particle approximation from calculations of the phase shift in the $T=\frac{3}{2}$, $l=1$, $j=\frac{3}{2}$ state in pion-nucleon scattering. Such calculations have been made by Chew[26] and they fit the results of phase analyses quite well. The result of such a calculation is also shown in Fig. 5. $V_0(\omega_0)$ is taken to be the difference between these two curves.

It is possible to estimate $V_0(\omega_0)$ from the results of phase-shift analyses of pion-nucleon scattering data. These give the phase shifts which are needed to evaluate the free particle $V_R(\omega_0,k_0)$ from Eqs. (37) and (43). The result of such a calculation is not even smooth, as shown in Fig. 5. Thus it is not possible to judge whether or not the assumption that the main contributions to $V_0(\omega_0)$ are made by S-states is valid in this way. However, we felt there was no better way of doing the calculation than to use Eq. (61) and estimate $V_0(\omega_0)$ in the manner described.

We may summarize our assumptions as follows:

(1) The cut-off theory is used in calculating the self-consistent value of the first term in Eq. (61).

(2) The contributions to $V_R(\omega_0,k)$ from states other than $T=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$ are not subject to "self-consistent" corrections discussed in this section, are independent of k, and have the values $V_0(\omega_0)$ shown in Fig. 5.

The two-body interaction in the $T=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$ state which results from a pseudoscalar symmetric theory of the pion-nucleon interaction is

$$(k'|V|k)=\frac{4f^2}{3\pi}\frac{k'k}{[\omega(k')\omega(k)]^{\frac{1}{2}}}\frac{1}{h(k')+h(k)-\omega_0}, \quad (62)$$

where nucleon recoil is neglected. It is necessary to cut this interaction off in order to get a convergent theory of the pion-nucleon interaction. Following Chew, we use a cutoff equal to the nucleon rest mass. We use a value of f^2 which makes the $T=\frac{3}{2}$, $j=\frac{3}{2}$, $l=1$ phase shift in pion-nucleon scattering 90° at 196-Mev pion energy in the laboratory system ($f^2=0.1$).

With Eq. (62) for $(k'|V|k)$ we must find a self-consistent solution of Eqs. (44), (50), and (61), with Eqs. (39) and (49) defining certain quantities appearing in these. To do this we use a straightforward iteration scheme. We first guess some $V_R(\omega_0,k)$ as a function of k. Equation (46) is not used to compute k_0, which is held fixed throughout a calculation; rather, it is used to compute ω_0. To get our next guess, we need $(k|t|k)$ in Eq. (61) as a function of k, and this must be calculated from Eq. (50). From Eq. (50) we see that we need b, $(k_0|K|k_0)$, $(k|K|k_0)=(k_0|K|k)$, and $(k|K|k)$. We get the value of b by evaluating the derivative of h at $k=k_0$ numerically. If we solve Eq. (44) with the value of ω_0 found from Eq. (46) and with $k_A=k_0$, we get $(k_B|K|k_0)$ as a function of k_B, from which we can get $(k_0|K|k_0)$ and $(k|K|k_0)$. Finally, if we solve Eq. (44) with $k_A=k$, we get $(k_B|K|k)$ as a function of k_B, from which we can get $(k|K|k)$. Thus we have all the quantities required to evaluate Eqs. (50) and (61), which gives us a new guess for $V_R(\omega_0,k)$.

A first guess of $V_R(\omega_0,k)=0$ leads to a solution in 10 or 20 iterations except in difficult cases to be mentioned shortly.

A few remarks on finer details are in order now. We load a table of the guessed $V_R(\omega_0,k)$ into the computer memory at intervals $k=0(0.4)6.4$. (Our unit is $\mu c/\hbar$, so that the cutoff is $k=6.557$.) The method described in detail by Gammel[27] was used to solve Eq. (44) for K. This replaces the integral Eq. (44) by a system of simultaneous linear equations. The integration over k'' is broken into two regions; the first extends from 0 to $2k_0$ and the second from $2k_0$ to 6.557. Eight mesh points are used in the first region; 20 in the second. A redefinition of variables described by Gammel was made in the first region to avoid the singularity in the integrand in Eq. (44) at $k''=k_0$. We get b by differentiating $V_R(\omega_0,k)$ with respect to k at $k=k_0$. Of course, it is easy to do this numerically with a table of $V_R(\omega_0,k)$ in the machine. For a given k_A, the values of $(k_B|K|k_A)$ were found for k_B equal to one of these mesh points. To evaluate K for other values of b, a 5-point interpolation method was used. This interpolating is necessary in computing Eq. (50). Briefly put, the iteration procedure consisted of computing a new V table from an old V table.

In those cases where no difficulty was encountered starting from $V_R(\omega_0,k)=0$ as the first guess, the new and old V tables agreed to 1 part in 10^4 after 10 to 20 iterations.

The amount of machine time required for this is not large; perhaps two hours suffices for all 10 to 20 iterations on an IBM 701.

Of course, difficulties were encountered in some cases. Equation (50) is nearly the same as

$$\tfrac{1}{2}\sin 2x=\tan x-\tan^3 x/(1+\tan^2 x),$$

which is awkward in the vicinity of $x=\pi/2$. We ran into this difficulty for $k_0=1.6$. The lowest energy we tried ($k_0=0.4$) also gave trouble because t was very sensitive to V; that is, the output was very sensitive to the input. This energy has a very low value ($\simeq$10 Mev in the lab system) so we did not try to overcome the difficulty. These same difficulties show up in the hand calculations described in Sec. IV A.

One would like to have existence and uniqueness theorems for such problems. While we realize that such things do not constitute a proof, we did succeed in finding a solution for each k_0, and failed in efforts to find more than one.

Graphs of the final $V_R(\omega_0,k)$ are shown in Fig. 5 for several ω_0 as functions of k.

The result of this calculation is that the final

[26] Geoffrey F. Chew, Phys. Rev. **95**, 285 (1954).

[27] J. L. Gammel, Phys. Rev. **95**, 209 (1954).

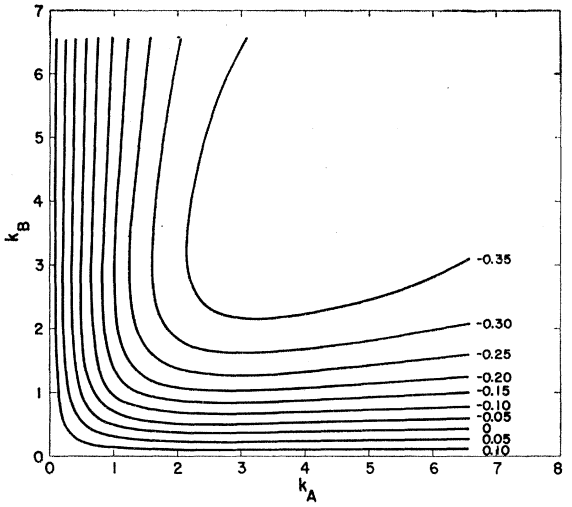

FIG. 6. Contour graph of $(k_A|t|k_B)$, from which the potential is to be computed according to Eq. (23). Center-of-mass values are plotted. These should be converted to lab values according to Eqs. (17) and (18).

$V_R(\omega_0,k_0)$ is not at all the same as that given in Fig. 2 (it is much weaker and much too weak to agree with experiment). This means that in this respect approximation 3 has failed.

The source of this failure is the singular nature of the potential Eq. (62); that is, its off-diagonal components are very large. This was verified in the hand calculations of Sec. IV. A and in the present calculation by requiring that $V_R(\omega_0,k)$ be equal to $V_R(\omega_0,k_0)$ for $k>k_0$ and be equal to its value from Eq. (61) for $k<k_0$. In this case the self-consistent $V_R(\omega_0,k)$ results on the first iteration and $V_R(\omega_0,k_0)$ is given by Fig. 2.

We view this result as evidence against the interaction Eq. (62) since the "principle of causality" $V_R(\omega_0,k_0)$ agrees with the experiments, which suggests that the corrections discussed in this section should be small.

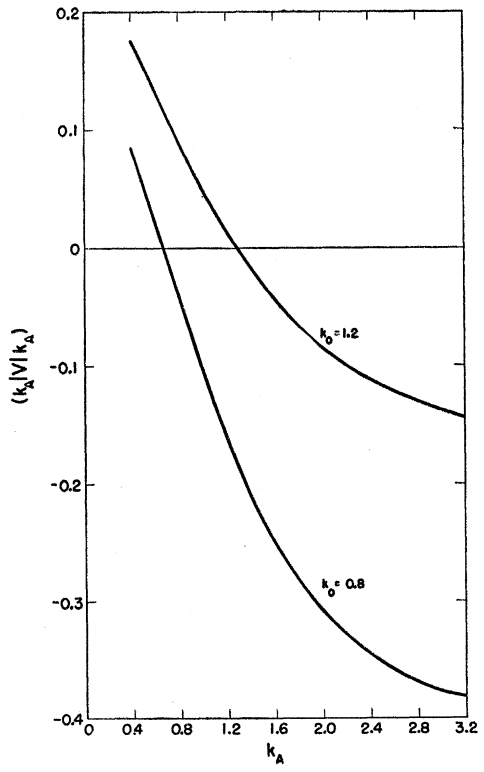

FIG. 7. The "self-consistent" potential $(k_A|V|k_A)$ for $k_0=0.8$ and 1.2.

IV. C

There is still a possibility that the large off-diagonal elements will still make the theory agree with experiment if one solves the Schrödinger equation correctly with the potential given by Eq. (23). A more general version of Eq. (50) is

$$\mathrm{Re}(k_B|t|k_A)=(k_B|K|k_A)-\frac{b^2(k_0|K|k_0)(k_B|K|k_0)(k_0|K|k_A)}{1+b^2(k_0|K|k_0)^2}, \quad (63)$$

from which we may calculate the complete potential matrix given by Eq. (23). For $\rho(z)$ in Eq. (24) we have used a rectangular distribution whose radius is given by Eq. (3). A contour plot of $(k_A|t|k_B)$, from which the potential is to be computed according to Eq. (23), is shown in Fig. 6.

We inadvertently did not use t_L (lab system) in Eq. (23), but used the center-of-mass t. The transformation should have been made using Eqs. (17) and (18). This error is not serious for an estimate of the effect of off-diagonal elements of the potential, and it was not thought worthwhile to correct the calculation.

We have solved the Schrödinger equation

$$(p'|K|p_0)=(p'|V_c|p_0)-\int d\mathbf{p}''\frac{(p'|V_c|p'')(p''|K|p_0)}{p_0^2-p''^2}. \quad (64)$$

The solution must be compared to a solution of Eq. (64) with $(p'|V_c|p'')$ given by Eq. (23) except that $(\mathbf{p}_0|t_L|\mathbf{p}_0)$ replaces $(p'|t_L|p'')$; that is, we must compare it to a solution of Schrödinger's equation for a potential whose range is given by Eq. (3) and whose depth is given by Eq. (2). [It ought to be stated that the neglect of the c.m. to lab transformation was made consistently, that is, it was also neglected in calculating $(p_0|t_L|p_0)$.]

The calculation was done for carbon and for $k_0=0.8$. The results are contained in Table III. They are that the off-diagonal elements do roughly double the depth of the equivalent rectangular potential; they are not sufficiently large to achieve agreement with experiment. It is interesting that the effects of the off-the-energy-shell corrections are not more important in this case for which t_L is more than ten times larger off the energy shell as on it. (See Fig. 7.)

V. OTHER NUCLEAR BINDING EFFECTS

In Sec. II we discussed corrections due to the nucleon velocity in the nucleus. There are still other corrections for nuclear binding, such as the error due to the impulse approximation. (This has been considered by many authors,[20] and will not be discussed by us.)

There are also corrections to Eq. (2) due to correlations in nucleon positions within the nucleus. (This effect is well known in optics—as for instance in the study of x-ray scattering by liquids and in critical opalescence.) The formal development of the optical model to include such effects was given by Francis and Watson.[8] A rough evaluation by them [Eq. (81) of reference 8] led to a correction, which is approximately

$$"V_R"=V_R[1+\tfrac{1}{2}(\hbar/\mu c)(1/\lambda_s)],$$

with λ_s given by Eq. (7). Here "V_R" is the value of V_R corrected for the nucleon correlation effect on the basis of a model for which the only correlations are due the Pauli principle. This formula is valid only when

TABLE III. The values of $\tan\delta/\pi\omega_0 k_0$ calculated for carbon and $k_0=0.8$ by solving the Schrödinger equation (64) with two potentials. The first potential is that calculated from Eq. (23) without approximation, and the second is that calculated from Eq. (23) with the approximation which lead eventually to Eqs. (32) and (2).

Potential I	Potential II
0.100	0.0539

$"V_R"\simeq V_R$. Except for $(E_\pi-\mu c^2)\simeq 200$ Mev, this does not seem to be a very significant effect.

VI. CONCLUSIONS

Except for the self-consistent evaluation using meson theory, the calculation of the well depth from the dispersion relations of Goldberger seems justifiable and in agreement with presently available experimental values. Further experimental study should provide considerable evidence for, or against, the basic assumption of two-body forces within nuclei.

The calculation using meson theory was in order-of-magnitude disagreement with the value of V_R as obtained from the dispersion relations. The source of the disagreement was traced to the very strong interactions at high momenta in meson theory.

It would appear that the experimental measurement of the well depth for both light and heavy nuclei would be of considerable interest, since the model which we have used implies that the well depth is independent of the nuclear size. If the model did not give a reasonable approximation to the actual well depth, we could not expect this treatment to be valid.

ACKNOWLEDGMENTS

It is a pleasure to acknowledge several very helpful conversations with Professor H. L. Anderson, who participated in the early phrases of this work.

APPENDIX. GENERAL RELATION BETWEEN THE t- AND K-MATRICES

In Sec. III, we set $V_0=V_R$, dropping the imaginary part. We now consider the modification of our equations which results when V_I is kept. The equation for t is now [see Eq. (38)]

$$t=V+V\frac{1}{\omega_0-h+iV_I}t, \tag{A-1}$$

the $i\eta$ term being superfluous. This may be rewritten as

$$t=V+V\left[\frac{(\omega_0-h)}{(\omega_0-h)^2+V_I^2}\right]t-iV\left[\frac{V_I}{(\omega_0-h)^2+V_I^2}\right]t. \tag{A-2}$$

TABLE IV. The potential V_R on the energy shell for the simple model of Sec. (IV-A). V_R is given in units of μc^2. The calculation of $\delta V_R/V_R$ is discussed in the Appendix.

	Case I [Eq. (60)]		Case II [Eq. (61)]		Case II $\delta V_R/V_R$
k_0	Free particle V_R	Self consistent V_R	Free particle V_R	Self consistent V_R	
0.4	−0.437	−0.066	−0.114	−0.144	0.002
0.8	−0.746	−0.079	−0.189	−0.214	0.007
1.2	−0.319	−0.088	−0.370	−0.366	0.150
1.6	−0.013	−0.013	−0.449	−0.380	
2.0	−0.043	+0.043			

This equation is, however, equivalent to the set of equations

$$K=V+V\left[\frac{(\omega_0-h)}{(\omega_0-h)^2+V_I^2}\right]K, \tag{A-3}$$

$$t=K-iK\left[\frac{V_I}{(\omega_0-h)^2+V_I^2}\right]t. \tag{A-4}$$

On going to the limit $V_I=0$, we obtain just Eqs. (44) and (45).

It is important to know the error introduced into Eqs. (A-3) and (A-4) by letting $V_I\to 0$, since this procedure formed the basis of the calculations made in Sec. IV. For the model of Sec. IV-A, Case II, calculations were made of the V_R given by Eqs. (A-3) and (A-4), and the V_R given by Eqs. (44) and (45). The results are presented in the last column of Table IV in terms of the ratio $\delta V_R/V_R$, where δV_R is the difference between the two V_R's. The correction does not seem to be very important, except near the peak in the pion-nucleon scattering cross section.