Vacuum Polarization in a Strong Coulomb Field*†

EYVIND H. WICHMANN‡ AND NORMAN M. KROLL
Columbia University, New York, New York
(Received July 8, 1955)

A study is carried out of the vacuum polarization in a strong Coulomb field. Radiative corrections are neglected. A perturbation calculation is avoided by making use of the explicit solutions of the Dirac equation in a Coulomb field. The Laplace transform of the polarization charge density times r^2 is found and used as a basis for further study. It is proved to be an analytic function of the strength of the inducing charge. It is verified that the first-order term in a power series expansion in the strength of the inducing charge just corresponds to the Uehling potential. The third-order term is studied in some detail. The leading term in the polarization potential close to the inducing charge and the space integral of the induced potential divided by r are found to all orders in the strength of the inducing charge. Ambiguities are handled by a method corresponding to regularization.

Some experimental applications are considered. The corrections to the Uehling term in these cases are found to be small.

I. INTRODUCTION

RECENT measurements of energy level differences in mu-mesonic atoms have raised the question as to whether quantum electrodynamical corrections to these level separations are of observable magnitude.[1–6] It is expected that the main quantum electrodynamical effect on the levels in mu-mesonic atoms would be the effect of vacuum polarization, arising from the coupling of the electron-positron field to the Coulomb field of the nucleus.[3]

Likewise quantum electrodynamical corrections to the x-ray fine structure separations in heavy elements may be of observable magnitude. In this latter case, vacuum polarization can be expected to be important, although it is not the only quantum electrodynamical effect expected to play a role.[7,8]

The phenomenon of vacuum polarization in an external field, to first order in a power series expansion in the strength of the inducing field, has been discussed previously.[9,10] Furthermore, in the case of a constant external field, it has been discussed to all orders in the strength of the inducing field.[11,12] For the case in which the inducing charge is a point charge, the first-order potential will be referred to as the Uehling potential. It is well known that it gives rise to a measurable contribution to the Lamb shift in hydrogen. The effect of the Uehling potential on the x-ray fine structure separation and on the levels in mu-mesonic atoms has also been considered.[2,3,5,6,8] It may here be remarked that the Uehling potential falls off exponentially at large distances from the inducing charge, and behaves as $(\ln r)/r$ at small distances.

The Uehling potential is the leading term in a perturbation expansion in which αZ is treated as a small expansion parameter. (α is the fine structure constant, and Z is the magnitude of the inducing charge, in units of the elementary charge.) The use of the first-order term only when considering the effects of vacuum polarization on hydrogen levels can thus be expected to be a very good approximation. This may not be the case when one considers mu-mesonic atoms or x-ray fine structure in heavy elements, since αZ is then of order unity. It is thus of interest to consider higher order effects.

With this in mind, we have undertaken a study of the vacuum polarization in a strong Coulomb field. We avoid a perturbation expansion by making use of the explicit solutions of the Dirac equation in a Coulomb field. Radiative corrections are neglected.

We first consider the general expression for the induced charge density and we show how the sum over states representing the charge density may be broken up into partial sums referring to different angular momenta. We next show how these partial sums may be expressed in terms of a contour integral of the Green's functions of the radial Dirac equations. The Green's functions are explicitly constructed, and their relevant properties discussed. Some further discussion is given in Appendix I.

The Laplace transform of the polarization charge density times r^2 is found and discussed. The expression is regulated and renormalized, and shown (in Appendix II), to be an analytic function of αZ inside the circle

* Work partially supported jointly by the Signal Corps, the Office of Naval Research, and the Air Research and Development Command.

† Work partially performed at Brookhaven National Laboratory under the auspices of the U. S. Atomic Energy Commission.

‡ Submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, in the Faculty of Pure Science, Columbia University, New York, New York.

[1] V. L. Fitch and J. Rainwater, Phys. Rev. **92**, 789 (1953).
[2] L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 801 (1953).
[3] H. C. Corben, Phys. Rev. **94**, 789 (1954).
[4] D. L. Hill and K. W. Ford, Phys. Rev. **94**, 1917 (1954).
[5] Koslow, Fitch, and Rainwater, Phys. Rev. **95**, 291 (1954).
[6] A. B. Mickelwait and H. C. Corben, Phys. Rev. **96**, 1145 (1954).
[7] A. L. Schawlow and C. H. Townes, "The Effect on X-Ray Fine Structure of Deviations from a Coulomb Field near the Nucleus" (to be published). We are indebted to the authors for the opportunity to read their manuscript.
[8] R. F. Christy and J. M. Keller, Phys. Rev. **61**, 147 (1942).
[9] R. Serber, Phys. Rev. **48**, 49 (1935).
[10] E. A. Uehling, Phys. Rev. **48**, 55 (1935).
[11] V. Weisskopf, Kgl. Danske Videnskab Selskab, Mat.-fys. Medd. **14**, 6 (1936).
[12] J. Schwinger, Phys. Rev. **82**, 664 (1951).

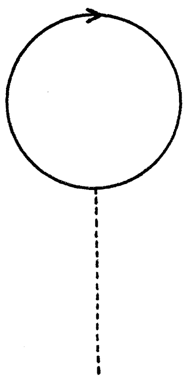

FIG. 1. Feynman diagram for the polarization potential using a Coulomb field interaction representation.

$|\alpha Z|=1$. The first-order term is extracted and shown to correspond to the Uehling term. The third-order term is discussed in some detail, and likewise the behavior of the polarization potential close to the origin is discussed. Except for the Uehling term, the charge renormalization turns out to be finite up to $\alpha Z=1$.

Higher-than-first-order vacuum polarization effects on the energy levels of mu-mesonic atoms are considered and found to be small.

We also study vacuum polarization effects on the x-ray fine structure separation, arising from the first- and third-order terms.

The contribution to the Lamb shift in hydrogen from third-order vacuum polarization is found to be negligible.

In Appendix III, the asymptotic behavior of the polarization potential is discussed briefly from the standpoint of the Euler-Heisenberg Hamiltonian.

In Appendix IV, some summation formulas are derived.

II. EXPRESSION OF THE POLARIZATION CHARGE DENSITY AS A SUM OF CONTRIBUTIONS FROM DIFFERENT ANGULAR MOMENTUM STATES

We are interested in the vacuum expectation value of the current operator;

$$j_\mu=\tfrac{1}{2}e\langle\mathrm{vac}|\bar{\psi}\gamma_\mu\psi-\psi\tilde{\gamma}_\mu\bar{\psi}|\mathrm{vac}\rangle, \tag{1}$$

where $-e$ is the charge of the electron, in the presence of a Coulomb field arising from a point charge of magnitude eZ located at the origin $\mathbf{r}=0$.

Only the timelike component of (1) is different from zero, and we may thus write the induced charge density;

$$\rho(\mathbf{r})=\tfrac{1}{2}e\sum_{(+)}\mathrm{trace}(\psi(\mathbf{r})\psi^*(\mathbf{r})),\quad -\tfrac{1}{2}e\sum_{(-)}\mathrm{trace}(\psi(\mathbf{r})\psi^*(\mathbf{r})), \tag{2}$$

where $\psi(\mathbf{r})$ is the solution to the time-independent Dirac equation in a Coulomb field, and $(+)$ indicates a sum over all positive energy states (=electron states), and $(-)$ indicates a sum over all negative energy states (=positron states), as defined by the Coulomb field.

This charge density gives rise to an electrostatic potential, $V_P(\mathbf{r})$, which may be represented by the Feynman diagram in Fig. 1, using an interaction representation based on the solutions to the Dirac equation in the Coulomb field.

The sum (2) is divergent as it stands. In the course of our study we shall try to give a more proper definition and to separate the physically meaningful quantities from meaningless infinities.

We study the Dirac equation in the Coulomb field, and consider simultaneous eigenstates to K, J_z and the Hamiltonian.[13] K has all nonzero integers as eigenvalues, and J_z has the eigenvalues;

$$m=-|k|+\tfrac{1}{2};\ -|k|+\tfrac{3}{2};\ \cdots;\ |k|-\tfrac{1}{2}. \tag{3}$$

We are then led to radial equations which we write in the form

$$(\mathfrak{D}_x+z)[w(x)]=0, \tag{4}$$

where

$$(\mathfrak{D}_x)=\begin{bmatrix}\dfrac{\gamma}{x}+1; & -\dfrac{d}{dx}-\dfrac{k}{x}\\[2ex] \dfrac{d}{dx}-\dfrac{k}{x}; & \dfrac{\gamma}{x}-1\end{bmatrix},$$

$$[w(x)]=[w_1(x);w_2(x)],$$

$$\gamma=\alpha Z=e^2Z/4\pi\epsilon_0\hbar c,$$

$$x=r(m_0c/\hbar),$$

$$z=E/m_0c^2$$

(E is here the energy).

The complete solutions can be written in terms of the radial eigenfunctions, and spherical harmonics as

$$\begin{aligned}
\psi_1(km;\mathbf{r})&=-\frac{i}{x}\frac{k}{|k|}w_1(x)\left[\frac{k+m-\frac{1}{2}}{2k-1}\right]^{\frac{1}{2}}Y_{|k-\frac{1}{2}|-\frac{1}{2}}{}^{m-\frac{1}{2}}(\theta;\varphi),\\
\psi_2(km;\mathbf{r})&=-\frac{i}{x}w_1(x)\left[\frac{k-m-\frac{1}{2}}{2k-1}\right]^{\frac{1}{2}}Y_{|k-\frac{1}{2}|-\frac{1}{2}}{}^{m+\frac{1}{2}}(\theta;\varphi),\\
\psi_3(km;\mathbf{r})&=-\frac{k}{|k|}\frac{1}{x}w_2(x)\left[\frac{k-m+\frac{1}{2}}{2k+1}\right]^{\frac{1}{2}}Y_{|k+\frac{1}{2}|-\frac{1}{2}}{}^{m-\frac{1}{2}}(\theta;\varphi),\\
\psi_4(km;\mathbf{r})&=\frac{1}{x}w_2(x)\left[\frac{k+m+\frac{1}{2}}{2k+1}\right]^{\frac{1}{2}}Y_{|k+\frac{1}{2}|-\frac{1}{2}}{}^{m+\frac{1}{2}}(\theta;\varphi).
\end{aligned} \tag{5}$$

The labels k and z are suppressed in w_1 and w_2. Using addition theorems for spherical harmonics, we find

$$\begin{aligned}
\sum_{m=-j}^{j}\mathrm{trace}(\psi(km;\mathbf{r})\psi^*(km;\mathbf{r}))
&=\sum_{m=-j}^{j}\sum_{\nu=1}^{4}\psi_\nu^*(km;\mathbf{r})\psi_\nu(km;\mathbf{r})\\
&=\frac{2|k|}{4\pi}\sum_{\nu=1}^{2}w_\nu(x)w_\nu^*(x),\quad (j=|k|-\tfrac{1}{2})
\end{aligned} \tag{6}$$

[13] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), first edition.

so that

$$\rho(\mathbf{r})=\sum_{k=1}^{\infty}[\rho_k(\mathbf{r})+\rho_{-k}(\mathbf{r})], \tag{7}$$

with

$$\rho_k(\mathbf{r})=\frac{e}{2}\frac{2|k|}{4\pi}\frac{1}{r^2}\left(\frac{m_0c}{\hbar}\right)$$
$$\times[\sum_{(+)}\{w_1(x)w_1^*(x)+w_2(x)w_2^*(x)\}$$
$$-\sum_{(-)}\{w_1(x)w_1^*(x)+w_2(x)w_2^*(x)\}], \tag{8}$$

where $[w(x)]$ are solutions to the radial equation (4) for a given k, and given E, and with appropriate normalization. (+) and (−) indicate respectively summations over positive and negative energy eigenstates to the radial equation. After the summation over m the spherical symmetry of the induced charge density is apparent.

We shall now focus our attention on the radial equation (4) and for convenience we shall suppress the index k from the solutions in the next paragraph, as we have done so far.

III. EXPRESSION OF THE SUMMATION OVER THE RADIAL EIGENSTATES AS A CONTOUR INTEGRAL OF THE RADIAL GREEN'S FUNCTION

Let $0<\gamma<1$. The boundary conditions,

$$\begin{aligned}&(a)\quad [w] \text{ finite at } x=0,\\ &(b)\quad [w] \text{ bounded at infinity,}\end{aligned} \tag{9}$$

define the eigenvalues and eigenfunctions of the radial equation (4). Let z be any complex number, *not* an eigenvalue to (4). We may then construct a Green's function, K, to the radial equation in the form of a bilinear sum;

$$K_{\mu\nu}(x_1,x_2;z)=\sum_{(\epsilon)}\frac{w_\mu(x_1;\epsilon)w_\nu^*(x_2;\epsilon)}{z-\epsilon}, \tag{10}$$

where the sum is over all eigenvalues ϵ. This Green's function has the property (provided $[w]$ is properly normalized);

$$(\mathfrak{D}_{x_1}+z)(K(x_1x_2;z))=\delta(x_1-x_2). \tag{11}$$

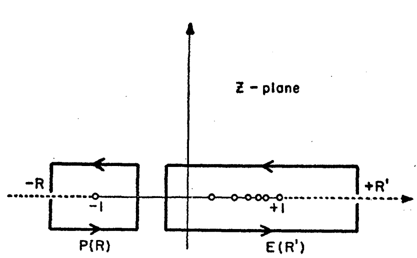

FIG. 2. Contours used in connection with the contour integral representation of the sum over the energy states.

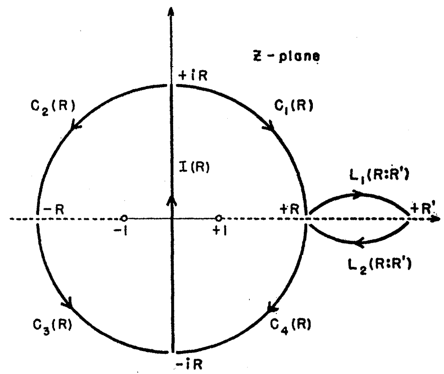

FIG. 3. Special contours used in the discussion of the contour integral representation of the sum over the energy states.

The Green's function is an analytic function of z, except possibly at the location of the eigenvalues. The set of eigenvalues consists of a point set on the real z-axis, between 0 and 1, and with $z=1$ as a point of accumulation, and of the half-lines

$$z\geqq 1 \quad\text{and}\quad z\leqq 1. \tag{12}$$

Let us introduce cuts in the z-plane along the half-lines defined by (12). It will be found that the Green's function has simple poles at the discrete eigenvalues and branch points at $z=1$ and $z=-1$, and that it is otherwise a single-valued analytic function in the cut plane.

Let us consider the contours $E(R)$ and $P(R)$ in the cut z-plane. (See Fig. 2.) $E(R)$ is a simple curve, starting at the point $+R+0i$, ending at the point $+R-0i$, which encircles all the discrete eigenvalues precisely once. $P(R)$ is a simple curve, starting at the point $-R-0i$, ending at the point $-R+0i$, which encircles none of the discrete eigenvalues.

After the introductory remarks on the Green's function as defined by (10), it is apparent that the sum over states in (8) that we are interested in can be represented by:

$$\sum_{(+)R'}\{w_1(x)w_1^*(x)+w_2(x)w_2^*(x)\}-\sum_{(-)R}\{w_1(x)w_1^*(x)+w_2(x)w_2^*(x)\}$$
$$=\frac{1}{2\pi i}\int_{E(R')}dz\,\text{trace}(K(x,x;z))-\frac{1}{2\pi i}\int_{P(R)}dz\,\text{trace}(K(x,x;z)) \tag{13}$$

in the limit R', $R\rightarrow\infty$. This limit will exist only after some regularization process has been performed.

In view of the analyticity of the Green's function, we may deform the contours $E(R')$ and $P(R)$. In particular, we consider the contours $C_1(R)$, $C_2(R)$, $C_3(R)$, $C_4(R)$, $I(R)$, $L_1(R;R')$ and $L_2(R;R')$ shown in Fig. 3.

Thus

$$\int_{E(R')} dz\cdot K = -\int_{I(R)} dz\cdot K - \int_{C_1(R)} dz\cdot K - \int_{C_4(R)} dz\cdot K - \int_{L_1(R;\,R')} dz\cdot K - \int_{L_2(R;\,R')} dz\cdot K, \quad (14)$$

$$\int_{P(R)} dz\cdot K = \int_{I(R)} dz\cdot K + \int_{C_2(R)} dz\cdot K + \int_{C_3(R)} dz\cdot K.$$

This is a convenient set of contours, since it will turn out that the physically meaningful contribution will appear as an integral along the contour $I(R)$, while the ambiguities are connected with the other contours. Upon regularization of the expression for the charge density, and passing to the limit $R\to\infty$ and $R'\to\infty$, only the integral along the imaginary z-axis will contribute.

The utility of (13) is a consequence of the fact that there exists an alternative expression for the Green's function in terms of the regular and irregular solutions to (4) for z not an eigenvalue, so that the summation over states in (10) can be avoided. Thus: For z not an eigenvalue, we construct two linearly independent solutions $[w^{(1)}(x;z)]$ and $[w^{(2)}(x;z)]$ to (4), by requiring that $[w^{(1)}]$ be finite at the origin and $[w^{(2)}]$ bounded at infinity. Let

$$K(z) = w_2^{(1)}(x;z)w_1^{(2)}(x;z) - w_1^{(1)}(x;z)w_2^{(2)}(x;z); \quad (15)$$

then

$$(\partial/\partial x)K(z)\equiv 0, \quad (16)$$

and $K(z)\neq 0$, if z is not an eigenvalue. Let

$$\theta(x)=\begin{cases}+1 & \text{if } x>0\\ +\tfrac{1}{2} & \text{if } x=0\\ 0 & \text{if } x<0.\end{cases}$$

Let

$$(K(x_1,x_2;z)) = \frac{\theta(x_2-x_1)}{K(z)}\cdot\begin{pmatrix} w_1^{(1)}(x_1;z)w_1^{(2)}(x_2;z); & w_1^{(1)}(x_1;z)w_2^{(2)}(x_2;z)\\ w_2^{(1)}(x_1;z)w_1^{(2)}(x_2;z); & w_2^{(1)}(x_1;z)w_2^{(2)}(x_2;z)\end{pmatrix} + \frac{\theta(x_1-x_2)}{K(z)}\cdot\begin{pmatrix} w_1^{(2)}(x_1;z)w_1^{(1)}(x_2;z); & w_1^{(2)}(x_1;z)w_2^{(1)}(x_2;z)\\ w_2^{(2)}(x_1;z)w_1^{(1)}(x_2;z); & w_2^{(2)}(x_1;z)w_2^{(1)}(x_2;z)\end{pmatrix}. \quad (17)$$

The diagonal elements of K are continuous functions of x_1 and x_2; the nondiagonal elements have a finite step-discontinuity at $x_1=x_2$. It is apparent that

$$(\mathfrak{D}_{x_1}+z)\cdot(K(x_1,x_2;z)) = \delta(x_1-x_2), \quad (18)$$

so that we have indeed found another expression for the Green's function. This construction is very similar in principle to the construction of a Green's function for an ordinary second order linear differential equation.[14]

The solutions $[w^{(1)}]$ and $[w^{(2)}]$ can be expressed in terms of the functions defined below. We define single valued functions in the cut plane:

$$\begin{aligned}&(z+1)^{\frac{1}{2}} \text{ by } (z+1)^{\frac{1}{2}}{}_{z=0}=1;\\ &(z-1)^{\frac{1}{2}} \text{ by } (z-1)^{\frac{1}{2}}{}_{z=0}=i;\\ &(z^2-1)^{\frac{1}{2}}=(z+1)^{\frac{1}{2}}(z-1)^{\frac{1}{2}}.\end{aligned} \quad (19)$$

Thus

$$\operatorname{Im}\{(z^2-1)^{\frac{1}{2}}\}\geqq 0.$$

Let

$$\begin{aligned}s&=(k^2-\gamma^2)^{\frac{1}{2}},\\ a&=s-i\gamma z/(z^2-1)^{\frac{1}{2}},\\ b&=2s+1,\\ \lambda&=k-i\gamma/(z^2-1)^{\frac{1}{2}}.\end{aligned} \quad (20)$$

We will use the following confluent hypergeometric functions:

$$F(p;q;t)\equiv\sum_{n=0}^{\infty}\frac{\Gamma(p+n)\Gamma(q)t^n}{\Gamma(q+n)\Gamma(p)n!}, \quad (21)$$

$$G(p;q;t)\equiv t^{1-q}\frac{\Gamma(p+1-q)\Gamma(q)}{\Gamma(2-q)\Gamma(p)}F(p+1-q;2-q;t), \quad (22)$$

$$H(p;q;t)\equiv F(p;q;t)-G(p;q;t), \quad (23)$$

where

$$\arg(t^{1-q})=(1-q)\arg t.$$

For some of the properties of these functions, see Appendix I. We note here the integral representations

$$F(p;q;t) = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)}\int_0^1 dx e^{xt}x^{p-1}(1-x)^{q-p-1}, \quad (24)$$

valid when

$$\operatorname{Re}\{p\}>0,\quad \operatorname{Re}\{q-p\}>0;$$

and

$$H(p;q;t) = \frac{\Gamma(p+1-q)}{\Gamma(p)\Gamma(1-q)}e^t\int_1^\infty dx e^{-xt}x^{q-p-1}(x-1)^{p-1}, \quad (25)$$

where $-\frac{1}{2}\pi<\arg t<\frac{1}{2}\pi$, valid when $\operatorname{Re}\{p\}>0$, and q not a positive integer. For further discussion, see Appendix I.

[14] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik* (Verlag Julius Springer, Berlin, 1931), Part I.

Having made these definitions, we are able to determine $[w^{(1)}]$ and $[w^{(2)}]$ in terms of the confluent hypergeometric functions, by examining the behavior of F and G at the origin and at infinity, The variables x_1 and x_2 are real and non-negative, and by (19) we may write

$$-\tfrac{1}{2}\pi \leq \arg\{-i\cdot x(z^2-1)^{\frac{1}{2}}\} \leq \tfrac{1}{2}\pi, \tag{26}$$

and thus get

$$\begin{aligned}
w_1^{(1)}(x;z) &= i(z-1)^{\frac{1}{2}}[2x(z^2-1)^{\frac{1}{2}}]^s \exp[ix(z^2-1)^{\frac{1}{2}}] \\
&\quad\times\{\lambda F(a;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\qquad + aF(a+1;b;-2ix(z^2-1)^{\frac{1}{2}})\}, \\
w_2^{(1)}(x;z) &= (z+1)^{\frac{1}{2}}[2x(z^2-1)^{\frac{1}{2}}]^s \exp[ix(z^2-1)^{\frac{1}{2}}] \\
&\quad\times\{\lambda F(a;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\qquad - aF(a+1;b;-2ix(z^2-1)^{\frac{1}{2}})\}, \\
w_1^{(2)}(x;z) &= i(z-1)^{\frac{1}{2}}[2x(z^2-1)^{\frac{1}{2}}]^s \exp[ix(z^2-1)^{\frac{1}{2}}] \\
&\quad\times\{\lambda\cdot H(a;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\qquad + aH(a+1;b;-2ix(z^2-1)^{\frac{1}{2}})\}, \\
w_2^{(2)}(x;z) &= (z+1)^{\frac{1}{2}}[2x(z^2-1)^{\frac{1}{2}}]^s \exp[ix(z^2-1)^{\frac{1}{2}}] \\
&\quad\times\{\lambda H(a;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\qquad - aH(a+1;b;-2ix(z^2-1)^{\frac{1}{2}})\}.
\end{aligned} \tag{27}$$

Thus,

$$K(z) = 4s\lambda(z^2-1)^{\frac{1}{2}} \frac{\Gamma(a+1-b)\Gamma(b)}{\Gamma(2-b)\Gamma(a)} \exp\left(\frac{i\pi b}{2}\right). \tag{28}$$

The last result is best obtained by letting x become very small in the solutions (27), and using (16).

Using (27) and (28) and the definition (17), we may construct the Green's functions explicitly in terms of the confluent hypergeometric functions. By examining the Green's function we may establish that (see Appendix I for some details):

(*a*) The Green's function has simple poles at the discrete eigenvalues. These, of course, lie on the real z-axis between $z=0$ and $z=1$.

(*b*) With the exception of the poles, the Green's function is a single-valued analytic function of z in the cut z-plane.

(*c*) The Green's function has branch points at $z=1$ and at $z=-1$.

(*d*) With x_1 (or x_2) fixed and finite, and z in the cut plane, not at a pole, the Green's function considered as a function of x_2 (or x_1) is finite at the origin, and bounded at infinity.

We discuss in Appendix I how various propagation functions may be defined in terms of contour integrals of the Green's function, and how these can be used to solve the time-dependent radial equation.

IV. THE LAPLACE TRANSFORM OF THE TRACE OF THE GREEN'S FUNCTION WHEN $x_1=x_2=x$

Using the results of the preceding section, we find:

$$\begin{aligned}
D_k(x;z) &= \mathrm{trace}(K_k(x,x;z)+K_{-k}(x,x;2)) \\
&= -2i\frac{\Gamma(a)\Gamma(1-b)}{\Gamma(b)\Gamma(a+1-b)} \frac{[-2ix(z^2-1)^{\frac{1}{2}}]^{2s}}{(z^2-1)^{\frac{1}{2}}} \\
&\quad\times\exp\{2ix(z^2-1)^{\frac{1}{2}}\} \\
&\quad\times\left\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\left[F(a;b;-2ix(z^2-1)^{\frac{1}{2}})\right.\right. \\
&\quad\times H(a;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\quad+\frac{a}{1+a-b}F(a+1;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\quad\left.\times H(a+1;b;-2ix(z^2-1)^{\frac{1}{2}})\right] \\
&\quad+az[F(a;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\quad\times H(a+1;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\quad+F(a+1;b;-2ix(z^2-1)^{\frac{1}{2}}) \\
&\qquad\left.\times H(a;b;-2ix(z^2-1)^{\frac{1}{2}})]\right\}.
\end{aligned} \tag{29}$$

Let

$$E_k(p;-iz) = \int_0^\infty e^{-px} D_k(x;z)dx. \tag{30}$$

This Laplace transform exists everywhere in the cut plane when z is not at a pole of K.

We denote:

$$u = ip/2(z^2-1)^{\frac{1}{2}}, \tag{31}$$

$$Q = [1-t]/[1+ut][1+t(u-1)], \tag{32}$$

$$g = \frac{\gamma z}{(z^2-1)^{\frac{1}{2}}} \ln\left\{\frac{[1-t][1+ut]}{[1+t(u-1)]}\right\}. \tag{33}$$

Using the integral representations (24) and (25), we get

$$\begin{aligned}
D_k(x;z) &= \frac{-2i}{(z^2-1)^{\frac{1}{2}}}[-2ix(z^2-1)^{\frac{1}{2}}]^{2s}\int_0^1 d\xi\int_1^\infty d\eta\,\exp\{-2ix(z^2-1)^{\frac{1}{2}}(\xi-\eta)\}\left\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\left[\frac{1}{\Gamma(a)\Gamma(b-a)}\right.\right. \\
&\quad\left.\times\xi^{a-1}(\eta-1)^{a-1}\eta^{b-a-1}(1-\xi)^{b-a-1}+\frac{1}{\Gamma(a+1)\Gamma(b-a-1)}\xi^a(\eta-1)^a\eta^{b-a-2}(1-\xi)^{b-a-2}\right] \\
&\quad\left.+z\left[\frac{(a+1-b)}{\Gamma(a)\Gamma(b-a)}\xi^{a-1}(1-\xi)^{b-a-1}\eta^{b-a-2}(\eta-1)^a+\frac{a}{\Gamma(a+1)\Gamma(b-a-1)}\xi^a(1-\xi)^{b-a-2}\eta^{b-a-1}(\eta-1)^{a-1}\right]\right\},
\end{aligned} \tag{34}$$

whence

$$E_k(p;-iz)=\frac{1}{z^2-1}\int_0^1 d\xi\int_1^\infty d\eta\Bigg\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\Bigg[\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\xi^{a-1}(1-\xi)^{b-a-1}\eta^{b-a-1}(\eta-1)^{a-1}+\frac{\Gamma(b)}{\Gamma(a+1)\Gamma(b-a-1)}$$

$$\times\xi^a(1-\xi)^{b-a-2}\eta^{b-a-2}(\eta-1)^a\Bigg]+z\Bigg[\frac{a\Gamma(b)}{\Gamma(a+1)\Gamma(b-a-1)}\xi^a(1-\xi)^{b-a-2}\eta^{b-a-1}(\eta-1)^{a-1}$$

$$+\frac{(a+1-b)\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\xi^{a-1}(1-\xi)^{b-a-1}\eta^{b-a-2}(\eta-1)^a\Bigg]\Bigg\}[u+\eta-\xi]^{-b}$$

$$=\frac{1}{z^2-1}\int_1^\infty d\eta\Bigg\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\Bigg[\frac{\eta^{b-a-1}(\eta-1)^{a-1}}{[u+\eta]^{b-a}[u+\eta-1]^a}+\frac{\eta^{b-a-2}(\eta-1)^a}{[u+\eta]^{b-a-1}[u+\eta-1]^{a+1}}\Bigg]$$

$$+z\Bigg[\frac{a\eta^{b-a-1}(\eta-1)^{a-1}}{[u+\eta]^{b-a-1}[u+\eta-1]^{a+1}}+\frac{(a+1-b)\eta^{b-a-2}(\eta-1)^a}{[u+\eta]^{b-a}[u+\eta-1]^a}\Bigg]\Bigg\}$$

$$=\frac{1}{z^2-1}\int_0^1 dt\Bigg\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\Bigg[\frac{[1-t]^{a-1}}{[1+ut]^{b-a}[1+t(u-1)]^a}+\frac{[1-t]^a}{[1+ut]^{b-a-1}[1+t(u-1)]^{a+1}}\Bigg]$$

$$+z\Bigg[\frac{a[1-t]^{a-1}}{[1+ut]^{b-a-1}[1+t(u-1)]^{a+1}}+\frac{(a+1-b)[1-t]^a}{[1+ut]^{b-a}[1+t(u-1)]^a}\Bigg]\Bigg\},$$

with $t=1/\eta$. Thus

$$E_k(p;-iz)=\frac{1}{z^2-1}\int_0^1 dtQ^s\exp\{-ig\}\Bigg\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\Bigg[\frac{1}{[1-t][1+ut]}+\frac{1}{[1+t(u-1)]}\Bigg]$$

$$+z\Bigg[\frac{a}{[1-t][1+t(u-1)]}+\frac{(a+1-b)}{[1+ut]}\Bigg]\Bigg\}.\qquad(35)$$

V. DIVERGENCE DIFFICULTIES AND REGULARIZATION OF THE LAPLACE TRANSFORM

According to our earlier considerations, we should now perform the contour integrations of (35) in accordance with (13) and pass to the limit, and finally sum the result over k as indicated by (8). We should then get an expression for the Laplace transform of the induced charge density times r^2. By inspection of (35), however, it is seen that the result would diverge. In addition, delicate considerations would arise as to what order should be followed in the integrations and summations. In our opinion, the theory does not give any answer to such questions.

To deal with this situation, we will proceed as follows: We first sum E_k over k in accordance with (8). We then carry out the integration over t (which will give an infinite term in first order in γ). We finally carry out the contour integrations over z and at the same time we will remove ambiguities by a regularization process.

For convenience, we shall first remove what will be shown to be the Uehling term from our expression. We write

$$E_k(p;-iz)=E_k'(p;-iz)+E_k''(p;-iz),\qquad(36)$$

$$E_k'(p;-iz)=\frac{1}{z^2-1}\int_0^1 dtQ^k\Bigg\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}$$

$$\times\Bigg[\frac{1}{[1-t][1+ut]}+\frac{1}{[1+t(u-1)]}\Bigg]+\frac{\gamma g}{(z^2-1)^{\frac{1}{2}}}$$

$$\times\Bigg[\frac{1}{[1-t][1+ut]}+\frac{1}{[1+t(u-1)]}\Bigg]$$

$$+zk\Bigg[1-ig+\tfrac{1}{2}g^2-\frac{\gamma^2}{2k}\ln Q\Bigg]\Bigg[\frac{1}{[1-t][1+t(u-1)]}$$

$$-\frac{1}{[1+ut]}\Bigg]-\frac{i\gamma z^2}{(z^2-1)^{\frac{1}{2}}}[1-ig]\Bigg[\frac{1}{[1-t][1+t(u-1)]}$$

$$+\frac{1}{[1+ut]}\Bigg]\Bigg\}.\qquad(37)$$

The expression (37) arises from the first three terms in a power series expansion of (35) in γ. Let

$$W''(p;-iz)=\sum_{k=1}^\infty kE_k''(p;-iz).\qquad(38)$$

Let $0<\epsilon<\frac{1}{2}$ and $0<\delta<\frac{1}{2}$.

$$W'(p;-iz;\epsilon;\delta)=\frac{1}{z^2-1}\int_{\epsilon}^{1-\delta}dt\sum_{k=1}^{\infty}kQ^k\left\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\right.$$

$$\times[1-ig]\left[\frac{1}{[1-t][1+ut]}+\frac{1}{[1+t(u-1)]}\right]$$

$$+zk\left[1-ig-\tfrac{1}{2}g^2-\frac{\gamma^2}{2k}\ln Q\right]\left[\frac{1}{[1-t][1+t(u-1)]}\right.$$

$$\left.-\frac{1}{[1+ut]}\right]-\frac{i\gamma z^2}{(z^2-1)^{\frac{3}{2}}}[1-ig]$$

$$\left.\times\left[\frac{1}{[1+ut]}+\frac{1}{[1-t][1+t(u-1)]}\right]\right\}. \quad (39)$$

It may be shown (see Appendix II) that for u finite, the order of summation over k and integration over t may be reversed in (38). The expression (38) is thus consistent with our program, and it is also gratifying to see that the ambiguity concerning the order of integration and summation exists only for E_k'.

The expression (39) is a polynomial of second degree in γ. Let us write this explicitly as

$$W'(p;-iz;\epsilon;\delta)=W^{(0)}(p;-iz;\epsilon;\delta)$$
$$+\gamma W^{(1)}(p;-iz;\epsilon;\delta)+\gamma^2W^{(2)}(p;-iz;\epsilon;\delta). \quad (40)$$

Consider the coefficients of the even powers. When integrated over the contour $I(R)$, the integrals will vanish. The only possible contribution actually comes from the contours $L_1(R:R')$ and $L_2(R:R')$, and so is dependent on the way we pass to the limit in the contour integration. We will set these terms equal to zero. It is also clear that the induced charge must change sign when the inducing charge does, and therefore even powers of γ cannot occur in our result. Evaluating $W^{(1)}$, we get

$$W^{(1)}(p;-iz;\epsilon;\delta)$$

$$=\frac{i}{(z^2-1)^{\frac{3}{2}}}\left\{\frac{1}{2u^2}\left[\frac{1}{\epsilon}-\frac{3}{2}\right]+\frac{1}{4u}-\frac{1}{2(1+u)}\right.$$

$$\left.-z^2\left[\frac{1}{2u^2}\left(\frac{1}{\epsilon}-\frac{1}{2}\right)-\frac{1}{4u}\right]+\frac{z^2}{u^3}\ln(1+u)\right\}$$

$$+O(\epsilon)+O(\delta), \quad (41)$$

$$W^{(1)}(p;-iz;\epsilon;\delta)$$

$$=\left[\frac{1}{1+u}-1\right]\left\{\frac{iz^4}{3(z^2-1)^{5/2}}-\frac{i}{2(z^2-1)^{\frac{3}{2}}}\right\}$$

$$+A(p;-iz;\epsilon;\delta), \quad (42)$$

$$A(p;-iz;\epsilon;\delta)$$

$$=\frac{\partial}{\partial z}\left[\frac{iz^3}{3u^3(z^2-1)^{\frac{3}{2}}}\ln(1+u)\right]_{p=\text{const}}+\frac{iz^4}{3(z^2-1)^{5/2}}$$

$$\times\left[1-\frac{1}{u}+\frac{1}{u^2}\right]+\frac{i}{(z^2-1)^{\frac{3}{2}}}\left[-\frac{1}{2}+\frac{1}{2u^2}\left(\frac{1}{\epsilon}-\frac{3}{2}\right)\right.$$

$$\left.+\frac{1}{4u}-z^2\left(\frac{1}{2\epsilon u^2}-\frac{1}{4u^2}-\frac{1}{4u}\right)\right]+O(\epsilon)+O(\delta). \quad (43)$$

The first sum in (42) is independent of ϵ and δ. Integrated around the contours and passing to the limit R, $R'\to\infty$, the only contribution comes from the integral along the imaginary z-axis. Let $z=iy$ in this integration. Finding the inverse Laplace transform, we thus get for the induced charge density and the potential, to first order in γ,

$$\gamma V_p^{(1)}(\mathbf{r})=\frac{e}{4\pi\epsilon_0 r}\left(\frac{\gamma}{3\pi}\right)\int_0^{\infty}dy\frac{y^2[3+2y^2]}{[y^2+1]^{5/2}}$$
$$\times\exp\left\{-\frac{2m_0cr}{\hbar}(y^2+1)^{\frac{1}{2}}\right\}, \quad (44)$$

$$\gamma\rho^{(1)}(\mathbf{r})=-\frac{e}{4\pi r}\left(\frac{\gamma}{3\pi}\right)\left(\frac{2m_0c}{\hbar}\right)^2\int_0^{\infty}dy\frac{y^2[3+2y^2]}{[y^2+1]^{\frac{3}{2}}}$$
$$\times\exp\left\{-\frac{2m_0cr}{\hbar}(y^2+1)^{\frac{1}{2}}\right\}. \quad (45)$$

In the expression for the charge density the infinite point charge at the origin, required to effect charge renormalization, has been omitted. The potential (44) is precisely the Uehling potential.

The contribution from the term A in (43) we set equal to zero. Its actual value depends on the way we perform the limiting process for the contour. We see that it is of the form

$$c_0+c_2(1/p)^2,$$

or, finding the inverse Laplace transform, this corresponds to a charge density

$$c_0'(1/r)^2\delta(r)+c_2'(1/r).$$

Such terms in the charge density vanish upon regularization, which is easily seen from a study of the dimensions of the coefficients.[15]

We now study $W''(p;-iz)$ and write

$$W'''(p;-iz)=W''(p;-iz)-\gamma^3W^{(3)}(p;-iz), \quad (46)$$

[15] W. Pauli and F. Villars, Rev. Modern Phys. 21, 434 (1949). c_0' has the dimension of a charge and is therefore independent of the pair field mass, while c_2', having the dimension of charge divided by an area must be proportional to the square of the mass. In the notation of Pauli and Villars, the conditions $\int\rho(x)dx=0$, $\int\rho(x)x^2dx=0$, remove these terms.

where

$$\gamma^3 W^{(3)}(p;-iz)$$

$$=\frac{1}{z^2-1}\int_0^1 dt\sum_{k=1}^{\infty} kQ^k\Bigg\{\frac{i\gamma}{(z^2-1)^{\frac{1}{2}}}\bigg[\frac{1}{[1+ut][1-t]}+\frac{1}{[1+t(u-1)]}\bigg]\bigg[-\frac{1}{2}g^2-\frac{1}{2}\frac{\gamma^2}{k}\ln Q\bigg]$$

$$+kz\bigg[\frac{1}{[1-t][1+t(u-1)]}-\frac{1}{[1+ut]}\bigg]$$

$$\times\bigg[\frac{i}{6}g^3+\frac{i}{2}\frac{\gamma^2}{k}g\ln Q+\frac{i}{2}\frac{\gamma^2}{k^2}g\bigg]-\frac{i\gamma z^2}{(z^2-1)^{\frac{1}{2}}}$$

$$\times\bigg[\frac{1}{[1-t][1+t(u-1)]}+\frac{1}{[1+ut]}\bigg]$$

$$\times\bigg[-\frac{\gamma^2}{2k}\ln Q-\frac{1}{2}g^2\bigg]\Bigg\}. \quad (47)$$

This expression arises from the third power of γ in a power series expansion in γ of E'' in (36).

Let us define the function $\psi(2;x)$, when $|x|<1$, by the power series:

$$\psi(2;x)=\sum_{n=1}^{\infty}\frac{x^n}{n^2}, \quad (48)$$

and let $\zeta(x)$ denote the Riemann zeta function. We carry out the integration over t in (47) and get

$$W^{(3)}(p;-iz)=\bar{W}^{(3)}(p;-iz)-\tfrac{1}{2}i(z^2-1)^{-\frac{1}{2}}$$

$$\times\bigg[\frac{4z^4-15z^2+6}{3(z^2-1)^2}+\frac{\pi^2}{6(z^2-1)}-\frac{2\ln u}{(z^2-1)^2}\bigg], \quad (49)$$

where

$$2i(z^2-1)^{\frac{1}{2}}\bar{W}^{(3)}(p;-iz)$$

$$=\frac{z^4}{(z^2-1)^2u^3}\bigg\{\frac{4}{3}\ln^3(1+u)+2\psi(2,u^2)\ln(1-u)$$

$$+\frac{2}{3}u^3-\frac{\pi^2}{3}[\ln(1-u^2)+u^2]+2\int_0^u\frac{dx}{x}\ln^2(1-x^2)$$

$$-2\bigg[\ln(1-u^2)\ln\frac{1+u}{1-u}+2u^3\bigg]\ln u\bigg\}$$

$$+\frac{z^2}{(z^2-1)u^2}\bigg\{\psi(2,u)\ln\frac{1+u}{1-u}-2u^2$$

$$-\frac{\pi^2}{6}\bigg[\ln\frac{1+u}{1-u}-2u\bigg]+u\ln^2(1+u)+\tfrac{1}{2}u\psi(2,u^2)$$

$$+\frac{1}{2}\int_0^u\frac{dx}{x}\ln(1-x^2)\ln\frac{1+x}{1-x}$$

$$-\bigg[\frac{1}{2}\ln^2\frac{1+u}{1-u}+u\ln\frac{1+u}{1-u}-4u^2\bigg]\ln u\bigg\}$$

$$-\frac{z^2}{(z^2-1)^2u^2(1-u^2)}\bigg\{2(1-u)\ln^2(1+u)$$

$$-(1+u)\psi(2,u^2)-u^2(1-u^2)+\frac{\pi^2}{3}u^3$$

$$-2\bigg[\ln(1-u^2)-u\ln\frac{1+u}{1-u}+3u^2(1-u^2)\bigg]\ln u\bigg\}$$

$$-\frac{1}{(z^2-1)u(1-u^2)}\bigg\{(1-u)\psi(2,u)$$

$$-(1+3u)\psi(2,-u)-2u(1-u^2)-\frac{\pi^2}{6}u^2(1+u)$$

$$-\bigg[\ln\frac{1+u}{1-u}+u\ln(1-u^2)-2u(1-u^2)\bigg]\ln u\bigg\}. \quad (50)$$

The expression $\bar{W}^{(3)}$ is a continuous function of u for positive u. It further satisfies the relation

$$|\bar{W}^{(3)}(p;-iz)|<c_1|u/(z^2-1)^{\frac{1}{2}}|$$

for some c_1 if $0\leqq u\leqq 1$. We thus see that $\bar{W}^{(3)}$ integrated over the contours C_1, C_2, C_3, C_4, L_1, L_2 vanishes in the limit R, $R'\rightarrow\infty$. We shall call $\bar{W}^{(3)}$ the "regulated $W^{(3)}$," and in general denote regulated functions by a bar. On the other hand, the bracketed expression in (49) gives a contribution to the contour integral depending on the manner in which we go to the limit. The result would be of the form

$$k_1+k_2\ln p.$$

Both these terms can be removed by regularization.[16]

The remaining terms, $W'''(p;-iz)$, can be handled in exactly the same way. In this case only the integral along the imaginary axis of this function contributes to the contour integral for the charge density, and the contribution is finite for all p. To renormalize the charge it is necessary to remove a term constant in p; the remainder, denoted by $\bar{W}'''(p;-iz)$, then vanishes at $p=0$. We have further proved that the integral of $\bar{W}'''$ is an analytic function of γ in the region $|\gamma|<1$ (see Appendix II).

[16] This corresponds to a charge density of the form: $k_1'\delta(r)/(r^2)+k_2'(1/r^3)$. This has, of course, no very precise meaning, since in order that the Laplace transform "exist" it is necessary that k_1' be infinite.

VI. FURTHER STUDY OF THE RENORMALIZED LAPLACE TRANSFORM

By the results of the preceding section, and of Appendix II, a power series expansion in γ can be carried out. The radius of convergence is $|\gamma|=1$, and only odd powers of γ will occur. Let us denote

$$q(p)=\int_0^\infty dx e^{-px}\rho(\mathbf{r})x^2, \tag{51}$$

$$q(p)=\sum_{n=0}^\infty \gamma^{2n+1}q^{(2n+1)}(p), \tag{52}$$

$$\begin{aligned} q'''(p)&=q(p)-\gamma q^{(1)}(p)-\gamma^3 q^{(3)}(p),\\ q'(p)&=q(p)-\gamma q^{(1)}(p). \end{aligned} \tag{53}$$

We thus have

$$\gamma^3 q^{(3)}(p)=-\gamma^3\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\int_{-\infty}^\infty dy\bar{W}^{(3)}(p;y), \tag{54}$$

$$q'''(p)=-\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\int_{-\infty}^\infty dy\bar{W}'''(p;y). \tag{55}$$

The induced charge gives rise to an electrostatic potential $V_P(r)$, according to

$$V_P(r)=-\frac{1}{r\epsilon_0}\int_r^\infty \rho(u)u^2du+\frac{1}{\epsilon_0}\int_r^\infty \rho(u)udu. \tag{56}$$

We also define

$$U(p)=\int_0^\infty V_p(r)e^{-px}xdx,\ r=(\hbar/m_0c)x, \tag{57}$$

and get (using Poisson's equation)

$$U(p)=\left(\frac{\hbar}{m_0c}\right)^2\frac{1}{\epsilon_0}\frac{1}{p^2}\int_0^p q(p')dp'. \tag{58}$$

From this we may find the Laplace transform of positive integral powers of x times the polarization potential by a differentiation with respect to p.

We expand (50) in powers of p, and carry out the integration over y. Using (54), we thus have

$$\begin{aligned}\gamma^3 q^{(3)}(p)=\gamma^3\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\Bigg\{&\left(\frac{p}{2}\right)\left(\frac{\pi}{4}\right)\left[-\frac{5\pi^2}{72}+\frac{13}{24}\right]\\ &+\left(\frac{p}{2}\right)^2\left[-\frac{2}{3}\frac{\pi^2}{6}+\frac{19}{15}\right]+\left(\frac{p}{2}\right)^3\left(\frac{\pi}{4}\right)\\ &\times\left[-\frac{31\pi^2}{360}+\frac{1}{2}\right]+\left(\frac{p}{2}\right)^4\left[-\frac{16}{135}\ln p\right.\\ &\left.-\frac{8\pi^2}{6\cdot 15}+\frac{3347}{2835}\right]+O(p^5)\Bigg\}\end{aligned} \tag{59}$$

converging when $0\leqq p<2$, or

$$\begin{aligned}\gamma^3 q^{(3)}(p)=\frac{e}{4\pi}\frac{\gamma^3}{\pi}\left(\frac{m_0c}{\hbar}\right)^3\{&-(0.112879)(p/2)\\ &+(0.170044)(p/2)^2-(0.274793)(p/2)^3\\ &-(0.118519)(p/2)^4\ln p\\ &+(0.303301)(p/2)^4+O(p^5)\}.\end{aligned} \tag{59'}$$

The term proportional to $p^4\ln p$ is not unexpected. It gives rise to the leading term in the third-order charge and potential at large distances:

$$\begin{aligned} V_P^{(3)}(r)&\sim-\frac{e}{4\pi\epsilon_0 r}\left(\frac{2}{225\pi}\right)\left(\frac{\hbar}{m_0cr}\right)^4,\\ \rho^{(3)}(r)&\sim\frac{e}{4\pi r^3}\left(\frac{40}{225\pi}\right)\left(\frac{\hbar}{m_0cr}\right)^4 \end{aligned} \tag{60}$$

(as $r\to\infty$). The same term can be derived from the Euler-Heisenberg Lagrangian. (See Appendix III.[17])

For large p, we get

$$\lim_{p\to\infty}q^{(3)}(p)=-\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\left[\frac{\pi^2}{6}-\frac{7}{9}-\frac{2}{3}\zeta(3)\right]. \tag{61}$$

We now return to $\bar{W}'''(p;y)$. After an integration by parts with respect to t, we may write it in the form

$$\begin{aligned}\bar{W}'''(p;y)&\\ =-\frac{\gamma}{[1+y^2]^{\frac{3}{2}}}&\int_0^1 dt\frac{[1-t][1+ut]+[1+t(u-1)]}{[1-t][1+ut][1+t(u-1)]}\\ &\times\sum_{k=1}^\infty\left\{\left[kQ^s\cos g-Q^k\left(k-\frac{k}{2}g^2-\tfrac{1}{2}\gamma^2\ln Q\right)\right]\right\}\\ &-\frac{2\gamma y^2}{[1+y^2]^{\frac{3}{2}}}\int_0^1\frac{dt}{[1+ut]}\sum_{k=1}^\infty\Big\{[kQ^s\cos g\\ &-Q^k(k-\tfrac{1}{2}kg^2-\tfrac{1}{2}\gamma^2\ln Q)]-\frac{(1+y^2)^{\frac{1}{2}}}{\gamma y}[ksQ^s\sin g\\ &-(k^2g-\tfrac{1}{6}k^2g^3-\tfrac{1}{2}k\gamma^2 g\ln Q-\tfrac{1}{2}\gamma^2 g)Q^k]\Big\}+\lambda'''(y).\end{aligned} \tag{62}$$

$\lambda'''(y)$ is determined so that, identically in y,

$$\lim_{u\to 0}\bar{W}'''(p;y)=0. \tag{63}$$

---

[17] The discussion in Appendix III implies $\rho'''(r)$ falls off faster than $(1/r^7)$ so that (60) indeed gives the leading term in the charge density at large distances. Since the Uehling term will be seen to dominate at small distances, one notes that the induced charge densities at large and small distances are always of opposite sign.

We find (see Appendix II)

$$\lambda'''(y)=\frac{\gamma}{[1+y^2]^{\frac{3}{2}}}\sum_{k=1}^{\infty}\left\{\frac{ks}{s^2+\gamma^2y^2[1+y^2]^{-1}}-1-\frac{\gamma^2}{2k^2}+\frac{\gamma^2y^2}{k^2(1+y^2)}\right\}+\frac{2\gamma}{[1+y^2]^{\frac{3}{2}}}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\times\left\{\frac{(n+s)k}{(n+s)^2+y^2\gamma^2[1+y^2]^{-1}}-\frac{k}{n+k}-\frac{\gamma^2}{2(n+k)^2}+\frac{\gamma^2ky^2}{(n+k)^3(1+y^2)}\right\}. \quad (64)$$

We may also find the limit of the integral of (62) with respect to y, when p goes to infinity. The result is (see Appendix II)

$$\lim_{p\to\infty}q'''(p)=\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\times\left\{-2\sum_{k=1}^{\infty}k\left[\tan^{-1}\left(\frac{\gamma}{s}\right)-\frac{\gamma}{k}-\frac{1}{6}\frac{\gamma^3}{k^3}\right]+4\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left[\frac{\gamma kn}{(n+s)^2+\gamma^2}-\frac{\gamma kn}{(n+k)^2}-\frac{\gamma^3n^2}{(n+k)^4}-k\tan^{-1}\left(\frac{\gamma}{n+s}\right)+\frac{\gamma k}{(n+k)}-\frac{k\gamma^3}{3(n+k)^3}+\frac{\gamma^3}{2(n+k)^2}\right]\right\}. \quad (65)$$

The fact that this limit is finite corresponds to the presence of a point charge at the origin, just as in the case of $q^{(3)}(p)$.

$q'''(p)$ can be expanded in powers of p, for small p, and expressions analogous to (65) obtained for each coefficient. We have carried out the calculation for the lowest terms only, obtaining

$$\left[\frac{\partial}{\partial p}q'''(p)\right]_{(p=0)}=\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3(\gamma\pi)\times\left\{\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{k[n^2+2ns]}{(n+s)^2+\gamma^2+(n+s)[(n+s)^2+\gamma^2]^{\frac{1}{2}}}-\frac{2k(n+s)}{(n+s)+[(n+s)^2+\gamma^2]^{\frac{1}{2}}}-f_{nk}^{(1)}-\gamma^2f_{nk}^{(2)}\right]+\sum_{k=1}^{\infty}\left[\frac{1}{2}k\left[s-k+\frac{\gamma^2}{2k}\right]-\frac{k}{2(k+s)}-\frac{ks}{(k+s)}-f_k^{(3)}-\gamma^2f_k^{(4)}\right]\right\}. \quad (66)$$

The f's are terms added to remove the (diverging) terms to first and third order in γ.

For small γ, we may expand

$$\left[\frac{\partial}{\partial p}q'''(p)\right]_{(p=0)}=\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\pi\times\left\{\gamma^5\left[\frac{5}{64}\zeta(2)-\frac{1}{8}\zeta(3)+\frac{1}{64}\zeta(4)\right]+\frac{\gamma^7}{32}\left[\frac{19}{2}\zeta(5)-\frac{95}{16}\zeta(4)-\frac{55}{16}\zeta(6)\right]+O(\gamma^9)\right\}\cong-\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\times\{\gamma^5(0.015191)+\gamma^7(0.007127)+O(\gamma^9)\}. \quad (67)$$

For $|\gamma|\leqq1$, we may write

$$\left[\frac{\partial}{\partial p}q'''(p)\right]_{(p=0)}=-\frac{e}{4\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\times\gamma^5[0.015191]F_1(\gamma^2). \quad (68)$$

The function F_1 is shown on the graph in Fig. 4.

VII. POLARIZATION POTENTIAL AT SMALL DISTANCES. CORRECTIONS TO ENERGY LEVELS IN MU-MESONIC ATOMS

We return to the question of the polarization charge at the origin. We combine (61) and (65) and write

$$\delta Q'=4\pi\left(\frac{\hbar}{m_0c}\right)^3\lim_{p\to\infty}[\gamma^3q^{(3)}(p)+q'''(p)]. \quad (69)$$

Expanding for small γ, we get

$$\delta Q'=\frac{e}{\pi}\left\{\gamma^3\left[\frac{2}{3}\zeta(3)+\frac{7}{9}-\frac{\pi^2}{6}\right]+\gamma^5\left[-\frac{3}{20}\zeta(4)+\frac{1}{4}\zeta^2(2)-\frac{27}{5}(\zeta(3)-\zeta(4))+\frac{2}{3}(\zeta(3)-\zeta(5))\right]+\gamma^7\left[\frac{34}{21}\zeta(5)+\frac{2}{3}\zeta(7)-\left(18+\frac{13}{24}\right)\zeta(6)+12\zeta(5)+\frac{1}{4}\zeta(2)\zeta(4)+\frac{5}{2}\zeta^2(3)\right]+O(\gamma^9)\right\}, \quad (70)$$

or

$$\delta Q'\cong-e\{\gamma^3(0.020940)+\gamma^5(0.007121)F_0(\gamma^2)\}. \quad (71)$$

The function F_0 has been rather roughly evaluated numerically and is represented graphically in Fig. 4. For small γ, we have [from (70)]

$$F_0(\gamma^2)\cong1+\gamma^2(0.5183)+O(\gamma^4). \quad (72)$$

$\delta Q'$ represents the part of the higher-than-first-order polarization charge which is located at the origin. Since the total polarization charge (to higher than first order) vanishes, $-\delta Q'$ is the part of the higher-than-first-order polarization charge located outside the origin. Thus, for

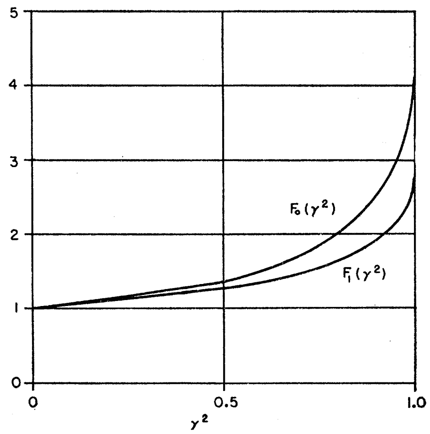

FIG. 4. The functions $F_0(\gamma^2)$ and $F_1(\gamma^2)$ defined in the text in connection with the polarization potential near the inducing charge and the space integral of the polarization potential divided by r.

r small compared to the mean radius of the outside charge distribution, we get for the higher-than-first-order potential

$$V_P'(r)=V_P(r)-\gamma V_P^{(1)}(r)\cong\delta Q'/(4\pi\epsilon_0 r). \quad (73)$$

A mean radius, r_P', for the charge distribution outside the origin may be defined by

$$r_P'=4\pi\int_0^\infty r^3\rho'(r)dr\bigg/-\delta Q'$$

$$=\lim_{p\to 0}-\left[\frac{\partial}{\partial p}q'''(p)+\frac{\partial}{\partial p}q^{(3)}(p)\right]4\pi\left(\frac{\hbar}{m_0c}\right)^4\bigg/-\delta Q'$$

$$\cong(0.8579)\left(\frac{\hbar}{m_0c}\right)\left[\frac{1+\gamma^2(0.2692)F_1(\gamma^2)}{1+\gamma^2(0.3401)F_0(\gamma^2)}\right]. \quad (74)$$

The mean radius is thus about equal to the Compton wavelength of the electron. It is also interesting to note that the mean radius does not change much in the range:

$$0\leqq\gamma\leqq 1.$$

We use (73) to estimate the displacement of energy levels in mesonic atoms, due to higher than first order terms in the vacuum polarization.

We use Schrödinger wave functions for a pure Coulomb field to describe the meson, and assume that Z is sufficiently high, and the principal quantum number sufficiently small, so that the meson is well inside the mean radius (74). We thus find the leading term in the displacement, $\Delta E_n'$, simply as the expectation value of the potential $V_P'(r)$. Let E_n denote the unperturbed energy. Then

$$\Delta E_n'=\langle n|V_P'(r)|n\rangle,$$

$$\Delta E_n'/E_n\cong-2\cdot\delta Q'/Ze=2\alpha\{\gamma^2(0.020940)+\gamma^4(0.007121)F_0(\gamma^2)\}. \quad (75)$$

Thus, for uranium, $Z=92$:

$$\Delta E_n'/E_n\cong 1.6\times 10^{-4}.$$

For low Z, the approximation (75), while not so well justified, is almost certainly an upper limit. The contribution is then particularly small due to the smallness of γ.

In this estimate, we have ignored the effect of the finite nuclear size. Nevertheless, we are confident that (75) gives a good picture of the order of magnitude involved. Our conclusion is thus that with present day experimental accuracy, level displacements in mesonic atoms, due to vacuum polarization in higher order than the first, are not detectable.

The effect of the Uehling potential (44) on level displacements in mesonic atoms has been considered elsewhere.[2,3,5,6] In this case, the effect must be considered susceptible to measurements. In some recent measurements[5] on x-rays from mesonic atoms, for the purpose of obtaining a value for the mass of the mu-meson, there is indeed some indication that the effect of vacuum polarization has to be considered in the interpretation of the results for consistency with other mass determinations. In the case of the first-order term, it is also easy to extend the result to the case of a nucleus of finite size.[6]

VIII. EFFECT OF VACUUM POLARIZATION ON X-RAY FINE STRUCTURE. THIRD-ORDER CONTRIBUTION TO THE LAMB SHIFT IN HYDROGEN

We consider the effect of vacuum polarization on the x-ray fine structure separation in heavy elements, in particular the $2p_{\frac{1}{2}}-2p_{\frac{3}{2}}$ separation. Our interest in this question derives from attempts that have been made to infer something about the nuclear size from an analysis of measured separations.

Schawlow and Townes[7,18] have pointed out the existence of a systematic deviation of the experimental separation from the theoretical prediction, which varies rapidly with Z. They demonstrate that such an effect could be attributed to short range departures from the Coulomb interaction. An attempt to attribute the effect entirely to the finite size of the nucleus leads to a nuclear radius considerably larger than that obtained from other experiments. It is clear that quantum electrodynamical modifications of the Coulomb interaction will also contribute to the effect, and indeed if one now regards the nuclear radius as known from other experiments, one may make use of this effect as a means of observing quantum electrodynamical effects in heavy elements.

One such effect, although by no means the only one, is the effect of vacuum polarization. Estimates of the contribution from the Uehling potential to the $2p_{\frac{1}{2}}-2p_{\frac{3}{2}}$ separation have been made previously, taking the leading term only in an expansion of the expectation

[18] A. L. Schawlow and C. H. Townes, Science 115, 284 (1952).

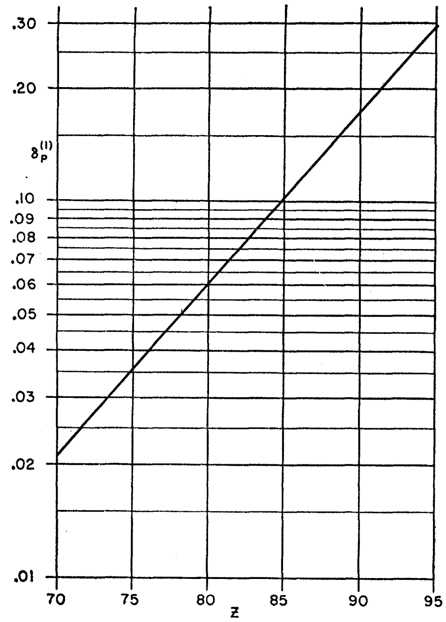

FIG. 5. The contribution of the Uehling potential to the $2p_{\frac{1}{2}}-2p_{\frac{3}{2}}$ separation in heavy elements in units of the Rydberg.

values in γ.[8] This approximation turns out to be inadequate when Z is large, and we have therefore carried out a more precise computation, as follows:

Using relativistic Coulomb wave functions for the $2p_{\frac{1}{2}}$ and $2p_{\frac{3}{2}}$ states, we have calculated the level shift due to the Uehling potential by evaluating the expectation values in the appropriate states. We express the result in the form of correction factors to the shifts one would obtain by taking the leading term only in an expansion in γ. Thus

$$
\begin{aligned}
-e\langle 2p_{\frac{1}{2}}|\gamma V_P^{(1)}|2p_{\frac{1}{2}}\rangle &= \frac{-9m_0c^2\alpha^7Z^6}{1120\pi}C(2p_{\frac{1}{2}}),\\
-e\langle 2p_{\frac{3}{2}}|\gamma V_P^{(1)}|2p_{\frac{3}{2}}\rangle &= \frac{-2m_0c^2\alpha^7Z^6}{1120\pi}C(2p_{\frac{3}{2}}).
\end{aligned}
\tag{76}
$$

The contribution to the fine structure separation thus becomes

$$
\begin{aligned}
\Delta_P^{(1)} = R_\infty\delta_P^{(1)};\quad \delta_P^{(1)} &= (\alpha^5Z^6/80\pi)C,\\
C &= [9C(2p_{\frac{1}{2}})-2C(2p_{\frac{3}{2}})]/7,
\end{aligned}
\tag{77}
$$

where R_∞ is the Rydberg constant. $C(2p_{\frac{1}{2}})$, $C(2p_{\frac{3}{2}})$, C, and $\delta_P^{(1)}$ are given in Table I, and $\delta_P^{(1)}$ has been represented graphically in Fig. 5 for some values of Z.

The large values of C shows that the lowest order approximation is not justified. Note in particular the rapid variation of $\delta_P^{(1)}$ with Z.

We have here neglected the presence of the other electrons in the atom. They give rise to two corrections which may properly be considered in this connection. The first correction is the effect of screening on the wave functions for the $2p$ electrons, and is the more important one. It could in a rough way be depicted as a diminuition of the parameter Z occurring in the wave functions for the $2p$ electrons. It would thus decrease the parameter C. Such an estimate would, however, necessarily be rough, since it is the very different behavior of the $2p_{\frac{1}{2}}$ and $2p_{\frac{3}{2}}$ wave functions near the origin that gives rise to the shift due to vacuum polarization, and only a calculation using "screened" wave functions could establish how screening affects this different behavior. The second effect is given by the difference in radiative corrections to the interactions between, respectively, the $2p_{\frac{1}{2}}$ and $2p_{\frac{3}{2}}$ electrons and the rest of the electrons in the atoms. This can be expected to be small.

Let us next consider the effect of the third-order polarization potential on the fine structure separation. From the Laplace transform (57) it is easy to obtain the expectation value of $V_P^{(3)}(r)$ over nonrelativistic wave functions in a Coulomb field, as an expansion in powers of p. It is considerably more difficult to find the expectation values over Dirac wave functions because of the occurrence of fractional powers of r in the expression for the square of the wave function.

As an orientation we shall first consider the contribution to the fine structure separation for small γ. This may be found as follows: We expand the square of the Dirac wave function in powers of γ and take only the leading term. We thus get (after an integration over angles)

$$
\begin{aligned}
\oint dw\cdot x^2|\psi(2p_{\frac{3}{2}})|^2 &= (1/24)\gamma(\gamma x)^4+O(\gamma^6),\\
\oint dw\cdot x^2|\psi(2p_{\frac{1}{2}})|^2 &= (1/24)\gamma(\gamma x)^2\\
&\quad\times[(\gamma x)^2+(9/4)\gamma^2]+O(\gamma^6).
\end{aligned}
\tag{78}
$$

Using (57), (58), and (59′), we thus get

$$
\begin{aligned}
&e\langle 2p_{\frac{1}{2}}|\gamma^3V_P^{(3)}|2p_{\frac{1}{2}}\rangle - e\langle 2p_{\frac{3}{2}}|\gamma^3V_P^{(3)}|2p_{\frac{3}{2}}\rangle\\
&\qquad\cong(3/32\pi)\alpha\gamma^8m_0c^2(0.01417)+O(\gamma^9),
\end{aligned}
\tag{79}
$$

giving the ratio of the third- and first-order shifts:

$$
\delta_P^{(3)}/\delta_P^{(1)}\cong-(0.212)\gamma^2+O(\gamma^3). \tag{80}
$$

[In obtaining this result, the constant C has been set equal to 1 in (77).]

We have tried to improve the estimate given by (80) by a method which cannot really be strictly justified.

TABLE I. Values of the parameters defined in Eqs. (76) and (77).

Z	$C(2p_{\frac{1}{2}})$	$C(2p_{\frac{3}{2}})$	C	$\delta_p^{(1)}$	$\delta_p^{(3)}/\delta_p^{(1)}$
95	3.82	0.65	4.73	0.287	−0.06
90	3.20	0.65	3.92	0.172	−0.06
85	2.74	0.65	3.33	0.104	−0.06
78	2.24	0.66	2.69	0.0500	−0.06
70	1.85	0.67	2.19	0.0213	−0.05

We carried out the following computation: Using (57), (58), and (59′) we find the averages of the polarization potential $V^{(3)}$ over xe^{-px}, x^2e^{-px}, x^3e^{-px} and x^4e^{-px}. We then use these averages to find, by interpolation, such averages as x^ge^{-px} where g represents the fractional exponents occuring in the square of the Dirac wave function. In this way, we arrive at the values of $\delta_p^{(3)}$ given in Table I. In view of the fact that the numbers in the last column are fairly small, it did not seem justified at this time to carry out a more elaborate evaluation. We estimate that the errors in the last column of Table I may be as high as 50%.

One may thus assume that the Uehling term gives the main contribution to the shift arising from vacuum polarization. The shift has the same rapid Z dependence as the deviation found by Schawlow and Townes,[7,18] and is of a similar order of magnitude. It has, however the wrong sign, which may be taken as an indication that other quantum electrodynamical effects play an important role.

The contribution from $V_P^{(3)}$ to the energy level displacement in hydrogen is easily computed, using (57), (58), and (59). The result is, for the 2s state, 308 cycles/sec, and therefore entirely negligible.

APPENDIX I

$F(p;q;t)$, $G(p;q;t)$, and $H(p;q;t)$ are special solutions to the differential equation

$$\left[t\frac{d^2}{dt^2}+(q-t)\frac{d}{dt}-p\right]\psi(t)=0. \quad (1)$$

G and H are defined so that they satisfy the same well-known recursion relations as F.

Using Pochhammer's contour P,[19] we may find an integral representation for F, valid for all p and q, except when p or $p-q$ is an integer:

$$F(p;q;t)=\frac{-\Gamma(q)\exp(-i\pi q)}{4\Gamma(p)\Gamma(q-p)\sin(\pi p)\sin[\pi(q-p)]}\times\int_P dz z^{p-1}(1-z)^{q-p-1}e^{zt}. \quad (2)$$

By deforming this contour, we get the asymptotic expansions:

$$F(p;q;t)=\frac{\Gamma(q)}{\Gamma(q-p)}(+t)^{-p}g(p;p-q+1;-t)e^{i\pi p\sigma}+e^t t^{p-q}\frac{\Gamma(q)}{\Gamma(p)}g(1-p;q;t), \quad (3)$$

$$H(p;q;t)=\frac{\Gamma(p+1-q)}{\Gamma(1-q)}t^{-p}g(p;p-q+1;-t), \quad (4)$$

when $-\frac{1}{2}\pi\leqq\arg t\leqq\frac{1}{2}\pi$, where

$$\sigma=\text{sign}(\arg t),$$

$$g(p;q;t)\sim\sum_{n=0}^{\infty}\frac{\Gamma(n+q)\Gamma(n+p)}{\Gamma(q)\Gamma(p)n!}t^{-n}.$$

The integral representations (24) and (25) are easily established: (24) by direct evaluation of the integral, and (25) by noting that the integral satisfies the differential equation (1), and that it has the same leading term in the asymptotic expansion as H, as given by (4).

Let us now study $K(x_1,x_2;z)$. If we write (for the case $x_1<x_2$):

$$K(x_1,x_2;z)=\sum_{n=0}^{1}\sum_{m=0}^{1}A_{mn}F(a+n,b;-2ix(z^2-1)^{\frac{1}{2}})\times H(a+m;b;-2ix(z^2-1)^{\frac{1}{2}}), \quad (5)$$

we may write

$$K(x_1,x_2;z)=K'(x_1,x_2;z)+K''(x_1,x_2;z), \quad (6)$$

$$K'(x_1,x_2;z)=\sum_{n=0}^{1}\sum_{m=0}^{1}A_{mn}F(a+n;b;-2ix(z^2-1)^{\frac{1}{2}})\times F(a+m;b;-2ix(z^2-1)^{\frac{1}{2}}), \quad (7)$$

$$K''(x_1x_2;z)=\sum_{n=0}^{1}\sum_{m=0}^{1}A_{mn}F(a+n;b;-2ix(z^2-1)^{\frac{1}{2}})\times G(a+m;b;-2ix(z^2-1)^{\frac{1}{2}}). \quad (8)$$

Let

$$[w^{(g)}(x;z)]\equiv[w^{(1)}(x;z)]-[w^{(2)}(x;z)]. \quad (9)$$

We shall study the behavior of K'' as we pass the cut in the z-plane. Let us denote: (where $f(z)$ is a function of z)

$$p>1 \quad\text{or}\quad p<-1,$$

$$f_+(p)=\lim_{z\to p+0\cdot i}f(z),\quad f_-(p)=\lim_{z\to p-0\cdot i}f(z), \quad (10)$$

$$(p^2-1)_+^{\frac{1}{2}}=-(p^2-1)_-^{\frac{1}{2}}, \quad (11)$$

$$b-1=a_++a_-, \quad (12)$$

$$|\lambda_+|=|\lambda_-|=|a_+|=|a_-|, \quad (13)$$

$$(p-1)_+^{\frac{1}{2}}/(p-1)_-^{\frac{1}{2}}=-p/|p|, \quad (14)$$

$$F_+(a;b;-2ix(p^2-1)^{\frac{1}{2}})=\exp[2ix(p^2-1)_-^{\frac{1}{2}}]F_-(a+1;b;-2ix(p^2-1)^{\frac{1}{2}}), \quad (15)$$

$$\begin{aligned}
w_{1-}^{(1)}&=-(-1)^s\frac{p}{|p|}\frac{|\lambda|^2}{\lambda_+a_+}\cdot w_{1+}^{(1)},\\
w_{2-}^{(1)}&=-(-1)^s\frac{p}{|p|}\frac{|\lambda|^2}{\lambda_+a_+}w_{2+}^{(1)},\\
w_{1-}^{(g)}&=(-1)^{-s}\frac{p}{|p|}\frac{|\lambda|^2}{\lambda_+a_+}\frac{\Gamma(1-a_+)\Gamma(a_+)}{\Gamma(1-a_-)\Gamma(a_-)}w_{1+}^{(g)},\\
w_{2-}^{(g)}&=(-1)^{-s}\frac{p}{|p|}\frac{|\lambda|^2}{\lambda_+a_+}\frac{\Gamma(1-a_+)\Gamma(a_+)}{\Gamma(1-a_-)\Gamma(a_-)}w_{2+}^{(g)},
\end{aligned} \quad (16)$$

[19] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis* (Cambridge University Press, Cambridge, 1950), fourth edition, p. 256.

$$K_-(p) = -\frac{a_-\lambda_-\Gamma(a_+)\Gamma(1-a_+)}{a_+\lambda_+\Gamma(a_-)\Gamma(1-a_-)}K_+(p). \quad (17)$$

Thus:

$$K_-''(x_1,x_2;\,p) = K_+''(x_1,x_2;\,p). \quad (18)$$

By inspection, we see that K'' has no poles, and that K'' is regular at $z=1$ and $z=-1$. Thus; $K''(x_1,x_2;z)$ is a single-valued analytic function in the whole complex plane. It follows that K'' when integrated along $E(R)$ or $P(R)$ gives zero. The charge density could thus have been defined only in terms of K'. However, an examination of the asymptotic behavior of K' using (3) shows that then the method of deforming the contours so that all physically significant contributions would come from the integral along the imaginary axis would fail.

Concerning K' we see that possible poles are located at the zeros of $K(z)$, i.e., at the points where a is a nonpositive integer. A closer examination shows that the poles are precisely at the bound-state eigenvalues, and the correctly normalized bound-state eigenfunctions may be obtained from the residues of K' at these poles. The correctly normalized continuum state eigenfunctions may likewise be obtained by considering the sum of the integrals of K' over the contours $L_1(R;R')$ and $L_2(R;R')$, letting R go to R'.

Let us now consider the time-dependent radial equation:

$$(\mathfrak{D}_{x_1}+i\partial/\partial\tau)[\phi(x_1;\tau)]=0. \quad (19)$$

Let C be some contour in the cut z-plane, which does not go through a pole of $K(x_1x_2;z)$. Let

$$(G_C(x_1,x_2;\tau)) = \frac{1}{2\pi i}\int_C dz(K(x_1,x_2;z))e^{-i\tau z}. \quad (20)$$

Consider in particular the contours $P(R)$ and $E(R)$. These contours define transformations $G_{E(R)}$ and $G_{P(R)}$ such that if $[f(x)]$ is some "physically well behaved" function, then:

$$[f(x;\tau)]_{E(R)} = \int_0^\infty dy(G_{E(R)}(x,y;\tau))[f(y)], \quad (21)$$

$$[f(x;\tau)]_{P(R)} = \int_0^\infty dy(G_{P(R)}(x,y;\tau))[f(y)], \quad (22)$$

are solutions of the time-dependent radial equation.

This result is easily verified formally by substituting (21) or (22) into (20), and inverting the order of integration and differentiation.

For certain functions f the transformed functions approach a limit as R goes to infinity. In this case,

$$[f(x;0)]_E+[f(x;0)]_P=[f(x)]. \quad (23)$$

We can see this in a somewhat unprecise way as follows: Let $f(x)$ be expanded in terms of the radial eigenfunctions, (both discrete and continuum eigenfunctions), and let f be such that there is a c_0 such that no eigenfunction occurs in the expansion whose eigenvalue is larger in absolute value than c_0. Let R be larger than c_0. Using the radial equation inside the integrals (21) and (22), and letting $\tau\to 0$, the result (23) follows readily, provided that the interchanges of order of differentiation and integration are allowed.

All these remarks are purely formal; to be more precise means essentially to prove an expansion theorem for the radial eigenfunctions.

We thus see that the transformations (20) play the role of time development transformations ("propagation functions"), and various types of such functions can be constructed by selecting suitable contours.

The procedure here outlined makes possible the construction of radial propagation functions for each k. From the matrix elements of the radial propagation functions and products of spherical harmonics, we may construct the corresponding propagation functions of the full Dirac equation in the case of a Coulomb field, in the form of a sum over k. If we let the strength of the Coulomb field go to zero, the sum over k can be explicitly carried out, and we obtain the usual propagation functions for a free Dirac field.

APPENDIX II

We prove here that

$$\int_0^\infty dyW'''(p;\,y)$$

is an analytic function of p and γ in the region

$$\mathrm{Re}[p]>0;\quad |\gamma|<1.$$

Writing

$$x=(1-t)/[1+t(u-1)], \quad (1)$$

then, after performing an integration by parts, we may write W''' in the form

$$W'''(p;\,y)=W_1'''(p;\,y)+W_2'''(p;\,y),$$

$$W_1'''(p;\,y)=\int_0^1 dx\sum_{k=1}^\infty T_1^{(k)}(p;\,y;\,x), \quad (2)$$

$$W_2'''(p;\,y)=\int_0^1 dx\sum_{k=1}^\infty T_2^{(k)}(p;\,y;\,x),$$

$$\rho=(1+y^2)^{\frac{1}{2}},$$

$$T_1^{(k)}(p;\,y;\,x) = -\frac{\gamma}{\rho^3}\left[\frac{1}{1+x(u-1)}+\frac{1}{x[1+u-x]}\right]\left\{kQ^s\cos g - [kQ^k-\tfrac{1}{2}kQ^kg^2-\tfrac{1}{2}\gamma^2Q^k\ln Q]\right\}, \quad (3)$$

$$T_2^{(k)}(p;\,y;\,x) = -\frac{2\gamma y^2}{\rho^3}\frac{u}{[1+u-x][1+x(u-1)]}\left\{[kQ^s\cos g - Q^k(k-\tfrac{1}{2}kg^2-\tfrac{1}{2}\gamma^2\ln Q)]-\frac{\rho}{\gamma y}[kQ^s s\sin g - (k^2g-\tfrac{1}{6}k^2g^3+\tfrac{1}{2}k\gamma^2g\ln Q-\tfrac{1}{2}\gamma^2g)Q^k]\right\}. \quad (4)$$

It is the behavior of the $T^{(k)}$ at $x=0$ and $x=1$ which prevents the analyticity properties of the integral over W''' from being almost self evident. We therefore split the x-integration into three intervals

$$0\leqq x\leqq \epsilon;\quad \epsilon\leqq x\leqq 1-\delta;\quad 1-\delta\leqq x\leqq 1,$$

with

$$0<\epsilon<\tfrac{1}{4};\quad 0<\delta<\tfrac{1}{4}. \tag{5}$$

We consider the region

$$\text{Re}\{p\}\geqq 0;\quad |\gamma|\leqq\gamma_0<1. \tag{6}$$

The idea of the proof is to show that the integral over the central interval has the desired analyticity properties, and the other two integrals tend to zero with ϵ and δ, uniformly in the region defined by (6).

We have

$$\text{Re}\left\{(1-\gamma^2)^{\frac{1}{2}}\pm\frac{i\gamma y}{\rho}\right\}\geqq\mu>0 \tag{7}$$

for some $0<\mu<\frac{1}{2}$, depending on γ_0 only.

$$Q=x[1+x(u-1)]/[1+u-x],$$

$$g=\frac{\gamma y}{\rho}\ln\left\{\frac{x[1+u-x]}{[1+x(u-1)]}\right\}.$$

We consider W_2'''. Let

$$A_2(p;y;\gamma)=\int_0^\epsilon dx\sum_{k=1}^\infty T_2^{(k)},$$

$$B_2(p;y;\gamma)=\int_\epsilon^{1-\delta} dx\sum_{k=1}^\infty T_2^{(k)},$$

$$C_2(p;y;\gamma)=\int_{1-\delta}^1 dx\sum_{k=1}^\infty T_2^{(k)}.$$

(*a*) Let $0\leqq x\leqq\epsilon$. Then there is a K_0, independent of p, y, γ, ϵ, x such that:

$$\begin{aligned}|Q^{(1-\gamma^2)^{\frac{1}{2}}}\cos g|&\leqq x^\mu K_0,\\ |Q^{(1-\gamma^2)^{\frac{1}{2}}}\sin g|&\leqq x^\mu K_0,\\ |Qg|&\leqq x^\mu K_0,\\ |Qg^2|&\leqq x^\mu K_0,\\ |Qg^3|&\leqq x^\mu K_0,\\ |Q\ln Q|&\leqq x^\mu K_0,\\ |Qg\ln Q|&\leqq x^\mu K_0.\end{aligned} \tag{8}$$

Furthermore,

$$\begin{aligned}&|Q|\leqq\tfrac{1}{4},\quad |Q^{(k^2-\gamma^2)^{\frac{1}{2}}}|\leqq x^\mu K_0(\tfrac{1}{4})^{k-1},\\ &\left|\frac{1}{1+u-x}\right|\leqq\frac{2}{|1+u|},\\ &\int_0^\epsilon\left|\frac{x^\mu}{1+x(u-1)}\right|dx\leqq\frac{1}{\mu}\epsilon^\mu\frac{1}{|1+u|}.\end{aligned} \tag{9}$$

Then,

$$A_2(p;y;\gamma)=\sum_{k=1}^\infty\int_0^\epsilon dx\cdot T_2^{(k)}, \tag{10}$$

and

$$|A_2(p;y;\gamma)|\leqq K_1\epsilon^\mu\frac{y}{1+y^2}\cdot\frac{|u|}{|1+u|^2}, \tag{11}$$

for some K_1 independent of p, y, γ, and ϵ. Also,

$$\left|\int_0^\infty dy\cdot A_2(p;y;\gamma)\right|\leqq K_2\epsilon^\mu\frac{|p|}{1+|p|}, \tag{12}$$

for some K_3 independent of p, γ, and ϵ.

(*b*) Let $\epsilon\leqq x\leqq 1-\delta$. Then

$$\begin{aligned}&|Q|\leqq 1-\delta,\\ &B_2(p;y;\gamma)=\sum_{k=1}^\infty\int_\epsilon^{1-\delta}dx\cdot T_2^{(k)},\end{aligned} \tag{13}$$

and $B_2(p;y;\gamma)$ is an analytic function of p and γ in the region

$$|\gamma|<\gamma_0,\quad \text{Re}\{p\}>0.$$

Furthermore,

$$|B_2(p;y;\gamma)|\leqq K_3\frac{y}{1+y^2}\frac{|u|}{|1+u|^2}, \tag{14}$$

for some K_3 independent of p, y, and γ. Then,

$$\int_0^\infty dy\cdot B_2(p;y;\gamma)$$

is an analytic function of p and γ in the region

$$|\gamma|<\gamma_0,\quad \text{Re}\{p\}>0,$$

and

$$\left|\int_0^\infty dyB_2(p;y;\gamma)\right|\leqq K_4\frac{|p|}{1+|p|}, \tag{15}$$

for some K_4 independent of p and γ.

(*c*) Let $1-\delta\leqq x\leqq 1$. Then,

$$|\ln Q|\leqq 3(1-x)\leqq\tfrac{3}{4},\quad |g|\leqq 3(1-x)\leqq\tfrac{3}{4},$$

$$|kQ^s\cos g-Q^k(k-\tfrac{1}{2}kg^2-\tfrac{1}{2}\gamma^2\ln Q|\leqq K_5(1-x)^4x^kk,$$

$$|ksQ^s\sin g-Q^k(k^2g-\tfrac{1}{6}k^2g^3+\tfrac{1}{2}k\gamma^2g\ln Q-\tfrac{1}{2}\gamma^2g|$$
$$\leqq K_5(1-x)^5x^kk^2,$$

for some K_5 independent of p, y, γ, and δ. Also,

$$\begin{aligned}&\left|\sum_{k=1}^\infty\int_{1-\delta}^1 dxT_2^{(k)}\right|\leqq K_6\frac{|u|}{|1+u|^2}\frac{y\delta}{[1+y^2]},\\ &|C_2(p;y;\gamma)|\leqq K_6\frac{|u|}{|1+u|^2}\frac{y\delta}{[1+y^2]},\end{aligned} \tag{16}$$

for some K_6 independent of p, γ, δ, and y. Then,

$$\left|\int_0^\infty C_2(p;y;\gamma)dy\right| \leqq K_7\delta\frac{|p|}{1+|p|} \tag{17}$$

for some K_7 independent of p, γ, δ.

(*d*) Thus

$$\int_0^\infty dyW_2'''(p;y)$$

is an analytic function of p and γ in the region:

$$|\gamma|<\gamma_0;\quad \mathrm{Re}\{p\}>0.$$

Furthermore

$$W_2'''(p;y)=\int_0^1 dx\sum_{k=1}^\infty T_2^{(k)}=\sum_{k=1}^\infty\int_0^1 dxT_2^{(k)},$$

and

$$\left|\int_0^\infty dyW_2'''(p;y)\right| \leqq K_8\frac{|p|}{1+|p|} \tag{18}$$

for some K_8, depending on γ_0 only.

The proof for W_1''' is entirely similar, only the estimate now becomes

$$\left|\int_0^\infty dyW_1'''(p;y)\right| \leqq \frac{K_9}{1+|p|} \tag{19}$$

for some K_9 depending on γ_0 only.

Using similar methods, we may prove that the order of integration over t and summation over k may be reversed in the expression for $W^{(3)}$, and the analytic properties of $W^{(3)}$ may be studied using (47) in the text.

Concerning the derivation of the expression (64) for $\lambda'''(y)$, we see that the contribution from W_2''' vanishes because of (12), (14) and (17). To find the contribution from W_1''' we may set $u=0$ in $T_1^{(k)}$ in (3) and expand the denominators in powers of x. Integrating the double series term by term we get (64).

In deriving (65), we first note that W_1''' gives no contribution because of (19). The constant comes from integrating $\lambda'''(y)$, and from W_2'''. To find the contribution from W_2''' we replace t and y by the new variables (which may be justified):

$$\zeta=\frac{1}{1+ut};\quad \eta=\frac{[1-t][1+ut]}{[1+t(u-1)]}. \tag{20}$$

We let $p\to\infty$ and get

$$\lim_{p\to\infty}\int_0^\infty dyW_2'''(p;y)$$

$$=-\gamma\int_0^1 d\xi\int_0^1 d\eta[1-\xi\eta]^{-2}\sum_{k=1}^\infty\Big\{k\xi^{2s}\eta^s\cos[\gamma\ln\eta]$$

$$-\frac{1}{\gamma}ks\xi^{2s}\eta^s\sin[\gamma\ln\eta]+k^2\xi^{2k}\eta^k[\ln\eta]-k\xi^{2k}\eta^k$$

$$-\tfrac{1}{6}k^2\xi^{2k}\eta^k[\ln^3\eta]+\tfrac{1}{2}k\xi^{2k}\eta^k[\ln\xi]$$

$$-\tfrac{1}{2}[\ln\xi][\ln\eta]\xi^{2k}\eta^k\Big\}. \tag{21}$$

We then expand $[1-\zeta\eta]^{-2}$ in a power series, and integrate the resulting double series term by term.

The expression (66) can be derived by similar methods. It may be noted that a certain care has to be exercised in all these integrations, especially in the regions

$$t\sim 0;\quad t\sim 1;\quad y\sim\infty.$$

APPENDIX III

We consider the Euler-Heisenberg Lagrangian density[20,21]:

$$L[x]=\tfrac{1}{2}\epsilon_0E^2+\frac{e^4E^4\hbar}{360\pi^2m_0^4c^7}+O(E^6).$$

E is the electric field strength, and the magnetic field has been put equal to zero. We assume that the field can be derived from a spherically symmetric potential, and thus get the field equations from the variational principle:

$$\mathbf{E}=-\nabla V(r),\quad \delta\int_0^\infty r^2drL\{V(r)\}=0,$$

or

$$\frac{d}{dr}\left\{r^2\epsilon_0\frac{dV}{dr}+\frac{e^4\hbar}{90\pi^2m_0^4c^7}\left(\frac{dV}{dr}\right)^3+O\left(\frac{dV}{dr}\right)^5\right\}=0.$$

Assuming the leading term to be a Coulomb potential, we immediately get the asymptotic form of the potential:

$$V(r)=\frac{eZ}{4\pi\epsilon_0r}\left\{1-\frac{2Z^2\alpha^3}{225\pi}\left(\frac{\hbar}{m_0cr}\right)^4+O\left(\frac{1}{r^8}\right)\right\}.$$

This agrees with (60).

APPENDIX IV

(*a*) We may define a function $\psi(n;x)$, when $|x|<1$ by the power series:

$$\psi(n;x)=\sum_{k=1}^\infty\frac{x^k}{k^n}.$$

We shall study only $\psi(2;x)$, which occurs in the expression for $\bar{W}^{(3)}$. We have the integral representation:

$$\psi(2;x)=-\int_0^1\frac{dt}{t}\ln[1-tx],$$

when $|x|<1$. Introducing an appropriate cut in the x-plane, we may continue ψ, using the integral repre-

[20] W. Heisenberg and H. Euler, Z. Physik **98**, 714 (1936).
[21] H. Euler, Ann. Physik **26**, 398 (1936).

sentation. In particular we get the relations:

$$\psi(2;1-x)=\psi(2;1)-[\ln x][\ln(1-x)]-\psi(2;x),$$

$$\psi(2;-x)=-\psi(2;1)-\tfrac{1}{2}\ln^2 x-\psi\left(2;-\frac{1}{x}\right),$$

$$\psi(2;-1)=-\tfrac{1}{2}\psi(2;1),$$

$$\psi(2;\tfrac{1}{2})=\tfrac{1}{2}\psi(2;1)-\tfrac{1}{2}\ln^2(2),$$

$$\psi(2;1)=\zeta(2).$$

(*b*) In deriving (66) and (67), we have to evaluate certain double series. We shall tabulate the sums we have evaluated, and indicate the method. Let $p>1$ and $q>1$. Then:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[\frac{1}{m^p(m+n)^q}+\frac{1}{m^q(m+n)^p}\right]=\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\left[\frac{1}{m^pn^q}+\frac{1}{m^qn^p}\right]=\zeta(p)\zeta(q)-\zeta(p+q).$$

Let $r\geqq 1$ (r an integer). Then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{mn(m+n)^r}$$

$$=(-1)^r[(r-1)!]^{-1}\int_0^1\frac{dx}{x}[\ln^{r-1}x][\ln^2(1-x)]$$

$$=(-1)^{r-1}[(r-1)!]^{-1}\lim_{y\to+0}\frac{\partial^{r-1}}{\partial y^{r-1}}\times\left\{\lim_{\epsilon\to 0}\frac{\partial^2}{\partial\epsilon^2}\int_0^1 dx\,x^{y-1}(1-x)^\epsilon\right\}$$

$$=(-1)^{r-1}[(r-1)!]^{-1}\lim_{y\to+0}\frac{\partial^{r-1}}{\partial y^{r-1}}\lim_{\epsilon\to 0}\frac{\Gamma(y)\Gamma(1+\epsilon)}{\Gamma(1+y+\epsilon)}\times\left\{\sum_{n=1}^{\infty}\left[\frac{1}{(n+\epsilon)^2}-\frac{1}{(n+\epsilon+y)^2}\right]\left[\sum_{n=1}^{\infty}\left(\frac{1}{n+\epsilon+y}-\frac{1}{n+\epsilon}\right)\right]^2\right\}$$

$$=(r+1)\zeta(2+r)-\sum_{l=0}^{r-2}\zeta(2+l)\zeta(r-l).$$

Using these results, we may easily construct the table:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m(m+n)^2}=\zeta(3),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m(m+n)^3}=\tfrac{3}{2}\zeta(4)-\tfrac{1}{2}\zeta^2(2),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2(m+n)^2}=\tfrac{1}{2}\zeta^2(2)-\tfrac{1}{2}\zeta(4),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m(m+n)^4}=2\zeta(5)-\zeta(2)\zeta(3),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2(m+n)^3}=3\zeta(2)\zeta(3)-(11/2)\zeta(5),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^3(m+n)^2}=(9/2)\zeta(5)-2\zeta(2)\zeta(3),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m(m+n)^5}=\tfrac{3}{4}\zeta(6)-\tfrac{1}{2}\zeta^2(3),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2(m+n)^4}=\zeta^2(3)-(4/3)\zeta(6),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^3(m+n)^3}=\tfrac{1}{2}\zeta^2(3)-\tfrac{1}{2}\zeta(6),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^4(m+n)^2}=\zeta(2)\zeta(4)-\zeta^2(3)+\tfrac{1}{3}\zeta(6),$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n)^p}=\zeta(p-1)-\zeta(p);\quad p\geqq 3,$$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{(m+n)^{p+1}}=\tfrac{1}{2}[\zeta(p-1)-\zeta(p)];\quad p\geqq 3.$$

The last two relations are trivial if one chosen $(m+n)$ as a new variable of summation.

To evaluate these sums numerically, we have used the tables of Davies for the Riemann zeta function for integral arguments.[22]

[22] H. T. Davies, *Tables of Higher Mathematical Functions*, Vol. II, p. 244.