vestigations of the fission yields of U^{238} when bombarded by 340-Mev protons have been carried out by Folger *et al.*[3] Upon integrating these yields they find a fission cross section of 2.0×10^{-24} cm^2, which is somewhat higher than the result reported here.

The bismuth fission cross section at 340 Mev as measured in this experiment is in fair agreement with the value of $0.239\pm0.03\times10^{-24}$ cm^2 obtained by Biller[4] by integration of the fission yields.

The following conclusions may be drawn from this experiment:

(a) The high-energy fission cross sections of uranium seem to be independent of whether U^{235} or U^{238} is used.

(b) The relative fission probabilities as well as the fission cross sections seem to decrease strongly with decreasing atomic number.

(c) The fission cross sections of uranium and thorium seem to be fairly constant as a function of proton energy in the energy region of 100 to 340 Mev.

(d) On the average approximately one-third of the proton's initial momentum is transferred to the fissioning nucleus at proton energies of 190 and 340 Mev.

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Calculations on the Cascade Theory of Showers*

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The solution of the cascade equations, obtained previously by Bhabha and Chakrabarty, is rearranged in a form suitable for numerical calculations. Although the solution is still in the form of an infinite series, the first term alone gives practically the entire contribution to the number of particles in a shower for all values of the energy of the shower particles. The results are compared with the values given previously by Bhabha and Chakrabarty and also by Snyder. The defects in the analysis of Snyder are discussed. Values of $N(E,t)$, the total number of particles in a shower having energies greater than E, are obtained for different values of E, t, and E_0. By the evaluation of a single integral it is now possible to obtain the values of $N(E,t)$ for any value of E in the entire range $(0,E_0)$, and also the nature of the energy spectrum of the shower electrons at different depths. Asymptotic values to which $P(E,t)$ and $N(E,t)$ merge, when E tends to zero and infinity, are derived from the general expression. It is shown that the values of $N_0(t)+N_2(t)$, derived previously by Bhabha and Chakrabarty, is a fair approximation to the value of $N(E,t)$ if we take $E=2mc^2$.

I. INTRODUCTION

THE development of the theory of cascade showers has been made by various authors at different times after the original works of Bhabha and Heitler[1] and Carlson and Oppenheimer.[2] An accurate estimate of the number of shower particles or photons and of their energy spectrum is very important for the interpretation of the different results of observations. In previous papers the solution of the cascade equations has been obtained by Bhabha and Chakrabarty[3] in the form of an infinite series, and it was shown that the first term in the series solution alone gives the major contribution. The subsequent terms are of importance only at large thicknesses and at the tail end of the shower where their contribution is mainly to the number of electrons whose energy is much smaller than the critical energy, a region where the cross sections for the radiation loss and pair creation are not well represented by their asymptotic forms assumed in the analysis. Thus the subsequent terms in the series are negligible, except in the region where the basic physical assumptions are not precise. The rate of ionization loss assumed in the analysis should also be modified in that region, which will again considerably reduce the number of particles in the actual shower. It was thus suggested that the figures given in (B) were sufficiently accurate unless improvements could be introduced in the basic physical assumptions made in deriving the cascade equations. Snyder[4] has modified the previous calcu-

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[1] H. J. Bhabha and W. Heitler, Proc. Roy. Soc. (London) **A159**, 432 (1937).

[2] J. F. Carlson and J. R. Oppenheimer, Phys. Rev. **51**, 220 (1937).

[3] H. J. Bhabha and S. K. Chakrabarty, Proc. Roy. Soc. (London) **181**, 267 (1943), hereafter denoted as (A); Phys. Rev. **74**, 1352 (1948), hereafter denoted as (B).

[4] H. S. Snyder, Phys. Rev. **76**, 1563 (1949).

lations of Snyder[5] and Serber[6] and has obtained a solution which, except for notation, is the same as derived in (B). He has, however, later rearranged the expression for the number of particles in a form which is not correct, as will be shown later, for any value of E, the energy of the shower particles, except when E is zero, in which case the infinite series reduces to the first term. In the derivation of the energy spectrum of the shower particles, all the other terms appear except the first in the series, and hence the results obtained by Snyder[4] differ entirely from those deduced in (B). The major contribution to $P(E,t)$, the number of particles at depth t with energies in the range $(E, E+dE)$, arises in Snyder's analysis from the second term in the series for $N(E,t)$, the total number of particles at depth t having energies greater than E. But it is easy to see that the first two terms in the series for $N(E,t)$ give a negative value for $N(E,t)$ even for E of the order of β/e.

In the present paper we rearrange the solution obtained in (B) in a form suitable for numerical calculations. Even now the solution is an infinite series, but the first term alone gives practically the entire contribution. In a later section we give the values of $N(E,t)$ for some values of E and t and for different energies of the primary particle. We also derive the values of $P(E,t)$ for some values of E and for a particular value of the energy of the primary particle and compare them with the results obtained by Snyder. We also give the reasons for the existence of the difference between the results of Snyder and that of the present paper.

II. MATHEMATICAL SOLUTION

As in (A) and (B), we denote by $P(E,t)dE$ the mean number of electrons and positrons in the energy range $(E, E+dE)$, to be found in a cascade shower at a depth t in radiation units, and by $Q(E,t)$ the corresponding expression for the number of quanta. It has been shown in [(B) Eq. (14)] that the solution of the cascade equations can be put in the form

$$P(E,t)=\frac{1}{2\pi i E_0}\int_C\left(\frac{E_0}{E}\right)^s \times\left\{\sum_{n=0}^{\infty}\left(-\frac{\beta}{E}\right)^n\frac{\Gamma(s+n)}{\Gamma(s)}\psi_n(s,t)\right\}ds, \quad (1)$$

where $\psi_n(s,t)$, λ_s, and μ_s have been defined in (B). By applying a Laplace transformation in t, it can be shown that

$$P(E,t)=-\frac{1}{4\pi^2\beta}\int ds\int dr\left(\frac{E_0}{\beta}\right)^{s-1}\sum_{n=0}^{\infty}(-1)^n \times\left(\frac{\beta}{E}\right)^{s+n}\frac{\Gamma(s+n)}{\Gamma(s)}e^{rt}\phi_n(s,r), \quad (2)$$

where

$$\phi_n(s,r)=\phi_{n-1}(s,r)\phi_0(s+n,\, r) =\prod_{i=0}^{n}\phi_0(s+i,\, r), \quad (3)$$

and

$$\phi_0(s,r)=(D+r)/(r+\lambda_s)(r+\mu_s).$$

A similar expression for $P(E,t)$ was also derived in a previous paper.[7]

We now define $\phi_p(s,r)$ as a function of p, s, r, such that

$$\phi_p(s,r)=\lim_{N\to\infty}\{\phi_0(s+N+1,\, r)\}^{p+1} \times\prod_{i=0}^{N}\frac{\phi_0(s+i,\, r)}{\phi_0(s+p+i+1,\, r)}, \quad (4)$$

where p is any number. $\phi_p(s,r)$ then satisfies the difference equation (3). We thus have

$$\sum_{n=0}^{\infty}(-1)^n\left(\frac{\beta}{E}\right)^{s+n}\Gamma(s+n)\phi_n(s,r) =\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma(s+p)\Gamma(p+1)\Gamma(-p) \times\left(\frac{\beta}{E}\right)^{s+p}\phi_p(s,r)dp, \quad (5)$$

where $-1<\gamma<0$. Substituting in (2) and simplifying, we get

$$P(E,t)=\sum_{m=0}^{\infty}P_m(E,t), \quad (6)$$

where

$$\begin{aligned}P_m(E,t)=&\frac{1}{2\pi i\beta}\int_{\sigma-i\infty}^{\sigma+i\infty}ds\left(\frac{E_0}{\beta}\right)^{s-1}\frac{1}{2\pi i} \times\int_{\gamma-i\infty}^{\gamma+i\infty}dp\frac{\Gamma(s+p)\Gamma(-p)}{\Gamma(s)}\left(\frac{\beta}{E}\right)^{s+p}\\&\times\left[\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}\exp(-\lambda_{s+m}t)G_m(s,p) +\frac{\mu_{s+m}-D}{\mu_{s+m}-\lambda_{s+m}}\exp(-\mu_{s+m}t)F_m(s,p)\right]\\&=P_m{}^\lambda(E,t)+P_m{}^\mu(E,t)\ \text{(say)},\end{aligned}$$

where

$$G_m(s,p)=\Gamma(p+1)\lim_{N\to\infty}\{\phi_0(s+N+1,\, -\lambda_{s+m})\}^{p+1} \times\prod_{\substack{i=0\\ \neq m}}^{N}\phi_0(s+i,\, -\lambda_{s+m})\prod_{i=0}^{N}1/\phi_0(s+p+i+1,\, -\lambda_{s+m}),$$

[5] H. S. Snyder, Phys. Rev. **53**, 960 (1938).
[6] R. Serber, Phys. Rev. **54**, 317 (1938).
[7] S. K. Chakrabarty, Nature **158**, 166 (1946).

and

$$F_m(s,p)=\Gamma(p+1)\lim_{N\to\infty}\{\phi_0(s+N+1,\,-\mu_{s+m})\}^{p+1}$$

$$\times\prod_{\substack{i=0\\ \neq m}}^{N}\phi_0(s+i,\,-\mu_{s+m})\prod_{i=0}^{N}1/\phi_0(s+p+i+1,\,-\mu_{s+m}).\quad(7)$$

The solution given by (6) of the diffusion equation of the cascade theory is also in the form of an infinite series, but it differs from the expansions used in previous papers, in particular in (B) [Eqs. (14) and (19)], in at least one important point. From the values of λ_s for real values of s given in (A) it is clear that the real part of λ_s increases with s, and hence the contribution of terms containing $\exp(-\lambda_{s+1}t)$ is much smaller than that containing $\exp(-\lambda_s t)$. Similarly the term containing $\exp(-\lambda_{s+2}t)$ is smaller than that containing $\exp(-\lambda_{s+1}t)$, and so on. In the expansions used in B(14) and B(19), each term of the series contained $\exp(-\lambda_s t)$; but in the form (6) and (7) given above $P_m(E,t)$ contains $\exp(-\lambda_{s+m}t)$, and hence $\exp(-\lambda_s t)$ is contained only in $P_0(E,t)$; thus the higher terms in the series are insignificant as compared to $P_0(E,t)$. But the form of the solution given in (A) and (B) shows clearly one of the physical aspects of the shower production, *viz.*, that "if we look for electrons with energy E at depth t, the majority of them will have been created at $t-g$ and then had an energy $E+\beta g$, where g is of the order of unity."

In the numerical evaluation of (6) we have evaluated the double integral by the saddle-point method. Snyder has carried out the p integration by the method of residues; each term $P_m(E,t)$ and similarly $N_m(E,t)$, where

$$N(E,t)=\int^{\infty}P(E,t)dE$$

$$=\sum_{m=0}^{\infty}N_m(E,t),\quad(8)$$

has been obtained in terms of an infinite series. In the next section we evaluate the values of $N(E,t)$ for different values of E and t and compare them with the results given by Snyder and also with the results given in (B).

III. NUMERICAL RESULTS

It has been pointed out in (A) that for all but small t, the part of $P_m(E,t)$ containing $\exp(-\mu_{s+m}t)$ makes a negligible contribution as compared to that containing $\exp(-\lambda_{s+m}t)$, although in (B) we have retained both these terms in evaluating $N_m(E,t)$. A comparison of the values of $N_0(E,t)$ given in (B) and that in the previous paper[8] will indicate the order of the contribution of the terms containing $\exp(-\mu_{s+m}t)$. In the present paper we have therefore neglected the terms containing $\exp(-\mu_{s+m}t)$ and have evaluated the double integral by the saddle-point method in the usual way.

From (6) we have easily

$$N(E,t)=\sum_{m=0}^{\infty}N_m(E,t),\quad(9)$$

where

$$N_m(E,t)$$

$$=\frac{1}{2\pi i}\int ds\left(\frac{E_0}{\beta}\right)^{s-1}\frac{1}{2\pi i}\int\frac{\Gamma(s+p-1)\Gamma(-p)}{\Gamma(s)}$$

$$\times\left(\frac{\beta}{E}\right)^{s+p-1}\left[\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}e^{-\lambda_{s+m}t}G_m(s,p)\right.$$

$$\left.+\frac{\mu_{s+m}-D}{\mu_{s+m}-\lambda_{s+m}}e^{-\mu_{s+m}t}F_m(s,p)\right]dp$$

$$\approx\frac{1}{2\pi i}\int ds\left(\frac{E_0}{\beta}\right)^{s-1}\frac{1}{2\pi i}\int\frac{\Gamma(s+p-1)\Gamma(-p)}{\Gamma(s)}$$

$$\times\left(\frac{\beta}{E}\right)^{s+p-1}\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}e^{-\lambda_{s+m}t}G_m(s,p)dp\quad(10)$$

$$=N_m^{\lambda}(E,t)\ \text{(say)}.\quad(11)$$

$G_m(s,p)$ is a smooth function of s and p for every value of m and its exact value for integral values of p, positive or negative can be easily obtained. For later calculations we need the values of $G_0(s,p)$; exact values for some integral values of p are given below:

$$G_0(s,0)=1;$$

$$G_0(s,1)=\frac{D-\lambda_s}{(\lambda_{s+1}-\lambda_s)(\mu_{s+1}-\lambda_s)};$$

$$G_0(s,2)=2\frac{D-\lambda_s}{(\lambda_{s+2}-\lambda_s)(\mu_{s+2}-\lambda_s)}G_0(s,1);$$

$$G_0(s,3)=3\frac{D-\lambda_s}{(\lambda_{s+3}-\lambda_s)(\mu_{s+3}-\lambda_s)}G_0(s,2);$$

$$G_0(s,-1)=\frac{\mu_s-\lambda_s}{D-\lambda_s}\lambda_s';$$

$$G_0(s,-2)=\frac{(\lambda_s-\lambda_{s-1})(\mu_{s-1}-\lambda_s)}{D-\lambda_s}G_0(s,-1);$$

$$G_0(s,-3)=\frac{1}{2}\frac{(\lambda_s-\lambda_{s-2})(\mu_{s-2}-\lambda_s)}{D-\lambda_s}G_0(s,-2).$$

For numerical computations we have utilized these exact values of $G_0(s,p)$ for integral values of p and different s such that $(s+p)>0$ and have obtained the values of $G_0(s,p)$ for intermediate fractional values of p

[8] H. J. Bhabha and S. K. Chakrabarty, Proc. Indian Acad. Sci. **15**, 464 (1942).

TABLE I. Values of $\log_e G_0(s,p)$ as functions of s and p; $\log_e G_0(s,0)=0$.

$s\diagdown p$	-2.0	-1.0	1.0	2.0	3.0	$-(s-1)$
1.3		2.3002	−0.9357	−1.3917	−1.5564	0.4883
1.5		1.6313	−0.6332	−0.8264	−0.7657	0.6359
1.8		1.0679	−0.3323	−0.3054	−0.0426	0.7689
2.0		0.8268	−0.1994	−0.0686	0.2846	0.8268
2.2	3.2023	0.6471	−0.0983	0.1109	0.5316	0.9423
2.5	2.0773	0.4532	0.0131	0.3065	0.8003	1.0434
2.8	1.4974	0.3181	0.0901	0.4409	0.9859	1.1549
3.0	1.2338	0.2487	0.1320	0.5131	1.0764	1.2338

by interpolation. It may be mentioned here that the function $K_\mu(y,-y)$ used by Snyder is the same as $G_0(s,-s+1)$, but in the evaluation of $K_\mu(y,-y)$ he has used a different function and the values are slightly different from what we have obtained, although their effect on the calculation of $N_0(0,t)$ is not very significant.

In Table I we have given the values of the function $G_0(s,p)$ for some different values of s and p in the region of the saddle point, from which it will appear that for the purpose of obtaining the saddle point in evaluating (11) the variation of $G_m(s,p)$ with s and p may be neglected, without introducing appreciable error in the final result.

We define a function $\omega_m(s,p)$ such that

$$\omega_m(s,p)=(s-1)y_0-zy-\lambda_{s+m}t+\log\frac{\Gamma(z)\Gamma(-p)}{\Gamma(s)}+\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}, \quad (12)$$

where

$$y_0=\log(E_0/\beta),\quad y=\log(E/\beta);\quad z=s+p-1,$$

and determine the double saddle point s_m, p_m through the equations

$$\partial\omega_m/\partial p=0,\quad \partial\omega_m/\partial s=0. \quad (13)$$

Then, as shown in (B), we get

$$N_m^\lambda(E,t)=\frac{1}{2\pi}\exp\{\omega_m(s_m,p_m)\}G_m(s_m,p_m)\times\left\{\frac{\partial^2\omega_m}{\partial p^2}\frac{\partial^2\omega_m}{\partial s^2}-\left(\frac{\partial^2\omega_m}{\partial p\partial s}\right)^2\right\}^{-\frac{1}{2}}, \quad (14)$$

where s_m, p_m, are the roots of the equations,

$$y_0-y-\lambda_{s+m}'t+\frac{d}{dz}\log\Gamma(z)-\frac{d}{ds}\log\Gamma(s)+\frac{d}{ds}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}=0, \quad (15)$$

$$-y+\frac{d}{dz}\log\Gamma(z)+\frac{d}{dp}\log\Gamma(-p)=0.$$

for assumed values of y_0, y, and t. We have calculated the values of $N_0(E,t)$ from (14) for some different values of y_0, t, and y and they are given in Table II. It can be easily shown that the subsequent terms in (9) make only an insignificant contribution to $N(E,t)$. From the above expressions the values of $N(E,t)$ when E tends to zero can also be obtained by taking the asymptotic values of ω_m, $\partial^2\omega_m/\partial p^2$, $\partial^2\omega_m/\partial s^2$, $\partial^2\omega_m/\partial p\partial s$, when z tends to zero; since when E tends to zero y tends to $-\infty$ and z must tend to zero, we then have

$$\frac{\partial\omega_m}{\partial p}=-y-(1/z)+\psi(z+1)-\psi(-p)=0,$$

$$\frac{\partial\omega_m}{\partial s}=y_0-\lambda_{s+m}'t-\frac{1}{(s-1)}+\frac{d}{ds}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}=0,$$

$$\frac{\partial^2\omega_m}{\partial p^2}=(1/z^2)+\psi'(z+1)+\psi'(-p); \quad (16)$$

$$\frac{\partial^2\omega_m}{\partial s^2}=(1/z^2)-\lambda_{s+m}''t+\psi'(z+1)-\psi'(s)+\frac{d^2}{ds^2}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}},$$

$$\frac{\partial^2\omega_m}{\partial p\partial s}=(1/z^2)+\psi'(z+1),$$

where

$$\psi(z)=\frac{d}{dz}\log\Gamma(z);\quad \psi'(z)=\frac{d}{dz}\psi(z),$$

so that we get

$$\left\{\frac{\partial^2\omega_m}{\partial p^2}\frac{\partial^2\omega_m}{\partial s^2}-\left(\frac{\partial^2\omega_m}{\partial p\partial s}\right)^2\right\}^{\frac{1}{2}}=\frac{1}{z}\left[-\lambda_{s+m}''t+\psi'(-p)-\psi'(s)+\frac{d^2}{ds^2}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}+O(z^2)\right]^{\frac{1}{2}}.$$

Consequently, when z tends to zero we get

$$N(0,t)=\sum_{m=0}^{\infty}N_m(0,t), \quad (17)$$

where

$$N_m(0,t)\approx N_m^\lambda(0,t)=(1/2\pi)\exp\{\omega_m^0(s_m^0,-s_m^0+1)\}\times G_m(s_m^0,-s_m^0+1)\{L_m(s_m^0,t)\}^{-\frac{1}{2}}, \quad (18)$$

and s_m^0 is the root of the equation

$$y_0-\lambda_{s+m}'t-\frac{1}{(s-1)}+\frac{d}{ds}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}=0,$$

and

$$\omega_m^0(s,-s+1)=(s-1)y_0+1-\lambda_{s+m}t-\log(s-1)+\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}, \quad (19)$$

$$L_m(s,t)=-\lambda_{s+m}''t+\frac{1}{(s-1)^2}+\frac{d^2}{ds^2}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}.$$

TABLE II. Values of $N_0(E,t)$ for different values of E_0, E, and t. $E_0=\beta e^{y_0}$; $E=\beta e^{y}$.

y_0	$y \diagdown t$	0.5	1.0	2.0	4.0	6.0	8.0	10.0	12.0	15.0	20.0
2	$-\infty$	1.79	1.96	1.74	0.701						
	-2	2.26	4.00	6.24	5.60	2.97	1.26				
4	-4	2.48	4.52	7.67	7.71	4.37	1.90				
	$-\infty$	2.57	4.89	8.51	9.44	5.69	2.66	0.797			
	-2	2.83	5.96	14.49	33.11	32.00	20.89	10.59	4.73	1.45	
6	-4	3.02	6.68	18.41	42.66	42.25	29.17	16.05	7.63	2.00	
	$-\infty$	3.55	8.71	24.00	52.59	55.70	40.74	22.39	10.47	2.85	
	-2	2.92	8.04	32.00	125.9	197.2	195.0	144.5	85.10	29.50	3.80
8	-4	3.39	8.91	35.50	160.3	261.0	263.0	201.8	125.0	43.70	5.75
	$-\infty$	4.03	11.22	50.00	201.8	346.7	358.9	285.1	173.8	64.60	8.71
	-2		7.28	45.80	338.6	891.3	1288	1259	952.0	436.5	80.30
10	-4		10.47	57.60	457.1	1175	1700	1718	1380	724.4	127.9
	$-\infty$		16.18	87.30	651.0	1549	2291	2388	1950	1072	197.2

The first term in the series (17), *viz.*, $N_0(0,t)$, is easily seen to be equal to

$$\frac{e}{(2\pi)^{\frac{1}{2}}}G_0(s_0^0,\,-s_0^0+1)N_\lambda(\beta,t), \tag{20}$$

where $N_\lambda(E,t)$ is the number of particles in a shower having energies greater than E if ionization loss is neglected, and was calculated previously.[9] Thus, since $N_0(0,t)$ gives practically the entire contribution of $N(0,t)$, it is evident that the effect of introducing the ionization loss is to multiply the number of particles with energies greater than β (if ionization loss is neglected in the calculation) at each depth by the factor $e(2\pi)^{-\frac{1}{2}}G_0(s_0^0,\,-s_0^0+1)$, which, however, will be different at different depth since s_0 depends on t according to (19). The values of $N_0(0,t)$ can be easily obtained from (18) and (19) using the last column of Table I. Some values have been given in Table II.

The energy spectrum of the particles in a cascade shower can be easily derived from (6) by evaluating the integral by the saddle-point method. We have as before

$$E_0P_m(E,t)\approx E_0P_m^{\lambda}(E,t)$$
$$=\frac{1}{2\pi}\exp\{\bar{\omega}_m(s_m,p_m)\}G_m(s_m,p_m)\times\left\{\frac{\partial^2\bar{\omega}_m}{\partial p^2}\frac{\partial^2\bar{\omega}_m}{\partial s^2}-\left(\frac{\partial^2\bar{\omega}_m}{\partial p\partial s}\right)^2\right\}^{-\frac{1}{2}}, \tag{21}$$

where

$$\bar{\omega}_m(s,p)=sy_0-(s+p)y-\lambda_{s+m}t+\log\frac{\Gamma(s+p)\Gamma(-p)}{\Gamma(s)}+\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}. \tag{19a}$$

s_m and p_m are the roots of the equations

$$\partial\bar{\omega}_m/\partial p=0;\quad \partial\bar{\omega}_m/\partial s=0.$$

It is easy to see that the saddle point lies within the extreme values $(s+p)\to 0$ and $(-p)\to 0$ and these extreme values correspond to the cases when E tends to zero and ∞, respectively. Different values of s_m will give different values of E_0 and t. When $(-p)\to 0$,

$$\frac{\partial\bar{\omega}_m}{\partial p}=-y+\psi(s+p)-\psi(1-p)+[1/(-p)]\approx y+(1/p)=0;\quad y(-p)=1,$$

$$\frac{\partial\bar{\omega}_m}{\partial s}=y_0-y-\lambda_{s+m}'t+\frac{d}{ds}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}=0;$$

$$\frac{\partial^2\bar{\omega}_m}{\partial p^2}=\psi'(s+p)+\psi'(1-p)+[1/(-p)^2];$$

$$\frac{\partial^2\bar{\omega}_m}{\partial s^2}=-\lambda_{s+m}''t+\psi'(s+p)-\psi'(s)+\frac{d^2}{ds^2}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}},$$

$$\frac{\partial^2\bar{\omega}_m}{\partial p\partial s}=\psi'(s+p),$$

$$\left\{\frac{\partial^2\bar{\omega}_m}{\partial p^2}\cdot\frac{\partial^2\bar{\omega}_m}{\partial s^2}-\left(\frac{\partial^2\bar{\omega}_m}{\partial p\partial s}\right)^2\right\}=\lim_{-p\to 0}\frac{1}{(-p)^2}\left\{-\lambda_{s+m}''t+\frac{d^2}{ds^2}\log\frac{D-\lambda_{s+m}}{\mu_{s+m}-\lambda_{s+m}}\right\}.$$

Also

$$\lim_{p_m\to 0}G_m(s_m,p_m)\begin{matrix}=1, & \text{if} & m=0,\\ =0, & \text{if} & m\neq 0.\end{matrix}$$

Hence, when E tends to infinity, i.e., $-p$ tends to 0, we have

$$E_0P_m^{\lambda}(E,t)=\frac{1}{2\pi}\frac{\exp\{sy_0-sy-\lambda_{s+m}t+1+\log[D-\lambda_{s+m}/\mu_{s+m}-\lambda_{s+m}]\}}{\{-\lambda_{s+m}''t+(d^2/ds^2)\log[D-\lambda_{s+m}/\mu_{s+m}-\lambda_{s+m}]\}^{\frac{1}{2}}}G_m(s_m,0).$$

[9] S. K. Chakrabarty, Proc. Natl. Inst. Sci. India 8, 331 (1942).

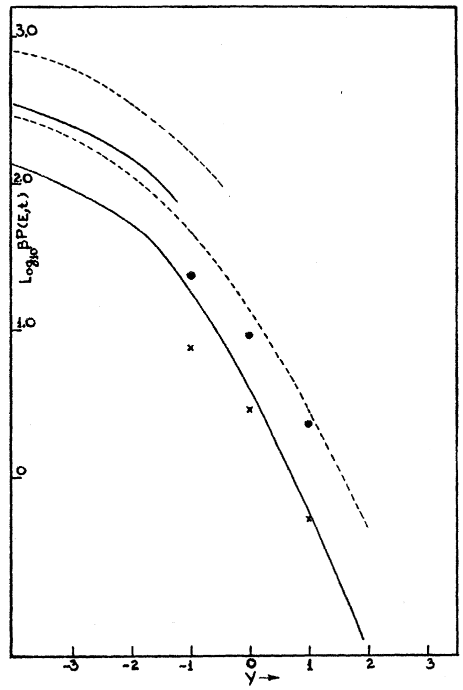

FIG. 1. Graphs of $\log_{10}\beta P_0(E,t)$ against y, for $y_0=6$. Solid curve —— for $t=10$; dashed curve - - - for $t=5$. × and ● are values obtained in (B); the upper curves are from Snyder's analysis.

Hence

$$P^\lambda(E,t)\approx P_0^\lambda(E,t)$$

$$=\frac{e}{(2\pi)^{\frac{1}{2}}}\frac{1}{2\pi i E_0}\int\left(\frac{E_0}{E}\right)^s\frac{D-\lambda_s}{\mu_s-\lambda_s}e^{-\lambda_s t}ds$$

$$\approx P_\lambda(E,t),$$

where

$$P_m(E,t)=P_m^\lambda(E,t)+P_m^\mu(E,t),$$

and

$$P(E,t)=P^\lambda(E,t)+P^\mu(E,t).$$

It can also be shown that the term containing

$$\exp(-\mu_{s+m}t)$$

in (6) reduces to $P_\mu(E,t)$ when E is large. In a similar way it can deduce from (16) that when E tends to infinity, i.e., $(-p)$ tends to zero,

$$N(E,t)=\frac{e}{(2\pi)^{\frac{1}{2}}}\frac{1}{2\pi i}\int\left(\frac{E_0}{E}\right)^{s-1}\frac{1}{s-1}\times\left\{\frac{D-\lambda_s}{\mu_s-\lambda_s}e^{-\lambda_s t}+\frac{\mu_s-D}{\mu_s-\lambda_s}e^{-\mu_s t}\right\}ds.$$

Thus when E is large $P(E,t)$ and $N(E,t)$ given by (6) and (9), nearly reduce to the values deduced in (A) [Eq. (24) and (34)] by neglecting ionization loss.

In a similar way it can be shown that when $E\rightarrow 0$, $(s+p)\rightarrow 0$, we have,

$$\beta P_m(E,t)\approx\left(\frac{4}{3}+\alpha\right)\frac{e}{2\sqrt{2}}\frac{B_1}{D-\lambda_{s+m}}N_m(0,t)\log_e\left(\frac{\beta}{E}\right),$$

and hence

$$\beta P(0,t)\approx\beta P_0(0,t)$$

$$\approx\left(\frac{4}{3}+\alpha\right)\frac{e}{2\sqrt{2}}\frac{B_1}{D-\lambda_s}N_0(0,t)\log_e\left(\frac{\beta}{E}\right),$$

where

$$y_0-\lambda_s't-\frac{1}{s-1}+\frac{d}{ds}\log\frac{D-\lambda_s}{\mu_s-\lambda_s}=0.$$

This shows that, when E tends to zero, $P(E,t)$ approaches infinity logarithmically. This is due to the fact that when $(s+p)\rightarrow 0$, the integrand contains a term $\lambda_{s+p+1}\mu_{s+p+1}$ which goes to $\lambda_1\mu_1$ and thus goes to infinity as B_sC_s when s tends to 1. If we look into the diffusion equation it will be apparent that this infinity is associated with the assumption that a photon of energy even less than $2mc^2$ can create pairs. If thus we restrict the lower limit of the photon energy for pair production to $2mc^2$, this singularity will be avoided and then $P(E,t)$ will tend to a finite limit when E tends to zero. The above expression for $P(E,t)$ represents its value only when $E=0$ and not for any other value of E, and as expected it is similar to the expression for $P(E,t)$ derived by Snyder. It is thus evident that our solution for the cascade equation in the form (6) is valid for the entire range of values of E, and merges to the two extreme values deduced earlier.

It can be shown easily that the solution obtained by Snyder [Eq. (40)], is, except for notation, identical with that given by (6) above. The difficulty in the analysis of Snyder arises in the evaluation of the integral (6) and (10) over p (in Snyder's notation it is s) by the method of residues. It is obvious from (7), (10), and (11), that the integral over p may be evaluated by the method of residues, the poles of the integrand being at $p=(-s-1-n)$ where n is zero or any positive integer, provided only that

$$\lim_{N\rightarrow\infty}\frac{\beta}{E}\phi_0(s+N+1,\,-\lambda_{s+m})>1,$$

and not only if $\beta>E$. But from (3) it is evident that

$$\lim_{N\rightarrow\infty}\phi_0(s+N+1,\,-\lambda_{s+m})$$

$$=\lim_{N\rightarrow\infty}\frac{D-\lambda_{s+m}}{(\lambda_{s+N+1}-\lambda_{s+m})(\mu_{s+N+1}-\lambda_{s+m})},$$

which tends to zero as $1/\log N$. The evaluation of the integral over p by the method of residues is thus possible only when E is zero. The expression for $N(E,t)$

given by Snyder [Eq. (42)] can be used only when $E=0$, in which case the series for $N(E,t)$ reduces to the first term, which is identical with the expression deduced from Eq. (18) given above by making $m=0$. Similarly, the expression for $P(E,t)$ given by Snyder represents its value only when E is zero.

In Fig. 1 we have plotted the values of $P_0(E,t)$ for some values of E, t, and $y_0=6$. For purpose of comparison we have plotted the values of $P_0(E,t)$ deduced from our previous analysis (B) and also that deduced from the results of Snyder. As expected, they indicate that our previous values are smaller than the present ones and the difference increases as E decreases; we also see that the contribution of the higher order terms, *viz.*, P_2, P_3, P_4, etc., in the previous analysis becomes more and more significant as we go much below the critical energy. The values derived from Snyder's expression are much larger than the present values even when $E<\beta$ and the expression is not valid when $E>\beta$. For comparison we have also evaluated the contribution of the second term in the series for $N_0(E,t)$ given by Snyder [Eq. (42)] and the results are given in Table III

TABLE III. Values of $N_{00}+N_{01}$ for different values of y, t, $y_0=6$, from Snyder's analysis.

$t\diagdown y$	-1	-2	-3	-4	$-\infty$
3.34	-48.84	-2.20	23.04	35.29	45.16
4.74	-73.35	-11.88	24.00	42.10	57.12
8.88	-47.02	-16.41	5.76	17.92	28.69

which indicates that the series given by (42), from which Snyder derived the form of the energy spectrum [Eq. (45)], cannot be used, at least without further justification. In Fig. 2 we have plotted the values of $N(E,t)$ against t for some different values of y and have also plotted the values of N_0+N_2 derived in (B) and that derived by Snyder. It is clear from these figures that significant differences exist between the results of the present paper and that given in (B) only in the region where the basic physical assumptions are not precise, and the differences will be much reduced even if the effect of the ionization loss is properly introduced. If we remember that with $E=2mc^2$ we have $-y$ equal to 1.93, 3.21, 4.00, 4.74, 4.63 for Pb, Fe, Al, H_2O, and air respectively, then, as long as we consider only particles with energies equal to or greater than $2mc^2$, our previous values are a very fair approximation for the total number of particles in a shower. In many problems we are required to use the value of $N(0,t)$, particularly in the calculations of the size frequency distribution of bursts produced by mesons under large thickness of material, and the adoption of the value of $N(E,t)$ when E tends to 0 instead of that when E tends to $2mc^2$ will considerably alter the result and the subsequent interpretation. We therefore feel that unless the basic physical assumptions are improved, particularly in the region of low energies where $E<2mc^2$, it will be better to use for $E=2mc^2$ the value of $N(E,t)$ for the total number of shower particle at any depth t, instead of $N(0,t)$.

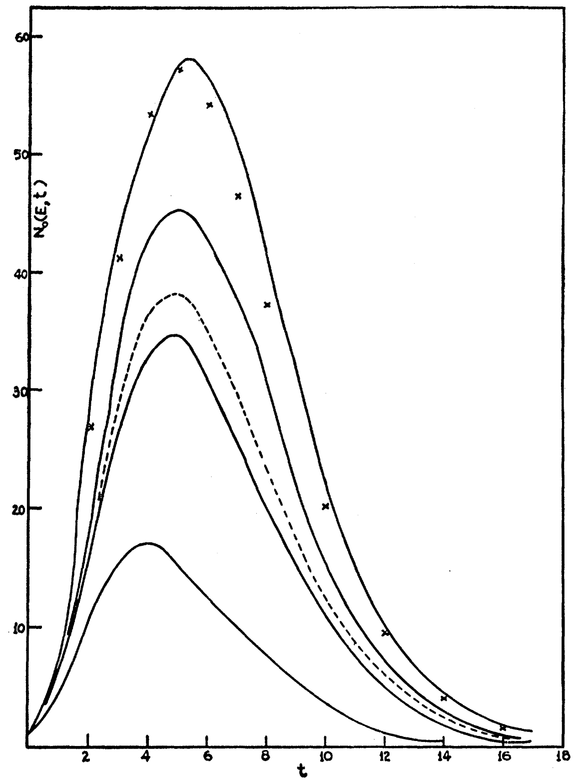

FIG. 2. Graph of $N_0(E,t)$ againt t for $y_0=6$. Solid curves are for $y=0$, -2, -4, $-\infty$. Dashed curve (- - -) represent values obtained in (B); crosses (×) represent points derived from Snyder's analysis.