# Effects of Short-Range Correlations between Two Protons in Elastic Scattering of High-Energy Electrons by Heavy Nuclei*

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An expression is obtained for the leading term in the two-proton contribution to the scattering cross section, with a two-proton charge density which is a Yukawa function of the separation between two protons. The cross section for scattering from more complicated charge distributions can be generated from this basic result by differentiation with respect to the Yukawa range parameter. The two-proton cross section is evaluated for a charge density in which short-range correlations in position between two protons appear. Results are given for the scattering of 600-Mev electrons by heavy nuclei.

## I. INTRODUCTION

THE effect of correlations in position among the nuclear protons in the elastic scattering of electrons by nuclei appears in the scattering theory when account is taken of the virtual intermediate states of the target nuclei. A general discussion of these correlations has been given by Lewis.[1] In the second-order perturbation theory, the cross section can be expressed as the sum of two terms, one of which contains the coordinates of the protons taken one at a time (one-proton cross section), while the other contains the coordinates of the protons taken two at a time (two-proton cross section). The two-proton cross section contains information about correlations in position between two protons. In a recent paper, Schiff[2] evaluated the one-proton cross section for the scattering of high-energy electrons by nuclei and estimated the magnitude of the two-proton cross section for the large-angle scattering of 200-Mev electrons by carbon and gold. For the latter calculation, Schiff made no attempt to specify the two-proton charge density or to take account of position correlations; instead, he relied on certain assumed average properties of the charge density.

The present paper supplements the work of Schiff by presenting a detailed evaluation of the leading term in the two-proton cross section for a particular two-proton charge density. It is the purpose of this paper to indicate the extent to which it may be possible to isolate correlation effects by an examination of the experimental scattering data.

The second-order scattering cross section arises from the product of the first-order and second-order matrix elements. Assuming that the electrons interact only with the electric charges of the nuclear protons, Schiff has obtained an expression for the second-order cross section $\sigma^{(2)}$ for the scattering of high-energy electrons by nuclei. For elastic scattering his result is[3]

$$\sigma^{(2)}=\frac{e^6E^2}{8\pi^2(\hbar c)^3(\hbar cq)^2}\bar{F}(\mathbf{q})$$
$$\times\sum_{n=0}^{\infty}\int\int\int\int\Big[\Big(\frac{E+\epsilon_0-\epsilon_n}{\hbar c}\Big)(1+\cos\theta$$
$$+\cos\theta_{0\rho}+\cos\theta_{f\rho})+\Big(\frac{i}{\rho}\Big)$$
$$\times(\cos\theta_{0\rho}+\cos\theta_{f\rho})\Big]e^{-i\mathbf{k}_f\cdot\mathbf{r}}\bar{\psi}_0(\mathbf{R})$$
$$\times\sum_{i=1}^{z}\frac{1}{|\mathbf{r}-\mathbf{R}_i|}\frac{e^{(i\rho/\hbar c)(E-\epsilon_0-\epsilon_n)}}{\rho}$$
$$\times\psi_n(\mathbf{R})e^{i\mathbf{k}_0\cdot\mathbf{r}'}\bar{\psi}_n(\mathbf{R}')\sum_{j=1}^{z}\frac{1}{|\mathbf{r}'-\mathbf{R}_j'|}\psi_0(\mathbf{R}')$$
$$\times d\mathbf{R}'d\mathbf{R}d\mathbf{r}'d\mathbf{r}+\text{complex conjugate (c.c.)}.\quad(1)$$

The $\psi_n(\mathbf{R})$ are the nuclear wave functions for states with energy $\epsilon_n$, $\psi_0(\mathbf{R})$ and $\epsilon_0$ being the wave function and energy of the ground state. The radius vector of the $i$th proton is $\mathbf{R}_i$, and $\mathbf{r}$ is the radius vector of the electron. The nuclear integrals extend over the coordinates of all the nucleons. The sum over $n$ extends over the virtual intermediate states of the nucleus. The electron energy is $E$, and $\mathbf{q}=\mathbf{k}_0-\mathbf{k}_f$; $\mathbf{k}_0$ is the wave number vector of the incident electron, and $\mathbf{k}_f$ is that of the electron scattered at an angle $\theta$ from the direction of $\mathbf{k}_0$. The angle between $\mathbf{k}_0$ and $\boldsymbol{\rho}=\mathbf{r}-\mathbf{r}'$ is $\theta_{0\rho}$, and $\theta_{f\rho}$ is the angle between $\mathbf{k}_f$ and $\boldsymbol{\rho}$. The first-order form factor is $F(\mathbf{q})$, the complex conjugate being indicated by a bar.

Schiff has evaluated the sum over $n$ in Eq. (1) with the closure relation for nuclear wave functions, replacing $\epsilon_n$ by some unspecified, average value of the energy $\bar{\epsilon}_n$ of the excited states which contribute to the sum over $n$. The evaluation of the sum over $n$ is discussed here in somewhat greater detail to indicate the extent to which

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[1] Robert R. Lewis, Jr., thesis, University of Michigan, 1954 (unpublished).
[2] L. I. Schiff, Phys. Rev. 98, 756 (1955).
[3] See reference 2, Eq. (13); a sign error is corrected.

variations in $\epsilon_n$ may influence the two-proton cross section. Equation (1) contains the energy $\epsilon_n$ in two factors. The factor $(1/\hbar c)(E+\epsilon_0-\epsilon_n)$ is probably not sensitive to changes in $\epsilon_n$ for large values of the electron energy; this factor is assumed to be constant, and is not discussed. The other factor, $e^{-(i\rho/\hbar c)(\epsilon_n-\epsilon_0)}$, may vary appreciably with $n$. In order to evaluate the sum over $n$, $\epsilon_n$ is replaced by the nuclear Hamiltonian $H_N(R)$, which is considered to operate only on the unprimed nuclear coordinates. The $\psi_n(\mathbf{R})$ are eigenfunctions of $H_N(R)$ with eigenvalues $\epsilon_n$. After $\epsilon_n$ has been replaced by $H_N(R)$, the exponential factor does not depend upon specific values of $n$; consequently, the sum can be evaluated directly by means of the closure relation for nuclear wave functions. The resulting nuclear integral is put into a convenient form for evaluation by replacing $e^{-(i\rho/\hbar c)(H_N-\epsilon_0)}$ by its series expansion. The rapidity with which the resulting series converges depends upon the size of $(\rho/\hbar c)(\bar{\epsilon}_n-\epsilon_0)$. If the average value of the energy is not unexpectedly high ($\bar{\epsilon}_n-\epsilon_0$ is expected to be about 10 Mev), $(\rho/\hbar c)(\bar{\epsilon}_n-\epsilon_0)$ should be a small number. The largest values of $\rho$ which make appreciable contributions to the $n>0$ terms of Eq. (1) are expected to be of the order of nuclear dimensions. The long-range electron-nucleus interaction is contained in the $n=0$ term of Eq. (1); and, for $n=0$, $e^{(i\rho/\hbar c)(\epsilon_n-\epsilon_0)}=1$. The first term in the series expansion does not contain $H_N(R)$; it can be obtained from Eq. (1) by setting $\epsilon_n=\epsilon_0$ before the sum over $n$ is performed. It can be shown that the second term in the series, which is linear in $H_N(R)$, contains only the coordinates of the protons taken one at a time.[4] This term does not contribute to the two-proton cross section. The two-proton cross section, then, arises from the first, third, and higher terms of the series in $H_N(R)$. The calculations of this paper are made with the leading term of the series, which is obtained from Eq. (1) by setting $\epsilon_n=\epsilon_0$.

## II. BASIC TWO-PROTON CROSS SECTION

The leading term in the two-proton part $\sigma_2^{(2)}$ of the second-order cross section is[5]

$$\sigma_2^{(2)}=\frac{e^6E^2}{8\pi^2(\hbar c)^3(\hbar cq)^2}\bar{F}(\mathbf{q})\times\int\int\int\int\left[k(1+\cos\theta)+2\left(k+\frac{i}{\rho}\right)\left(\frac{\mathbf{K}_0\cdot\boldsymbol{\varrho}}{k\rho}\right)\right]\times\exp\{i[\mathbf{q}\cdot\mathbf{S}+k\rho-\mathbf{K}_0\cdot\boldsymbol{\varrho}+\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s}+\tfrac{1}{2}\mathbf{q}\cdot\mathbf{s}']\}\times\frac{1}{\rho ss'}\rho_{00}^{(2)}(\mathbf{R}_a,\mathbf{R}_b)d\mathbf{S}d\mathbf{s}'d\mathbf{s}d\boldsymbol{\varrho}+\text{c.c.}\quad(2)$$

[4] $\int\bar{\psi}_0\{|\mathbf{r}-\mathbf{R}_i|^{-1}(H_N-\epsilon_0)|\mathbf{r}-\mathbf{R}_j|^{-1}+|\mathbf{r}-\mathbf{R}_j|^{-1}(H_N-\epsilon_0)|\mathbf{r}-\mathbf{R}_i|^{-1}\}\psi_0d\tau$ $=\frac{1}{2}\int\bar{\psi}_0\{[|\mathbf{r}-\mathbf{R}_i|^{-1},[H_N,|\mathbf{r}-\mathbf{R}_j|^{-1}]]-[[H_N,|\mathbf{r}-\mathbf{R}_j|^{-1}],|\mathbf{r}-\mathbf{R}_i|^{-1}]\}\psi_0d\tau$ $=0$, if $i\neq j$.

[5] See reference 2, Eq. (23).

Schiff obtained Eq. (2) from the two-proton part of Eq. (1) by changing the variables of integration from $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{R}_a$, and $\mathbf{R}_b$ to $\mathbf{s}=\mathbf{r}-\mathbf{R}_a$, $\mathbf{s}'=\mathbf{r}'-\mathbf{R}_b$, $\mathbf{S}=\frac{1}{2}(\mathbf{R}_a+\mathbf{R}_b)$ and $\boldsymbol{\varrho}=\mathbf{r}-\mathbf{r}'$ after having defined the two-proton charge density

$$\rho_{00}^{(2)}(\mathbf{R}_a,\mathbf{R}_b)=\int\psi_0(\mathbf{R})\sum_{i=1}^{z}\sum_{j\neq i}^{z}\delta(\mathbf{R}_a-\mathbf{R}_i)\times\delta(\mathbf{R}_b-\mathbf{R}_j)\psi_0(\mathbf{R})d\mathbf{R}.\quad(3)$$

It is noted that $\mathbf{S}$ is the center of gravity of $\mathbf{R}_a$ and $\mathbf{R}_b$; $k=k_0=k_f$ is the electron energy divided by $\hbar c$; $\mathbf{K}_0=\frac{1}{2}(\mathbf{k}_0+\mathbf{k}_f)$; and $2\mathbf{K}_0\cdot\boldsymbol{\varrho}/k\rho=\cos\theta_{0\rho}+\cos\theta_{f\rho}$.

It is assumed that the two-proton charge density has the form

$$\rho_{00}^{(2)}(\mathbf{R}_a,\mathbf{R}_b)=f(\mathbf{S})g(D),\quad(4)$$

where $\mathbf{D}=\mathbf{R}_b-\mathbf{R}_a=\mathbf{s}-\mathbf{s}'-\boldsymbol{\varrho}$. With Eq. (4) the four integrals of Eq. (2) can be separated; this is the advantage of the form just chosen for the two-proton charge density. The disadvantage is related to edge effects in the nucleus; it is discussed in Sec. III, where a specific expression for $g(D)$ is selected. With Eq. (4) the $\mathbf{S}$ integral of Eq. (2) can be evaluated at once to give

$$\int\exp(i\mathbf{q}\cdot\mathbf{S})f(\mathbf{S})d\mathbf{S}=\mathfrak{F}(\mathbf{q}).\quad(5)$$

In order to separate the remaining integrals of Eq. (2), it is convenient to express $g(D)$ as a Fourier integral:

$$g(D)=\int\exp[-i\boldsymbol{\gamma}\cdot\mathbf{D}]G(\gamma)d\boldsymbol{\gamma}.\quad(6)$$

Since $g(D)$ is assumed to be a function only of the magnitude of $D$, the Fourier transform $G(\gamma)$ is a function only of the magnitude of $\boldsymbol{\gamma}$. With Eqs. (4), (5), and (6), Eq. (2) becomes

$$\sigma_2^{(2)}=\frac{e^6E^2}{8\pi^2(\hbar c)^3(\hbar cq)^2}\bar{F}(\mathbf{q})\mathfrak{F}(\mathbf{q})\int G(\gamma)d\boldsymbol{\gamma}\times\int\left[k(1+\cos\theta)+2\left(k+\frac{i}{\rho}\right)\left(\frac{\mathbf{K}_0\cdot\boldsymbol{\varrho}}{k\rho}\right)\right]\times\frac{\exp[i(k\rho-\mathbf{K}_0\cdot\boldsymbol{\varrho}+\boldsymbol{\gamma}\cdot\boldsymbol{\varrho})]}{\rho}d\boldsymbol{\varrho}\times\int\frac{\exp[i(\frac{1}{2}\mathbf{q}-\boldsymbol{\gamma})\cdot\mathbf{s}]}{s}d\mathbf{s}\times\int\frac{\exp[i(\frac{1}{2}\mathbf{q}+\boldsymbol{\gamma})\cdot\mathbf{s}']}{s'}d\mathbf{s}'+\text{c.c.}\quad(7)$$

In order to evaluate the $\mathbf{s}$ and $\mathbf{s}'$ integrals of Eq. (7), convergence factors $e^{-bs}$ are inserted in the integrands with the understanding that the limit $b\rightarrow0$ will be taken at the appropriate time. The use of these factors is justified by the (hitherto neglected) screening which the atomic electrons provide between the nucleus and the scattering electron. The $\mathbf{s}$ and $\mathbf{s}'$ integrals yield

$$\int d\mathbf{s}=\frac{4\pi}{(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2};\quad \int d\mathbf{s}'=\frac{4\pi}{(\boldsymbol{\gamma}+\frac{1}{2}\mathbf{q})^2}. \tag{8}$$

It is noted that the $\mathbf{s}$ integral is infinite when $\boldsymbol{\gamma}=\frac{1}{2}\mathbf{q}$ and the $\mathbf{s}'$ integral is infinite when $\boldsymbol{\gamma}=-\frac{1}{2}\mathbf{q}$. These infinities could be formally removed by retaining the convergence parameters $b$ until after the $\boldsymbol{\gamma}$ integral of Eq. (7) has been evaluated. Retention of the convergence parameters is unnecessary, however, because the contributions to the $\boldsymbol{\gamma}$ integral from the neighborhoods of $\boldsymbol{\gamma}=\pm\frac{1}{2}\mathbf{q}$ are finite.

In order to evaluate the $\boldsymbol{\varrho}$ integral of Eq. (7), a convergence factor $e^{-b\rho}$ is also used. The first term in the $\boldsymbol{\varrho}$ integral $(A)$ has the value

$$(A)=\frac{4\pi k(1+\cos\theta)}{(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2}. \tag{9}$$

The convergence parameter $b$ in Eq. (9) is not discarded at this stage as were the corresponding parameters in the $\mathbf{s}$ and $\mathbf{s}'$ integrals. The parameter $b$ serves two purposes: It prevents the denominator on the right-hand side (rhs) of Eq. (9) from vanishing at $\boldsymbol{\gamma}=\pm\frac{1}{2}\mathbf{q}$ where the denominators in Eqs. (8) vanish, and it defines a contour for the eventual $\gamma$ integration. In order to simplify the angle integrals in the second term in the $\rho$ integral $(B)$, the $\mathbf{K}_0$ in the factor $(\mathbf{K}_0\cdot\boldsymbol{\rho})/k\rho$ is expressed as the sum of a vector parallel to $\mathbf{K}_0-\boldsymbol{\gamma}$ and one perpendicular to $\mathbf{K}_0-\boldsymbol{\gamma}$. That part of $(B)$ which arises from the latter integrates to zero in the integration over the azimuthal angle when the polar axis is chosen to lie along the vector $\mathbf{K}_0-\boldsymbol{\gamma}$. The integral $(B)$ has the value

$$\begin{aligned}(B)=&\frac{4\pi}{k}\left[1+\frac{k^2+K_0^2-\gamma^2}{(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2}\right]+\frac{4\pi}{k}\left[\frac{K_0^2-\mathbf{K}_0\cdot\boldsymbol{\gamma}}{(\boldsymbol{\gamma}-\mathbf{K}_0)^2}\right]\\ &\times\left[\frac{2b^2-2ikb}{(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2}-\frac{ib}{|\boldsymbol{\gamma}-\mathbf{K}_0|}\lim_{\epsilon\to0}\right.\\ &\times\int_\epsilon^\infty\frac{\exp\{-[b-i(k-|\boldsymbol{\gamma}-\mathbf{K}_0|)]\rho\}}{\rho}d\rho\\ &+\frac{ib}{|\boldsymbol{\gamma}-\mathbf{K}_0|}\lim_{\epsilon\to0}\\ &\left.\times\int_\epsilon^\infty\frac{\exp\{-[b-i(k+|\boldsymbol{\gamma}-\mathbf{K}_0|)]\rho\}}{\rho}d\rho\right].\end{aligned} \tag{10}$$

The lower limit of the integrals in Eq. (10) was changed from 0 to $\epsilon$ because both integrands are infinite at $\rho=0$ while their sum is finite. The second term on the rhs of Eq. (10) is dropped at this point; it is of order $b$, and it does not serve either of the purposes mentioned for the retention of the convergence parameter $b$. With the relations $K_0=k\cos(\theta/2)$ and $q=2k\sin(\theta/2)$, Eqs. (9) and (10) are combined to complete the evaluation of the $\rho$ integral. The result is

$$\int d\boldsymbol{\varrho}=\frac{4\pi}{k}\left[1+\frac{4k^2-\frac{3}{4}q^2-\gamma^2}{(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2}\right]. \tag{11}$$

With the values of the $\mathbf{s}$, $\mathbf{s}'$, and $\boldsymbol{\varrho}$ integrals given by Eqs. (8) and (11), Eq. (7) becomes

$$\begin{aligned}\sigma_2^{(2)}=&\frac{8\pi e^6E^2}{(\hbar c)^3(\hbar cq)^2k}\bar{F}(\mathbf{q})\mathfrak{F}(\mathbf{q})\left\{\int\frac{G(\gamma)d\boldsymbol{\gamma}}{(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2(\boldsymbol{\gamma}+\frac{1}{2}\mathbf{q})^2}\right.\\ &\left.+\int\frac{(4k^2-\frac{3}{4}q^2-\gamma^2)G(\gamma)d\boldsymbol{\gamma}}{(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2(\boldsymbol{\gamma}+\frac{1}{2}\mathbf{q})^2[(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2]}\right\}\\ &+\text{c.c.}\end{aligned} \tag{12}$$

The angular integrals of Eq. (12) could be evaluated first, leaving $G(\gamma)$ unspecified for maximum generality. This procedure is undesirable, however, because the resulting integrands for the $\gamma$ integrals contain logarithmic factors the arguments of which vanish for some values of $\gamma$; the evaluation of the $\gamma$ integrals is then unnecessarily complicated. It is more convenient to introduce a specific function $G(\gamma)$ at this point. The $G(\gamma)$ which is used is the Fourier transform of a basic Yukawa function $g_0(D)$,

$$g_0(D)=Ne^{-\beta D}/D,\quad \beta>0. \tag{13}$$

The function $g_0(D)$ just chosen offers flexibility. By differentiation with respect to $\beta$ and by addition, one can generate from $g_0(D)$ any function of the form $g(D)=P(D)e^{-\beta D}$; $P(D)$ is a polynomial in $D^n$ with $n\geqq-1$. From Eqs. (6) and (13), $G_0(\gamma)$ is obtained:

$$G_0(\gamma)=\frac{N}{2\pi^2}\frac{1}{(\gamma^2+\beta^2)}. \tag{14}$$

With $G(\gamma)$ replaced by $G_0(\gamma)$ in Eq. (12), the cross section is designated by $\sigma_{2,0}^{(2)}$.

The next step in the evaluation of Eq. (12) is a partial-fraction decomposition of the first two denominators in each term:

$$\frac{1}{(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2(\boldsymbol{\gamma}+\frac{1}{2}\mathbf{q})^2}=\frac{1}{2(\gamma^2+\frac{1}{4}q^2)}\left[\frac{1}{(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2}+\frac{1}{(\boldsymbol{\gamma}+\frac{1}{2}\mathbf{q})^2}\right]. \tag{15}$$

The contributions to $\sigma_{2,0}^{(2)}$ arising from the two terms on the rhs of Eq. (15) are identical; therefore, the rhs of Eq. (15) is replaced by twice the first term. With Eqs.

(14) and (15), Eq. (12) becomes

$$\sigma_{2,0}^{(2)}=\frac{4Ne^6E^2}{\pi(\hbar c)^3(\hbar cq)^2k}\bar{F}(\mathbf{q})\mathfrak{F}(\mathbf{q})\times\left\{\int\frac{d\boldsymbol{\gamma}}{(\gamma^2+\beta^2)(\gamma^2+\frac{1}{4}q^2)(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2}+\int\frac{(4k^2-\frac{3}{4}q^2-\gamma^2)d\boldsymbol{\gamma}}{(\gamma^2+\beta^2)(\gamma^2+\frac{1}{4}q^2)(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2\times[(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2]}\right\}+\text{c.c.}\quad(16)$$

The integrals in Eq. (16) are evaluated with the help of two additional partial-fraction decompositions:

$$\frac{1}{(\gamma^2+\beta^2)(\gamma^2+\frac{1}{4}q^2)}=\frac{1}{(\beta^2-\frac{1}{4}q^2)}\left[\frac{1}{(\gamma^2+\beta^2)}-\frac{1}{(\gamma^2+\frac{1}{4}q^2)}\right];\quad(17)$$

$$\frac{(4k^2-\frac{3}{4}q^2-\gamma^2)}{(\gamma^2+\beta^2)(\gamma^2+\frac{1}{4}q^2)}=\frac{1}{(\beta^2-\frac{1}{4}q^2)}\left[\frac{(4k^2-\frac{1}{2}q^2)}{(\gamma^2+\frac{1}{4}q^2)}-\frac{(4k^2-\frac{3}{4}q^2+\beta^2)}{(\gamma^2+\beta^2)}\right].\quad(18)$$

It is noted that Eqs. (17) and (18) are not valid when $q=2\beta$; however, use of these equations does not introduce a discontinuity in the final result at $q=2\beta$.

With Eq. (17) the first integral $(C)$ in Eq. (16) becomes

$$(C)=\frac{1}{(\beta^2-\frac{1}{4}q^2)}\int\frac{d\boldsymbol{\gamma}}{(\gamma^2+\frac{1}{4}q^2)(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2}-\frac{1}{(\beta^2-\frac{1}{4}q^2)}\int\frac{d\boldsymbol{\gamma}}{(\gamma^2+\beta^2)(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2}.\quad(19)$$

In order to simplify the integrals in Eq. (19), an auxiliary integration is introduced by the identity[6,7]

$$\frac{1}{ab}=\frac{1}{2}\int_{-1}^{1}\frac{4dz}{[(1+z)a+(1-z)b]^2}.\quad(20)$$

With the help of Eq. (20), the angular integrals of Eq. (19) can be evaluated at once with a shift of the origin to $[(1-z)/4]\mathbf{q}$. The $\gamma$ integrals are evaluated by means of a contour integration in the complex $\gamma$ plane, the contour being the real axis and an infinite semicircle in the upper half-plane. This contour encloses one second-order pole for each integrand of Eq. (19). The $z$ integrals are evaluated last to give

$$(C)=\frac{2\pi^2}{q(\beta^2-\frac{1}{4}q^2)}\arcsin\frac{\beta^2-\frac{1}{4}q^2}{\beta^2+\frac{1}{4}q^2},\quad(21)$$

where the principal value of the arcsine is to be taken.

[6] R. H. Dalitz, Proc. Roy. Soc. (London) **A206**, 509 (1951).
[7] R. P. Feynman, Phys. Rev. **76**, 769 (1949).

With Eq. (18) the second integral $(D)$ in Eq. (16) becomes

$$(D)=\frac{(4k^2-\frac{1}{2}q^2)}{(\beta^2-\frac{1}{4}q^2)}\times\int\frac{d\boldsymbol{\gamma}}{(\gamma^2+\frac{1}{4}q^2)(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2[(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2]}-\frac{(4k^2-\frac{3}{4}q^2+\beta^2)}{(\beta^2-\frac{1}{4}q^2)}\times\int\frac{d\boldsymbol{\gamma}}{(\gamma^2+\beta^2)(\boldsymbol{\gamma}-\frac{1}{2}\mathbf{q})^2[(\boldsymbol{\gamma}-\mathbf{K}_0)^2-(k+ib)^2]}.\quad(22)$$

The angular integrals in Eq. (22) are evaluated with the help of Eq. (20), which is used to expand the last two factors in each denominator. The angle integrals can then be evaluated at once. The $\gamma$ integrals are evaluated by means of contour integrations, the contour being the same as that described above. This contour encloses three first-order poles for each integrand in Eq. (22): One of these poles lies to the right of the origin near the real axis; one, to the left of the origin near the real axis; and one, on the imaginary axis. The first two of these poles move to the real axis in the limit $b\to 0$. In the limit $b\to 0$, the residues at these two poles give imaginary contributions to $(D)$. It is now assumed that both of the form factors $F(\mathbf{q})$ and $\mathfrak{F}(\mathbf{q})$, which appear in Eq. (16), are real. This is a reasonable assumption because the first-order form factors $F(\mathbf{q})$ usually considered[8] are real, and the form factor $\mathfrak{F}(\mathbf{q})$ is not expected to be much different from $F(\mathbf{q})$. With this assumption, the imaginary part of $(D)$ is cancelled when the complex conjugate is added in Eq. (16). The real part of $(D)$ arises from the residue at the pole of each integrand which lies on the imaginary axis. After the $\gamma$ integrals have been evaluated, the limit $b\to 0$ is taken. The $z$ integrals are evaluated last to give the real part of $(D)$

$$\text{Re}(D)=-\frac{\pi^3(4k^2-\frac{1}{2}q^2)}{kq^2(\beta^2-\frac{1}{4}q^2)}+\frac{\pi^2(4k^2-\frac{3}{4}q^2+\beta^2)}{k(\beta^4-\frac{1}{16}q^4)}\arctan\frac{2\beta k}{(\beta^2-\frac{1}{4}q^2)}.\quad(23)$$

In Eq. (23), the arctangent lies between 0 and $\pi$.

With Eqs. (21) and (23) and the assumption that the form factors are real, Eq. (16) becomes

$$\sigma_{2,0}^{(2)}=\frac{16\pi Ne^6E^2}{(\hbar c)^3(\hbar cq)^2k}F(\mathbf{q})\mathfrak{F}(\mathbf{q})\left[\frac{1}{q(\beta^2-\frac{1}{4}q^2)}\arcsin\frac{\beta^2-\frac{1}{4}q^2}{\beta^2+\frac{1}{4}q^2}-\frac{(4k^2-\frac{1}{2}q^2)\pi}{2kq^2(\beta^2-\frac{1}{4}q^2)}+\frac{(4k^2-\frac{3}{4}q^2+\beta^2)}{2k(\beta^4-\frac{1}{16}q^4)}\times\arctan\frac{2\beta k}{(\beta^2-\frac{1}{4}q^2)}\right].\quad(24)$$

[8] See, for example, L. I. Schiff, Phys. Rev. **92**, 988 (1953).

### III. TWO-PROTON CROSS SECTION FOR A PARTICULAR CHARGE DENSITY

The particular function $g(D)$ with which the two-proton cross section is calculated is

$$g(D)=\frac{(Z-1)}{8\pi r_0^3(0.7904)}\left[e^{-7D/8r_0}-e^{-9D/8r_0}+\frac{1}{40}e^{-10D/r_0}\right]. \quad (25)$$

The normalizing factor $N$, which multiplies the square brackets in Eq. (25), was determined by the condition $\int g(D)d\mathbf{D}=(Z-1)$. This normalization of $g(D)$ implies that $f(S)$ is normalized by the condition $\int f(\mathbf{S})d\mathbf{S}=Z$.[9] The value of $g(0)$ is 1/3.69 times the peak value $g(r_0)$; $g(D)$ has zero slope at $D=0$. The ratio $g(0)/g(r_0)$ was chosen in an attempt to take some account of the fact that, if the protons in the nucleus are taken in pairs, some pairs will be in a triplet spin state and some will be in a singlet spin state. On account of the exclusion principle, $g(0)$ is zero for the triplet pairs; for the singlet pairs, $g(0)$ is not zero. The function $g(D)$ falls off fairly rapidly as $D$ approaches the magnitude of the nuclear radius in order to take some account of the possibility that either $\mathbf{R}_a$ or $\mathbf{R}_b$ or both may be outside the charge distribution. Whether or not $\mathbf{R}_a$ or $\mathbf{R}_b$ is outside the charge distribution depends on the vectors $\mathbf{D}=\mathbf{R}_b-\mathbf{R}_a$ and $\mathbf{S}=\frac{1}{2}(\mathbf{R}_a+\mathbf{R}_b)$. If the center of gravity of $\mathbf{R}_a$ and $\mathbf{R}_b$ is near the edge of the nucleus, some orientations of $\mathbf{D}$ may mean that both $\mathbf{R}_a$ and $\mathbf{R}_b$ are inside the nucleus, while other orientations may mean that either $\mathbf{R}_a$ or $\mathbf{R}_b$ is outside the nucleus. The function $g(D)$ makes no distinction between these two cases. The two-proton cross section is calculated here for $r_0=1.315\times10^{-13}$ cm.

The basic two-proton cross section $\sigma_{2,0}^{(2)}$, which is given by Eq. (24), was calculated with $g_0=(N/D)e^{-\beta D}$. The cross section $\sigma_2^{(2)}$ for the function $g(D)$ given by Eq. (25) is generated from $\sigma_{2,0}^{(2)}$ by differentiation with respect to $\beta$.

$$\sigma_2^{(2)}=\left[-\frac{\partial}{\partial\beta}\sigma_{2,0}^{(2)}\right]_{7/8r_0}-\left[-\frac{\partial}{\partial\beta}\sigma_{2,0}^{(2)}\right]_{9/8r_0}+\frac{1}{40}\left[-\frac{\partial}{\partial\beta}\sigma_{2,0}^{(2)}\right]_{10/r_0}. \quad (26)$$

The subscripts on the square brackets indicate the values of $\beta$ for which the functions are to be evaluated after differentiation. With the normalizing factor $N=(Z-1)/[8\pi r_0^3(0.7904)]$, Eqs. (25) and (26) are combined to give

$$\sigma_2^{(2)}=\frac{4e^4E^2}{(\hbar cq)^4}F(\mathbf{q})\mathfrak{F}(\mathbf{q})\left[\left(\frac{e^2}{\hbar c}\right)\frac{(Z-1)}{1.5808kr_0}Q(qr_0)\right], \quad (27)$$

where

$$Q(qr_0)=U(7/8)-U(9/8)+(1/40)U(10). \quad (28)$$

The function $U(\beta)$ is

$$U(\beta)=q^2\left\{\frac{1}{[\beta^2-\frac{1}{4}q^2]}\left(\frac{[4k^2-\frac{1}{2}q^2]+[\beta^2-\frac{1}{4}q^2]}{4\beta^2k^2+[\beta^2-\frac{1}{4}q^2]^2}\right)+\frac{1}{[\beta^2-\frac{1}{4}q^2]^2}\left(-\frac{\pi\beta[4k^2-\frac{1}{2}q^2]}{kq^2}+\frac{2\beta}{q}\arcsin\frac{\beta^2-\frac{1}{4}q^2}{\beta^2+\frac{1}{4}q^2}\right)-\frac{1}{[\beta^4-\frac{1}{16}q^4]}+\frac{4\beta^3[4k^2-\frac{1}{2}q^2]+2\beta[\beta^2-\frac{1}{4}q^2]^2}{2k[\beta^4-\frac{1}{16}q^4]^2}\times\arctan\frac{2\beta k}{\beta^2-\frac{1}{4}q^2}\right\}. \quad (29)$$

In Eq. (29), the principal value of the arcsine is to be taken, and the arctangent lies in the range 0 to $\pi$. In the function $U(\beta)$, $q$, $k$, and $\beta$ are expressed in units or $r_0$; $U(\beta)$ is dimensionless.

The function $Q(qr_0)$ has been evaluated for $kr_0=4.00$; for $r_0=1.315\times10^{-13}$ cm, this means an electron energy of 600 Mev. The maximum energy presently available

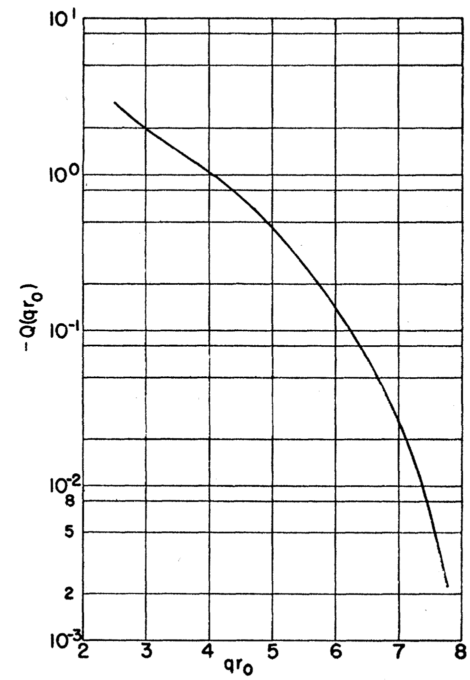


FIG. 1. The negative of the function $Q(qr_0)$ which appears as a factor in the two-proton cross section.

[9] See reference 2, Eqs. (6) and (18).

for electron-scattering experiments is about 600 Mev. For $kr_0=4.00$, the de Broglie wavelength is roughly the same size as the correlation distance $r_0$, and the cross section is expected to be sensitive to changes in the charge density over a distance of this size. The function $-Q(qr_0)$ is graphed in Fig. 1. For $qr_0=2.50$, the scattering angle $\theta$ is 36.5°; for $qr_0=7.75$, $\theta=151.3°$. The contribution to $Q(qr_0)$ from $U(10)$ is less than three percent, being less than one percent for $qr_0\leqq 7.00$; the function $U(10)$ arises from that part of $g(D)$ which fixes the value of $g(0)$.

## IV. DISCUSSION

The first-order cross section $\sigma^{(1)}$ and the one-proton part of the second-order cross section $\sigma_1^{(2)}$ depend upon the square of the first-order form factor $F(\mathbf{q})$.[2] The two-proton part of the second-order cross section $\sigma_2^{(2)}$ depends upon the product of the form factors $F(\mathbf{q})$ and $\mathfrak{F}(\mathbf{q})$. If $F(\mathbf{q})$ and $\mathfrak{F}(\mathbf{q})$ were the same, then $\sigma^{(1)}$, $\sigma_1^{(2)}$, and $\sigma_2^{(2)}$ would have the common factor $[4e^4E^2/(\hbar cq)^4][F(\mathbf{q})]^2$; differences in magnitude and in dependence on the scattering angle among the three parts of the cross section would be apparent. In general, however, $F(\mathbf{q})$ and $\mathfrak{F}(\mathbf{q})$ are not the same. The difference to be expected in one case is indicated below.

From Eq. (5) it is apparent that $\mathfrak{F}(\mathbf{q})$ is related to $f(\mathbf{S})$ in the same way that $F(\mathbf{q})$ is related to the one-proton charge density $\rho_{00}^{(1)}(\mathbf{R}_a)$.[10] Once the function $g(D)$ has been chosen, the relation between $\rho_{00}^{(1)}(\mathbf{R}_a)$ and $f(\mathbf{S})$ and, therefore, between $F(\mathbf{q})$ and $\mathfrak{F}(\mathbf{q})$ is fixed by the condition[11]

$$\int f(\mathbf{S})g(D)d\mathbf{R}_b=(Z-1)\rho_{00}^{(1)}(\mathbf{R}_a). \tag{30}$$

If a form for $f(\mathbf{S})$ is chosen, $\rho_{00}^{(1)}(\mathbf{R}_a)$ is determined by Eq. (30). To give an indication of the relation between $f(\mathbf{S})$ and $\rho_{00}^{(1)}(\mathbf{R}_a)$ fixed by Eqs. (25) and (30), $f(\mathbf{S})$ is taken to be a Gaussian function whose volume integral is normalized to $Z$.

$$f(\mathbf{S})=\frac{Z\alpha^3}{(\pi)^{\frac{3}{2}}}\exp(-\alpha^2S^2). \tag{31}$$

With Eqs. (25) and (31), Eq. (30) yields

$$\rho_{00}^{(1)}(\mathbf{R}_a)=\frac{Z}{2\pi r_0^3(0.7904)\alpha R_a}\times\left[T\left(\frac{7}{8r_0}\right)-T\left(\frac{9}{8r_0}\right)+\frac{1}{40}T\left(\frac{10}{r_0}\right)\right], \tag{32}$$

where

$$T(\beta)=\exp(\beta^2/\alpha^2)\left\{e^{2\beta R_a}\left[\frac{\beta}{\alpha}+\alpha R_a\right]\left[1-\operatorname{erf}\left(\frac{\beta}{\alpha}+\alpha R_a\right)\right]-e^{-2\beta R_a}\left[\frac{\beta}{\alpha}-\alpha R_a\right]\left[1-\operatorname{erf}\left(\frac{\beta}{\alpha}-\alpha R_a\right)\right]\right\}. \tag{33}$$

$\operatorname{Erf}x=(2/\sqrt{\pi})\int_0^x \exp(-t^2)dt$ is the error function. The one-proton charge density given by Eq. (32) is finite at $R_a=0$ and behaves like $(1/R_a)\exp(-\alpha^2R_a^2)$ for large $R_a$. The choice of the simple functions $g(D)$ and $f(\mathbf{S})$, which are given by Eqs. (25) and (31), does not imply an unreasonable one-proton charge density.

On account of errors arising from the edge effects discussed below Eq. (25), Eq. (27) cannot be applied with any certainty to light nuclei. If the difference between $F(\mathbf{q})$ and $\mathfrak{F}(\mathbf{q})$ is neglected for heavy nuclei, $\sigma_2^{(2)}$ can be compared with $\sigma^{(1)}$ and $\sigma_1^{(2)}$, which are given by[2]

$$\begin{aligned}\sigma^{(1)}&=\frac{4e^4E^2}{(\hbar cq)^4}|F(\mathbf{q})|^2\cos^2(\theta/2);\\ \sigma_1^{(2)}&=\frac{4e^4E^2}{(\hbar cq)^4}|F(\mathbf{q})|^2\left\{\frac{\pi e^2}{\hbar c}[\sin(\theta/2)-\sin^2(\theta/2)]\right\}.\end{aligned} \tag{34}$$

In the range $2.50\leqq qr_0\leqq 7.75$, the absolute value of the ratio $\sigma_2^{(2)}/\sigma^{(1)}$ is a decreasing function of $qr_0$. The ratio $\sigma_2^{(2)}/\sigma^{(1)}$ for lead at $qr_0=2.50$ is $-0.30$; the ratio $\sigma_2^{(2)}/\sigma_1^{(2)}$ is $-56$. Since the two-proton cross section is negative, its addition to the first-order cross section substantially reduces the cross section at small scattering angles. Perhaps part of this reduction can be observed near the maxima of the first-order cross section, where this cross section describes the scattering fairly well.[12] In the usual phase-shift analysis of electron scattering,[12] that part of $\sigma_2^{(2)}$ which arises from the $n=0$ term in Eq. (1) is included in the cross section; consequently $\sigma_2^{(2)}$ cannot be added directly to the phase-shift cross section.

In conclusion, the correlation contribution to the scattering, at least as estimated in this paper, does not have a distinctive or easily recognized form. It is possible, although unlikely, that this situation would be altered if $g(D)$ were chosen to have a different shape. It is also possible that the absence of characteristic effects is the result of the choice of the separable form (4) for the two-proton charge density.

## V. ACKNOWLEDGMENTS

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[10] See reference 2, Eq. (7).
[11] See reference 2, Eq. (18).
[12] Yennie, Ravenhall, and Wilson, Phys. Rev. 95, 500 (1954).