PHYSICAL REVIEW VOLUME 101, NUMBER 2 JANUARY 15, 1956

Quantum Field Theory in Terms of Vacuum Expectation Values

A. S. WIGHTMAN
Palmer Physical Laboratory, Princeton University, Princeton, New Jersey
(Received July 18, 1955)

Vacuum expectation values of products of neutral scalar field operators are discussed. The properties of these distributions arising from Lorentz invariance, the absence of negative energy states and the positive definiteness of the scalar product are determined. The vacuum expectation values are shown to be boundary values of analytic functions. Local commutativity of the field is shown to be equivalent to a symmetry property of the analytic functions. The problem of determining a theory of a neutral scalar field given its vacuum expectation values is posed and solved.

1. INTRODUCTION

RECENT work in relativistic quantum field theory has made heavy use of certain basic singular functions defined as vacuum expectation values of products of fields taken at various space-time points. In this paper, we present some results of a systematic study of relativistic field theory based on such vacuum expectation values. For simplicity, we treat the case of a neutral scalar field interacting with itself. The methods used have generalizations in any field theory.

The objects of our attentions are the singular functions:

$$F^{(n)}(x_1,\cdots x_n)=(\Psi_0,\phi(x_1)\phi(x_2)\cdots\phi(x_n)\Psi_0),$$

where Ψ_0 is the vacuum state, assumed to be the unique state of energy and momentum zero, and $\phi(x)$ is a neutral scalar field. As is well known, $F^{(n)}$ has to be understood as a distribution in the sense of L. Schwartz. It is a linear functional which gives a complex number for each infinitely differentiable function $f(x_1,\cdots x_n)$ which vanishes outside a bounded region of space time:

$$F^{(n)}(f)=\int d^4x_1\cdots d^4x_n f(x_1,\cdots x_n)F^{(n)}(x_1,\cdots x_n).$$

(We call such f testing functions.) Furthermore, $F^{(n)}(f_k)\rightarrow 0$ if a sequence of testing functions $f_k(x_1,\cdots x_n)$ (vanishing outside a fixed bounded region) and all their derivatives converge to zero uniformly in space time. We shall study the structure of $F^{(n)}$, exploiting systematically the Lorentz transformation properties of $\phi(x)$ which are given by

$$U(a,\Lambda)\phi(x)U(a,\Lambda)^{-1}=\phi(\Lambda x+a). \tag{1}$$

Here, $\{a,\Lambda\}$ is an element of the inhomogeneous Lorentz group meaning the operation of transforming by the homogeneous Lorentz transformation, Λ, followed by translating by a. $U(a,\Lambda)$ is the corresponding unitary or antiunitary operator which yields the transformed wave functions. We shall also determine the consequences for $F^{(n)}$ of the assumptions that no negative energy states exist in the theory and that $\phi(x)$ is a local field. Finally, we shall show how, given a set $F^{(n)}$, $n=1, 2, \cdots$, one can construct a theory of a neutral scalar field which has these $F^{(n)}$ as its vacuum expectation values. We do not show that any set of $F^{(n)}$ outside of those determined by the free field actually exists.

2. CONSEQUENCES OF LORENTZ INVARIANCE

For Lorentz transformations without time inversion, we know that $U(a,\Lambda)$ is unitary. Thus,

$$\begin{aligned}F^{(n)}(x_1,\cdots x_n)&=(U(a,\Lambda)\Psi_0,U(a,\Lambda)\phi(x_1)\cdots\phi(x_n)\Psi_0)\\&=(\Psi_0,\ \phi(\Lambda x_1+a)\cdots\phi(\Lambda x_n+a)\Psi_0),\end{aligned}$$

and therefore

$$F^{(n)}(x_1,\cdots x_n)=F^{(n)}(\Lambda x_1+a,\ \cdots\Lambda x_n+a) \tag{2}$$

for $\{a,\Lambda\}$ without time inversion. Here we have used (1) and the Lorentz invariance of the vacuum state:

$$U(a,\Lambda)\Psi_0=\Psi_0. \tag{3}$$

For Lorentz transformations with time inversion, we have, on the other hand,[1]

$$\begin{aligned}F^{(n)}(x_1,\cdots x_n)&=(U(a,\Lambda)^{-1}\Psi_0,\phi(x_1)\cdots\phi(x_n)\Psi_0)\\&=[(\Psi_0,U(a,\Lambda)\phi(x_1)\cdots\phi(x_n)\Psi_0)]^*\\&=[(\Psi_0,\phi(\Lambda x_1+a)\cdots\phi(\Lambda x_n+a)\Psi_0)]^*.\end{aligned}$$

Thus,

$$F^{(n)}(x_1,\cdots x_n)=[F^{(n)}(\Lambda x_1+a,\cdots\Lambda x_n+a)]^* \tag{4}$$

for $\{a,\Lambda\}$ with time inversion.

A somewhat similar relation is derived from the hermiticity of $\phi(x)$:

$$\begin{aligned}F^{(n)}(x_1,\cdots x_n)&=[(\Psi_0,(\phi(x_1)\cdots\phi(x_n))^*\Psi_0)]^*\\&=[(\Psi_0,\phi(x_n)\cdots\phi(x_1)\Psi_0)]^*.\end{aligned}$$

Thus,

$$F^{(n)}(x_1,\cdots x_n)=[F^{(n)}(x_n,\cdots x_1)]^*. \tag{5}$$

3. CONSEQUENCES OF THE ABSENCE OF NEGATIVE ENERGY STATES

From (2), we see that $F^{(n)}$ is a function only of the differences of the $x_1,\cdots x_n$. We shall therefore write

$$F^{(n)}=F^{(n)}(\xi_1,\cdots\xi_{n-1}),$$

[1] For convenience in printing we shall often denote the complex conjugate, $\bar{a}$, of a number a, by $[a]^*$; this latter notation is the same as the notation we use to denote adjoint of an operator.

where

$$\xi_1 = x_1 - x_2, \quad \xi_2 = x_2 - x_3, \cdots \quad \xi_{n-1} = x_{n-1} - x_n.$$

We shall assume that $F^{(n)}$ has a Fourier transform and shall show that if

$$F^{(n)}(\xi_1, \cdots \xi_{n-1}) = \int \exp(-i \sum_{j=1}^{n-1} p_j \cdot \xi_j) \times G^{(n)}(p_1, \cdots p_{n-1}) d^4p_1 \cdots d^4p_{n-1},$$

then $G^{(n)}$ vanishes unless the p_j satisfy

$$p_j^2 \equiv (p_j^0)^2 - \mathbf{p}_j^2 \geq 0, \quad p_j^0 \geq 0,$$

by virtue of the assumption that no negative-energy states exist.

If Ψ is an arbitrary state, its component of momentum p is

$$\int e^{-ip\cdot a} d^4a U(a,1)\Psi.$$

Thus, by our hypothesis that no negative energy states exist,

$$\begin{aligned} &\int e^{ip\cdot a} d^4a F^{(n)}(\xi_1, \cdots \xi_{j-1}, \xi_j + a, \xi_{j+1} \cdots \xi_{n-1}) \\ &= \int e^{ip\cdot a} d^4a (\Psi_0, \phi(x_1) \cdots \phi(x_j) \times \phi(x_{j+1} - a) \cdots \phi(x_n - a)\Psi_0) \\ &= \left(\Psi_0, \phi(x_1) \cdots \phi(x_j) \int e^{ip\cdot a} d^4a U(-a, 1) \times \phi(x_{j+1}) \cdots \phi(x_n)\Psi_0\right) \end{aligned}$$

vanishes unless p is within or on the forward light cone. Therefore, $G^{(n)}(p_1, \cdots p_{n-1})$ vanishes if any of its arguments lie outside the forward light cone. In the special case $n=2$, this result is well known.

It is an important consequence of this property of the $G^{(n)}$ that the distributions $F^{(n)}$ are boundary values of analytic functions. This result is displayed in and simultaneously proved by the formula

$$\begin{aligned} &F^{(n)}(\xi_1 - i\eta_1, \cdots \xi_{n-1} - i\eta_{n-1}) \\ &= \int \exp(-i \sum_{j=1}^{n-1} p_j \cdot (\xi_j - i\eta_j)) \times G^{(n)}(p_1, \cdots p_{n-1}) d^4p_1 \cdots d^4p_{n-1}, \end{aligned}$$

where the four-vectors η_j are restricted to lie in the future light cone. The $8(n-1)$-dimensional open region thus defined in the $8(n-1)$-dimensional space of the components of the $\xi_1, \cdots \xi_{n-1}, \eta_1, \cdots \eta_{n-1}$ is called the future tube. Thus, $F^{(n)}(\xi_1, \cdots \xi_{n-1})$ is a boundary value of a function analytic in the future tube.

We introduce the notation

$$z_{j,j+1} = \xi_j - i\eta_j.$$

Equation (2) implies the Lorentz invariance of $G^{(n)}$:

$$G^{(n)}(p_1, \cdots p_{n-1}) = G^{(n)}(\Lambda p_1, \cdots \Lambda p_{n-1}),$$

for Λ without time inversion, and Eq. (4):

$$G^{(n)}(p_1, \cdots p_{n-1}) = [G^{(n)}(-\Lambda p_1, \cdots -\Lambda p_{n-1})]^*,$$

for Λ with time inversion. Consequently, we have throughout the tube

$$F^{(n)}(z_{12}, \cdots z_{n-1,n}) = F^{(n)}(\Lambda z_{12}, \cdots \Lambda z_{n-1,n}) \tag{6}$$

for Λ without time inversion and

$$F^{(n)}(z_{12}, \cdots z_{n-1,n}) = [F^{(n)}(\Lambda \bar{z}_{12}, \cdots \Lambda \bar{z}_{n-1,n})]^* \tag{7}$$

for Λ with time inversion.

We use the following theorem, whose proof will be published in another paper.

Theorem.—A function $f(z_1, \cdots z_n)$ of n four-vector variables $z_1, \cdots z_n$ analytic in the tube:

$$-\infty < \text{Re } z_{i\mu} < \infty, \ \text{Im } z_i \text{ in the future cone},$$

and invariant under the homogeneous Lorentz group without time inversion:

$$f(z_1, \cdots z_n) = f(\Lambda z_1, \cdots \Lambda z_n)$$

is a function of the scalar products $z_j \cdot z_k$, $j, k = 1, 2, \cdots n$. It is analytic in the complex manifold over which the scalar products vary when the vectors $z_1, \cdots z_n$ vary over the future tube.

If we introduce the complex vectors

$$z_{ij} = \sum_{k=i}^{j-1} z_{k,k+1} \quad \text{for} \quad i < j,$$

then the theorem tells us that the $F^{(n)}$ are analytic functions of the squares of the lengths of these vectors

$$F^{(n)} = F^{(n)}(z_{ij}^2). \tag{8}$$

As the variables $z_{j,j+1}$ vary over the tube each variable z_{ij}^2 varies in an open set of the complex plane. From the explicit form

$$z^2 = (\xi - i\eta)^2 = \xi^2 - \eta^2 - 2i\xi\cdot\eta$$

one can easily see that this set fills the entire plane except for the positive real axis and the origin. The situation is indicated in detail in Fig. 1. Of course, the z_{ij}^2 are not independent, so that the manifold they define is not just the topological product of cut planes. The theory of a scalar field can be regarded as a boundary value problem for this set of analytic functions. We give an example of such a set, those arising from the theory of a scalar field $\phi(x)$ satisfying $(\square + m^2) \times \phi(x) = 0$ and the commutation rules $[\phi(x), \phi(x')]_-$

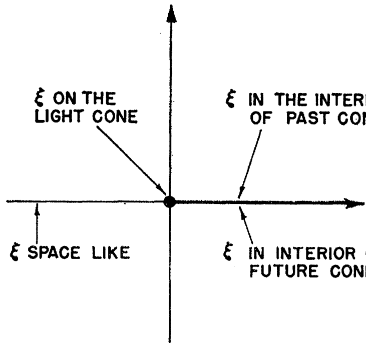

FIG. 1. Domain of variation of the complex variable $z^2\equiv(\xi-i\eta)^2=\xi^2-\eta^2-2i\xi\cdot\eta$, when ξ varies over all space time and η in the future cone. The labeled points indicate values of z^2 in the physical limit $\eta\to 0$ for typical positions of ξ.

$=i^{-1}\Delta(x-x')$:

$$F^{(n)}=0, \qquad n \text{ odd}$$

$$F^{(n)}(z_{ij}^2)=\left(\frac{m^2}{8\pi i}\right)^{n/2}\sum\frac{H_1^{(1)}(m(z_{12}^2)^{\frac{1}{2}})}{m(z_{12}^2)^{\frac{1}{2}}}\times\frac{H_1^{(1)}(m(z_{34}^2)^{\frac{1}{2}})}{m(z_{34}^2)^{\frac{1}{2}}}\cdots\frac{H_1^{(1)}(m(z_{n-1,n}^2)^{\frac{1}{2}})}{m(z_{n-1,\,n}^2)^{\frac{1}{2}}}, \quad n \text{ even.} \tag{9}$$

The sum in (9) goes over all "pairings" of the sequence $(1\cdots n)$, a "pairing" being a division of the set into $(n/2)$ disjoint subsets. The formulas assume a more familiar aspect if one goes over to the physical limit in which

$$\left(\frac{m^2}{8\pi}\right)\frac{H_1^{(1)}(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}\to\Delta^{(+)}(\xi).$$

4. CONSEQUENCES OF THE COMMUTATION RULES

The local commutation rules

$$[\phi(x),\phi(y)]_-=0, \quad (x-y)^2<0,$$

imply that

$$F^{(n)}(x_1,\cdots x_j,x_{j+1},\cdots x_n)=F^{(n)}(x_1,\cdots x_{j+1},x_j\cdots x_n) \tag{10}$$

as long as x_j and x_{j+1} are space-like separated points. These relations can be extended by analytic continuation to relations of the analytic functions $F^{(n)}(z_{ij}^2)$:

$$F^{(n)}(z_{ik}^2)=F^{(n)}(Pz_{ik}^2), \tag{11}$$

where P stands for the operation of permuting the subscripts j and $j+1$, and, by definition, $z_{ik}^2=z_{ki}^2$. The proof is simple. (11) coincides with (10) for all z_{ik}^2 on the negative real axis. Consequently, it holds everywhere by analytic continuation. Thus, the local property of a field $\phi(x)$ is characterized by a global symmetry relation on the analytic functions $F^{(n)}$.

Unlike the local commutation rules, the more special canonical commutation rules require a certain amount of heuristic juggling before a translation in terms of of analytic functions is possible.

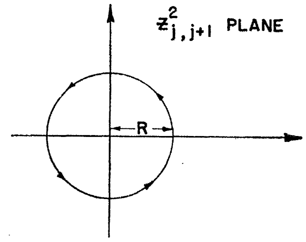

FIG. 2. Contour for the integration of Eq. (14).

We start from the equation

$$\left[\frac{\partial\phi}{\partial(ct)}(t,\mathbf{x}_1),\phi(t,\mathbf{x}_2)\right]_-=\frac{1}{i}\delta(\mathbf{x}_1-\mathbf{x}_2). \tag{12}$$

Integrating over all space, we have

$$\int d^3\mathbf{x}_1\left[\frac{\partial\phi}{\partial(ct)}(t,\mathbf{x}_1),\,\phi(t,\mathbf{x}_2)\right]_-=i^{-1}$$

or

$$\lim_{\Delta t\to 0}(2c\Delta t)^{-1}\left\{\int d^3\mathbf{x}_1[\phi(t+\Delta t,\mathbf{x}_1),\,\phi(t,\mathbf{x}_2)]_- -\int d^3\mathbf{x}_1[\phi(t-\Delta t,\,\mathbf{x}_1),\,\phi(t,\mathbf{x}_2)]_-\right\}=i^{-1}. \tag{13}$$

Now we assume that, in the limit, the only contribution to this integral will come from the singularity on the light cone, an assumption whose validity will be discussed at the end of this section. For the first term in the first commutator of (13), the integral then can be written

$$\int d^3x_1\cdots=4\pi\int_{(c\Delta t)^2}^{0}d\xi^2\{-\tfrac{1}{2}[(c\Delta t)^2-\xi^2]^{\frac{1}{2}}\}\cdots,$$

where $\xi=(c\Delta t,\,\mathbf{x}_1-\mathbf{x}_2)$. In the complex plane of Fig. 1, this corresponds to an integration just below the real axis from $(c\Delta t)^2$ to the origin. From the first term in the second commutator we get a similar integral just above the real axis. This suggests that the two terms contribute a contour integral around the branch cut. The second terms combine to given an equal contribution. Thus, it seems plausible that the canonical commutation relations are equivalent to the requirement that

$$\lim_{R\to 0}\int_{C(R)}dz_{j,\,j+1}{}^2F^{(n)}(z_{kl}^2)=\frac{1}{2\pi i}F^{(n-2)}(z_{kl}^2), \tag{14}$$

where the contour, $C(R)$, is that indicated in Fig. 2, and it is understood that on the right-hand side the point x_j is identified with x_{j+1}.

It is not difficult to verify that the $F^{(n)}$ of the free-field case satisfy (14). One writes

$$\frac{H_1^{(1)}(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}=\frac{J_1(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}+i\frac{Y_1(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}.$$

Now $J_1(m(z^2)^{\frac{1}{2}})/m(z^2)^{\frac{1}{2}}$ is an entire function of z^2, so that the contour integral of (14) vanishes by Cauchy's theorem. On the other hand

$$\frac{Y_1(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}=\frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{(-m^2z^2/4)^r}{r!(r+1)!}\times\left\{2\log\left[\frac{m(z^2)^{\frac{1}{2}}}{2}\right]-\psi(r+1)-\psi(r+2)\right\}-\frac{2}{\pi m^2 z^2}.$$

The precise definition of the $\psi(r)$ is irrelevant since their contribution to the integrand is an entire function. The contribution from the last term is

$$\frac{m^2}{8\pi i}\int_{C(R)}dz^2\left[\frac{-2i}{\pi m^2 z^2}\right]=\frac{1}{2\pi i}.$$

From the terms involving the logarithm, we get

$$\frac{1}{2\pi}\sum_{r=0}^{\infty}\frac{(-m^2R/4)^{r+1}}{r!(r+1)!(r+1)},$$

which approaches zero in the limit $R\rightarrow 0$. The resultant formula,

$$\lim_{R\rightarrow 0}\int_{C(R)}\frac{H_1^{(1)}(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}}=\frac{1}{2\pi i}$$

shows that the $F^{(n)}$ of Eq. (9) satisfy (14).

In current renormalization theory, the canonical commutation relation (12) holds for the unrenormalized fields, but the renormalized fields are supposed to satisfy

$$\left[\frac{\partial\phi}{\partial x^0}(x),\,\phi(y)\right]_-\Big|_{x^0=y^0}=\frac{1}{Zi}\delta(\mathbf{x}-\mathbf{y}), \tag{15}$$

where $0\leq Z\leq 1$. If $Z>0$, our previous analysis requires only trivial modifications: a $1/Z$ appears as a factor on the right-hand side of (14). However, for $Z=0$, the commutator is essentially more singular than in the case of a free field. In order to understand this case, we study $F^{(2)}$ in more detail.

It is an immediate consequence of the Lorentz invariance of $G^{(2)}$, that the most general $F^{(2)}$ is of the form

$$F^{(2)}(z^2)=\int_0^{\infty}dg(m)\frac{m^2}{8\pi i}\frac{H_1^{(1)}(m(z^2)^{\frac{1}{2}})}{m(z^2)^{\frac{1}{2}}} \tag{16}$$

(apart from a constant), where dg is a weight function which will be shown to be positive in Sec. 5. It would be tempting to argue that the operation indicated in Eq. (14) can be carried out under the integral sign in (16) so that

$$1/Z=\int_0^{\infty}dg(m). \tag{17}$$

This is indeed correct if $\int_0^\infty dg<\infty$, but if $\int_0^\infty dg=\infty$, then (17) is somewhat misleading, as the following example shows.

Let

$$dg(m)=dm, \quad \text{for} \quad m\geq 0.$$

Then, using a standard integral representation of the Hankel function, we get

$$\begin{aligned}F^{(2)}(z)&=\frac{1}{(2\pi)^2}\int_1^{\infty}d\xi(\xi^2-1)^{\frac{1}{2}}\int_0^{\infty}m^2dg(m)\times\exp[-m\xi(-z^2)^{\frac{1}{2}}]\\&=(-z^2)^{-\frac{3}{2}}\Gamma(3)(2\pi)^{-2}\int_1^{\infty}d\xi(\xi^2-1)^{\frac{1}{2}}\xi^{-3}\\&=(8\pi i)^{-1}\Box[(z^2)^{-\frac{1}{2}}].\end{aligned} \tag{18}$$

The corresponding singularity of the commutation relation is obtained by examining the distribution:

$$(\Psi_0,[\phi(x_1),\phi(x_2)]_-\Psi_0)=\lim_{\eta\rightarrow 0}\{F^{(2)}(z)-[F^{(2)}(z)]^*\}=\begin{cases}0 & \xi^2=(x_1-x_2)^2<0\\ \dfrac{1}{4\pi i}\Box\left(\dfrac{\epsilon(\xi)}{(\xi^2)^{\frac{1}{2}}}\right) & \xi^2\geq 0,\end{cases} \tag{19}$$

where $\epsilon(\xi)=\operatorname{sign}(\xi^0)$.

$\epsilon(\xi)(\xi^2)^{-\frac{1}{2}}$ is that very special kind of distribution, an integrable function. However, its derivatives are not functions nor can they be expressed in terms of δ functions and their derivatives except in the following rather singular way. Let f be a testing function. Then

$$\int d^4\xi\epsilon(\xi)(\xi^2)^{-\frac{1}{2}}\Box f=\lim_{\zeta\rightarrow 0}\left\{-\frac{1}{\zeta^2}\int d\Omega(\xi)\xi^{\mu}\frac{\partial f}{\partial\xi^{\mu}}(\xi)\epsilon(\xi)-\frac{1}{\zeta^2}\int d\Omega(\xi)f(\xi)\epsilon(\xi)-\int_{R_\zeta}f(\xi)\epsilon(\xi)(\xi^2)^{-}\,d^4\xi\right\} \tag{20}$$

In (20), the first and second integrals on the right-hand side are over the entire mantle of the light cone, while the third is over the four-volume later and earlier than the hyperboloids $\xi^2=\zeta^2$, $\xi^0\geq 0$ and $\xi^2=\zeta^2$, $\xi^0\leq 0$, respectively. The second integral can be written

$$\frac{1}{\zeta^2}\int d\Omega(\xi)\epsilon(\xi)f(\xi)=\frac{2}{\zeta^2}\int d^4\xi\epsilon(\xi)\delta(\xi^2)f(\xi),$$

which is just of the right type to yield the right-hand side of (15).

To obtain the commutation rules at a fixed time, one takes $f=\partial h/\partial x^0$, where h is positive and symmetrical in time, and shrinks to zero the time interval in which h is nonzero. The first term of (20) is then zero and the last two different from zero. If f is held fixed and $\xi\rightarrow 0$, each of the last two terms becomes arbitrarily large, although their sum nearly cancels. The expression as a whole is proportional to $h(0)$ in an approximation which becomes better and better as h shrinks toward the origin of time. Thus, the right-hand side of (15) should indeed be $i^{-1}\infty\delta(\mathbf{x}-\mathbf{y})$ but it does *not* arise from a term in $(\Psi_0,[\phi(x),\phi(y)]_-\Psi_0)$ of the form $(2\pi i)^{-1}\infty\epsilon(x-y)\times\delta((x-y)^2)$ as the derivation of (17) would indicate. We conclude that the treatment of the canonical commutation rules in the case, $\int_0^\infty dg=\infty$, requires a direct consideration of the singularities of $F^{(n)}(z)$. In such a case, to replace (14) by an appropriate alternative is not difficult, but to avoid complications unessential to the present paper we defer the discussion. In particular, in Sec. 6 we consider only the local commutation rules.

5. POSITIVE DEFINITENESS CONDITIONS

Since the length of a vector is greater than or equal to zero, we have

$$\left\|\alpha_0 f_0\Psi_0+\alpha_1\int d^4x_1 f_1(x_1)\phi(x_1)\Psi_0 + \alpha_2\int\int d^4x_1d^4x_2 f_2(x_1,x_2)\phi(x_1)\phi(x_2)\Psi_0+\cdots\right\|^2\geq 0$$

for all $\alpha_0,\alpha_1,\alpha_2,\cdots$ and all testing functions $f_1(x_1)$, $f_2(x_1,x_2),\cdots$. Therefore,

$$\sum_{i,j=0}\bar{\alpha}_i\alpha_j\int\cdots\int\bar{f}_i(x_1,\cdots x_i)\times F^{(i+j)}(x_i,x_{i-1},\cdots x_1,y_1,\cdots y_j)f_j(y_1,\cdots y_j)\times d^4x_1\cdots d^4x_id^4y_1\cdots d^4y_j\geq 0. \quad (21)$$

We will refer to (21) as the positive-definiteness conditions. The simplest consequences of (21) are obtained by setting all but one of the α_i equal to zero. For example, for $\alpha_1\neq 0$, we find

$$\int\bar{f}_1(x_1)F^{(2)}(x_1-x_2)f_1(x_2)d^4x_1d^4x_2\geq 0. \quad (22)$$

It can be shown that (22) is a necessary and sufficient condition that $F^{(2)}$ be a Fourier transform of a positive measure of not too fast increase[2] (generalized Bochner theorem). For $\alpha_2\neq 0$, we have

$$\int\bar{f}_2(x_1,x_2)F^{(4)}(x_1-x_2,\,x_2-y_1,\,y_1-y_2)\times f_2(y_1,y_2)d^4x_1d^4x_2d^4y_1d^4y_2, \quad (23)$$

an inequality whose consequences have not been characterized as neatly as those of (22).

[2] L. Schwartz, *Theorie des Distributions* (Hermann & Cie, Paris, 1951), Vol. 2, p. 132, Theorem 18.

We shall not study the relations (21) in detail in this paper but we want to point out their general significance for field theory. $F^{(n)}$ is intimately connected with the S-matrix elements for those processes in which the sum of the number of ingoing and outgoing particles is n, e.g.,

$$(\Psi_0,\phi^{\text{out}}(p_1')\phi^{\text{out}}(p_2')\phi^{\text{in}}(p_1)\phi^{\text{in}}(p_2)\Psi_0)$$

is the S-matrix element for the scattering of two mesons of momenta p_1 and p_2, to produce two mesons of momenta p_1' and p_2'. The positive-definiteness conditions will therefore imply relations not only for the S-matrix elements of a given process, but also relations between the S-matrix elements of different processes. It might be argued that such relations can yield nothing more than those results arising from the conservation of probability. We do not believe that to be true. The positive-definiteness conditions deserve a detailed investigation.

6. INVERSE PROBLEM—DETERMINATION OF A THEORY FROM ITS VACUUM EXPECTATION VALUES

If one is given analytic functions $F^{(n)}(z_{ij}^2)$, $n=0,1,2,\cdots$, then one can construct a theory of a scalar field, $\phi(x)$, in which the n-fold vacuum expectation value of $\phi(x)$ is just the boundary value of $F^{(n)}$, provided that the $F^{(n)}$ satisfy certain conditions, most of which have been discussed in the preceding sections. The construction will be carried out explicitly, the conditions on $F^{(n)}$ being introduced as they are needed.

First, we reconstruct the Hilbert space from the $F^{(n)}$. Certain vectors of it will be defined as conjugate linear functionals on the spaces of testing functions. Then addition, multiplication by a scalar factor and formation of the scalar product will be defined for these vectors. The full Hilbert space is then obtained by a standard completion process.

Let Ψ_0 denote the conjugate linear functional which yields for the testing function $f^{(n)}(x_1,\cdots x_n)$ the value

$$\int\bar{f}^{(n)}(x_1,\cdots x_n)d^4x_1\cdots d^4x_nF^{(n)}(x_1,\cdots x_n).$$

Let $\Psi_{g^{(m)}}$ denote the conjugate linear functional which yields for the testing function $f^{(n)}(x_1,\cdots x_n)$, the value

$$\int\bar{f}^{(n)}(x_1,\cdots x_n)F^{(n+m)}(x_n,\cdots x_1,y_1,\cdots y_m)\times g^{(m)}(y_1,\cdots y_m)d^4x_1\cdots d^4x_nd^4y_1\cdots d^4y_m.$$

The $\Psi_{g^{(m)}}$ and Ψ_0, together with the null functional 0, can be made into the basis vectors of a linear vector space if we define multiplication by a scalar and addition as follows: $\alpha\Psi_{g^{(m)}}$ is defined as $\Psi_{\alpha g^{(m)}}$, while $\Psi_{g^{(m)}}+\Psi_{h^{(n)}}$ is the conjugate linear functional which yields for the testing function $f^{(l)}(x_1,\cdots x_l)$ the value

$$\int \bar{f}^{(l)}(x_1,\cdots x_l)F^{(l+m)}(x_l,\cdots x_1,y_1,\cdots y_m)$$
$$\times g^{(m)}(y_1,\cdots y_m)d^4x_1\cdots d^4x_l d^4y_1\cdots d^4y_m$$
$$+\int \bar{f}^{(l)}(x_1,\cdots x_l)F^{(l+n)}(x_l,\cdots x_1,y_1,\cdots y_n)$$
$$\times h^{(n)}(y_1,\cdots y_n)d^4x_1\cdots d^4x_l d^4y_1\cdots d^4y_n.$$

The sum $\Psi_0+\Psi_{g^{(m)}}$ is defined similarly. It is obvious indeed that addition is associative and commutative and has a zero, the null functional. Further, inverses (negatives) exist and scalar multiplication is commutative, associative, and distributive. In short, the set of all linear combinations of the basis vectors $\Psi_{g^{(m)}}$ and Ψ_0 forms a linear vector space, $\mathfrak{H}$.

We introduce a scalar product into $\mathfrak{H}$ by means of the following formulas:

$$(\Psi_{g^{(n)}},\Psi_{h^{(m)}})=\int \bar{g}^{(n)}(x_1,\cdots x_n)F^{(n+m)}$$
$$\times(x_n,\cdots x_1,y_1,\cdots y_m)h^{(m)}(y_1,\cdots y_m)$$
$$\times d^4x_1\cdots d^4x_n d^4y_1\cdots d^4y_m, \quad (24)$$
$$(\Psi_0,\Psi_{g^{(m)}})=\int F^{(m)}(y_1,\cdots y_m)$$
$$\times g^{(m)}(y_1,\cdots y_m)d^4y_1\cdots d^4y_m. \quad (25)$$

Equations (24) and (25) define the scalar product for the basis of our linear vector space. It then turns out that for linear combinations of the basis vectors, it is consistent to define it as linear in its second argument and antilinear in its first.

The scalar product so defined will have the desirable property, $(\Phi,\Psi)=[(\Psi,\Phi)]^*$, by virtue of the hermiticity condition (5) on $F^{(n)}$. It will be positive definite by virtue of the positive definiteness conditions (21) on $F^{(n)}$. Thus, with this definition, $\mathfrak{H}$ has all the properties of a Hilbert space except completeness. So we complete it by the standard method.

Next we define the unitary operators $U(a,\Lambda)$ which give the representation of the inhomogeneous Lorentz group. Consider first the action of $U(a,\Lambda)$ for $\{a,\Lambda\}$ without time inversion on the basis elements Ψ_0 and $\Psi_{g^{(n)}}$:

$$U(a,\Lambda)\Psi_0=\Psi_0,$$
$$U(a,\Lambda)\Psi_{g^{(n)}}=\Psi_{g'^{(n)}},$$

where

$$g'^{(n)}(x_1,\cdots x_n)=g^{(n)}(\Lambda^{-1}(x_1-a),\cdots\Lambda^{-1}(x_n-a)).$$

The $U(a,\Lambda)$ preserve scalar products of the basis vectors,

$$(U(a,\Lambda)\Psi_{g^{(n)}},U(a,\Lambda)\Psi_{h^{(m)}})=(\Psi_{g^{(n)}},\Psi_{h^{(m)}}),$$

by virtue of the Lorentz invariance property of the $F^{(n)}$ given in Eq. (2). Furthermore, on the basis vectors,

$$U(a,\Lambda)U(b,\mathrm{M})=U(a+\Lambda b,\Lambda\mathrm{M}). \quad (26)$$

The operators $U(a,\Lambda)$ can be extended to all vectors in $\mathfrak{H}$ by linearity and, so defined, are unitary.

For the $\{a,\Lambda\}$ containing time inversion we define

$$U(a,\Lambda)\Psi_0=\Psi_0,$$
$$U(a,\Lambda)\Psi_{g^{(n)}}=\Psi_{g^{\times(n)}}, \quad (27)$$

where

$$g^{\times(n)}(x_1,\cdots x_n)=[g^{(n)}(\Lambda^{-1}(x_1-a),\cdots\Lambda^{-1}(x_n-a))]^*.$$

This definition has as a consequence

$$(U(a,\Lambda)\Psi_{g^{(m)}},U(a,\Lambda)\Psi_{h^{(m)}})=[(\Psi_{g^{(n)}},\Psi_{h^{(m)}})]^*,$$

by virtue of the Lorentz invariance property of the $F^{(n)}$ given in Eq. (4). Just as for the operators for $\{a,\Lambda\}$ without time inversion, (26) is satisfied, and the $U(a,\Lambda)$ can be extended to all vectors, and, so defined, are antiunitary.

The $U(a,\Lambda)$ which are determined in this way are guaranteed to contain no negative energies by virtue of the analyticity of the $F^{(n)}$ in the future tube. The details of their momentum spectrum can be ascertained from the set on which the Fourier transforms of the $F^{(n)}$ do not vanish. The real four-vector, p, is in the momentum spectrum of $U(a,\Lambda)$ if for some $F^{(n)}$, the Fourier transform with respect to at least one of the variables does not vanish at p.

The action of the field operator $\phi(x)$ [or better its average, $\phi(f)$, with the testing function $f(y)$] on the vector $\Psi_{g^{(n)}}$ is defined as follows: $\phi(f)\Psi_{g^{(m)}}$ is the conjugate linear functional which, for the testing function $f^{(n)}(x_1,\cdots x_n)$ yields the number

$$\int \bar{f}^{(n)}(x_1,\cdots x_n)F^{(n+m+1)}(x_n,\cdots x_1,y,y_1,\cdots y_m)$$
$$\times f(y)g^{(m)}(y_1,\cdots y_m)d^4x_1\cdots d^4x_n d^4y_1\cdots d^4y_m.$$

This is just the vector $\Psi_{fg^{(m)}}$, where $f(y)g^{(m)}(y_1,\cdots y_m)$ is regarded as a testing function in $(m+1)$ variables. $\phi(f)\Psi_0$ is defined as Ψ_f. $\phi(f)$ is defined as the closed linear extension of this operator.

On the basis vectors and their finite linear combinations $\phi(f)$ possesses the property

$$\phi(f)^*=\phi(\bar{f}),$$

where $\bar{f}(x)$ is the complex conjugate of $f(x)$. Thus, $\phi(f)$ will be formally Hermitean if f is real.

The transformation properties of the operator $\phi(f)$ follow immediately from those of the $\Psi_{g^{(n)}}$. For $U(a,\Lambda)$ unitary, we have

$$U(a,\Lambda)\phi(f)\Psi_{g^{(n)}}=U(a,\Lambda)\Psi_{fg^{(n)}}$$
$$=\Psi_{f'g'^{(n)}}=\phi(f')U(a,\Lambda)\Psi_{g^{(n)}}.$$

Therefore,

$$U(a,\Lambda)\phi(f)=\phi(f')U(a,\Lambda),$$

with $f'(x)=f[\Lambda^{-1}(x-a)]$ in agreement with (1). For the anti-unitary $U(a,\Lambda)$ we have similarly:

$$U(a,\Lambda)\phi(f)\Psi_{g^{(n)}}=U(a,\Lambda)\Psi_{fg^{(n)}} =\Psi_{f^\times g^{\times(n)}}=\phi(f^\times)U(a,\Lambda)\Psi_{g^{(n)}}$$

Therefore,

$$U(a,\Lambda)\phi(f)=\phi(f^\times)U(a,\Lambda),$$

with $f^\times$ defined as in (27) in agreement with (1).

The local commutation rules of $\phi(f)$ can be verified as follows:

$$\phi(f_1)\phi(f_2)\Psi_{g^{(n)}}=\Psi_{f_1f_2g^{(n)}}=\Psi_{f_2f_1g^{(n)}}=\phi(f_2)\phi(f_1)\Psi_{g^{(n)}},$$

when $f_1(x)f_2(y)$ for all $(x-y)^2\geq 0$. The proof depends on the fact that for such f_1 and f_2,

$$f_1(x)f_2(y)[F^{(\cdots)}(\cdots x,y,\cdots)-F^{(\cdots)}(\cdots y,x,\cdots)]=0,$$

by virtue of Eq. (10).

To complete the reconstruction of the theory we need only show that the vacuum expectation values of products of $\phi(f)$'s are the $F^{(n)}$. This is an easy consequence of the formula $\phi(f_1)\phi(f_2)\cdots\phi(f_n)\Psi_0=\Psi_{f_1\cdots f_n}$. It implies

$$(\Psi_0,\phi(f_1)\cdots\phi(f_n)\Psi_0) =\int f_1(x_1)\cdots f_n(x_n)F^{(n)}(x_1,\cdots x_n)d^4x_1\cdots d^4x_n,$$

which was to be proved.

Our reconstruction remains valid in a theory in which the field $\phi(x)$ is not a complete description of the system, e.g., in a theory of interacting neutral mesons and nucleons. However, in such a case the reconstruction process given here will not recover the entire Hilbert space. If one were to define a theory by its analytic functions $F^{(n)}$, rather than by its field equations and commutation rules, then, to be sure that the theory was one of a single field $\phi(x)$, one would have to impose some kind of "completeness" requirement. For example one could require that the set of vectors $\phi(f_1)\cdots\phi(f_n)\Psi_0$ for $n=0, 1, 2, \cdots$ span the whole Hilbert space, where the f_j are testing functions which vanish outside of a spacelike slice of space time of arbitrarily small thickness Δt in the time direction.

7. CONCLUSION

A theory of a neutral scalar field can be reformulated as a theory of a denumerable set of analytic functions of complex variables, $F^{(n)}$, $n=0, 1, 2, \cdots$. Relativistic invariance implies that the $F^{(n)}$ are invariant under Lorentz transformation without time inversion and are therefore functions of certain complex variables z_{ij}^2. Local commutation rules of $\phi(x)$ imply $F^{(n)}(z_{ij}^2) =F^{(n)}(Pz_{ij}^2)$, where P is any of a certain set of permutations of the labels i, j; the positive definiteness of the scalar product implies a set of inequalities connecting the boundary values of the $F^{(n)}$. Given a set of $F^{(n)}$ satisfying the conditions listed, one can reconstruct a theory of a neutral scalar field.

PHYSICAL REVIEW VOLUME 101, NUMBER 2 JANUARY 15, 1956

Radiative Corrections to Decay Processes

R. E. Behrends, R. J. Finkelstein, and A. Sirlin
Department of Physics, University of California, Los Angeles, California
(Received August 1, 1955)

Radiative corrections associated with the electromagnetic field have been determined for the decay of a fermion of arbitrary mass into a lighter one with the emission of a single boson or of two other fermions; no special assumptions have been made about the nature of the interaction responsible for the instability. The particular example of the muon-electron decay has been worked through in detail. Sufficiently accurate experimental determination of the muon spectrum would permit the observation of a Lamb term without vacuum polarization. Modified formulas for the Michel parameter ρ are given.

INTRODUCTION

All instabilities of the elementary particles are somewhat modified by fluctuations of their electromagnetic fields.[1,2] These fluctuations are responsible first for the emission of real photons, simultaneous with the decay and independent of the surrounding matter (inner bremsstrahlung) and second for damping effects associated with the unradiated field. This damping may be described in terms of virtual photons and is exactly similar to the processes responsible for the Lamb-Retherford shift. The total probability of decay with and without inner bremsstrahlung would of course exceed the probability of unperturbed decay, were it not for the damping effect of the virtual photons; however both effects are of the same order and must be considered together.

We have considered the decay of an arbitrary charged fermion into a lighter one with the emission of a single boson or of two other fermions, without making any

[1] S. Hanawa and T. Miyazima, Progr. Theoret. Phys. (Japan) 5, 459 (1950).
[2] T. Nakano *et al.*, Progr. Theoret. Phys. (Japan) 5, 1014 (1950).