Phase-Space Integrals for Multiparticle Systems*

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(Received October 7, 1955)

The total phase-space integrals, occurring in the Fermi statistical theory of meson production, for 2-, 3-, 4-, and 5-body final states are reduced to forms suitable for numerical evaluation by hand computation. In addition, the momentum spectrum of any one particle, as derived from the statistical factor, is evaluated. For the 3- and 4-body systems, the Q distribution for any two of the particles (effective mass minus the rest masses of the 2-particle subsystem) is derived. Numerical results are presented for nucleon-nucleon collisions at 0.8-, 1.5-, and 2.7-Bev bombarding kinetic energies.

APPLICATION of Fermi's statistical theory[1-5] of meson production to Cosmotron and Bevatron energies requires an exact relativistic evaluation of the phase-space integral for the final multiparticle state. Because of the complexity of the integrations, various approximations[1-4] have been employed, but none has yielded sufficiently accurate solutions in this intermediate energy region. In this note, we present formulas for general 2-, 3-, 4-, and 5-body systems, from which one can obtain accurate numerical results for the following: (a) total volume in momentum space per unit energy; (b) differential momentum spectrum for any one of the particles; (c) distribution in the Q value between any two particles, i and j, of the system $\{Q_{ij}=[(E_i+E_j)^2-(\mathbf{p}_i+\mathbf{p}_j)^2]^{\frac{1}{2}}-(m_i+m_j)\}$.[6]

The fundamental integral to be evaluated for an n-body system is ρ_n, the volume in momentum space per unit energy, given by

$$\rho_n=\frac{d}{dE}\prod_{i=1}^{n-1}\int d\mathbf{p}_i, \tag{1}$$

where E is the total energy of the system, and $\mathbf{p}_i$ is the momentum of the ith particle, of mass m_i and total energy E_i. In writing (1), we have utilized momentum conservation to eliminate one particle. All integrals will be evaluated in the center-of-mass system defined by $\sum_{i=1}^n \mathbf{p}_i=0$. The momentum spectrum of particle 1 is given by

$$\frac{d\rho_n}{dp_1}=4\pi p_1^2dp_1\frac{d}{dE}\prod_{i=2}^{n-1}\int d\mathbf{p}_i, \tag{2}$$

where $p_1=|\mathbf{p}_1|$, and the 4π comes from the angular integration over $\mathbf{p}_1$. The distribution in Q_{ij} will be obtained in II and III for the 3- and 4-body cases.

I. TWO-BODY SYSTEM

The integration of (1) is well known for the two-body system, yielding

$$\rho_2=\pi\frac{E^2}{2}\left[1-\frac{(m_1^2-m_2^2)^2}{E^4}\right]\times\left[1-\frac{2(m_1^2+m_2^2)}{E^2}+\frac{(m_1^2-m_2^2)^2}{E^4}\right]^{\frac{1}{2}}. \tag{3}$$

II. THREE-BODY SYSTEM

The solution of (2) for the 3-body system was found[7] to be

$$\left(\frac{d\rho_3}{dp_1}\right)=\frac{2\pi^2}{3}p_1^2\times\left\{\left[1-\frac{2(m_2^2+m_3^2)}{(E-E_1)^2-p_1^2}+\frac{(m_2^2-m_3^2)^2}{[(E-E_1)^2-p_1^2]^2}\right]^{\frac{1}{2}}\right\}\times\left\{3(E-E_1)^2\left[1-\frac{(m_2^2-m_3^2)^2}{[(E-E_1)^2-p_1^2]^2}\right]-p_1^2\left[1-\frac{2(m_2^2+m_3^2)}{(E_1-E_1)^2-p_1^2}+\frac{(m_2^2-m_3^2)^2}{[(E-E_1)^2-p_1^2]^2}\right]\right\}. \tag{4}$$

The total momentum space volume is given by the integration of (4) over p_1 between the limits

$$(p_1)_{\max}=\frac{\{[E^2-(m_1+m_2+m_3)^2][E^2-(m_1-m_2-m_3)^2]\}^{\frac{1}{2}}}{2E}$$

and

$$(p_1)_{\min}=0.$$

Unless the rest mass of one of the particles is zero, this integral involves elliptic integrals of the first, second, and third kind.

At this point, let us determine the "effective-mass" distribution[8] for the system containing particles 2 and 3.

* This work was supported by a joint Office of Naval Research and U. S. Atomic Energy Commission contract.

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[5] C. Yang and R. Christian, Brookhaven Cosmotron Internal Report (unpublished).

[6] Units are chosen so that $c=1$.

[7] A complete derivation of the formulas contained here is given in a report by M. M. Block, Duke University, 1955 (unpublished); which will be available upon request.

[8] For a discussion and application of the concept of Q values to particle pairs emitted from nuclear collisions see Fowler, Shutt, Thorndike, and Whittemore, Phys. Rev. **95**, 1026 (1954).

If we define the "effective mass" $\mathfrak{M}$ to be $[(E_2+E_3)^2-(\mathbf{p}_2+\mathbf{p}_3)^2]^{\frac{1}{2}}$, then $\mathfrak{M}=Q_{23}+m_2+m_3$, and thus the distribution in $\mathfrak{M}$ is essentially the distribution in Q. We note however that from energy and momentum conservation, $E_2+E_3=E-E_1$, and $\mathbf{p}_2+\mathbf{p}_3=-\mathbf{p}_1$. Therefore, $\mathfrak{M}=[E^2+m_1^2-2E(p_1^2+m_1^2)^{\frac{1}{2}}]^{\frac{1}{2}}$, and it is clear that the distribution in p_1 will specify the Q distribution. Now,

$$\left(\frac{d\rho_3}{dQ_{23}}\right)dQ_{23}=\left(\frac{d\rho_3}{d\mathfrak{M}}\right)d\mathfrak{M}=\left(\frac{d\rho_3}{dp_1}\right)\left|\frac{\partial p_1}{\partial\mathfrak{M}}\right|d\mathfrak{M}=\left(\frac{d\rho_3}{dp_1}\right)\left|\frac{\partial p_1}{\partial\mathfrak{M}}\right|dQ_{23}, \quad (5)$$

where

$$\left|\frac{\partial p_1}{\partial\mathfrak{M}}\right|=\frac{\mathfrak{M}}{E}\frac{E_1}{p_1}. \quad (6)$$

Thus, combining (4), (5), and (6), we obtain

$$\left(\frac{d\rho_3}{dQ_{23}}\right)dQ_{23}=\frac{\pi}{24}\frac{\mathfrak{M}}{E^5}(E^2+m_1^2-\mathfrak{M}^2)\times\{[E^2-(\mathfrak{M}+m_1)^2][E^2-(\mathfrak{M}-m_1)^2]\}^{\frac{1}{2}}\times\left\{\left[1-\frac{2(m_2^2+m_3^2)}{\mathfrak{M}^2}+\frac{(m_2^2-m_3^2)^2}{\mathfrak{M}^4}\right]^{\frac{1}{2}}\right\}\times\left\{3(E^2-m_1^2+\mathfrak{M}^2)^2\left[1-\frac{(m_2^2-m_3^2)^2}{\mathfrak{M}^4}\right]-[E^2-(\mathfrak{M}-m_1)^2][E^2-(\mathfrak{M}+m_1)^2]\times\left[1-\frac{2(m_2^2+m_3^2)}{\mathfrak{M}^2}+\frac{(m_2^2-m_3^2)^2}{\mathfrak{M}^4}\right]\right\}dQ_{23}. \quad (7)$$

III. FOUR-BODY SYSTEM

For the 4-body system, (1) and (2) are simple conceptually, but become extremely complicated because of the manifold integrations. The results[7] are given in two forms, in terms of the momentum spectrum of particle 1, and in terms of the distribution of the effective mass $\mathfrak{M}$ of particles 3 and 4. The latter distribution is given by

$$\left(\frac{d\rho_4}{d\mathfrak{M}}\right)d\mathfrak{M}=16\pi^3\mathfrak{M}\left\{\int_{\mathfrak{M}}^{[E^2+\mathfrak{M}^2-(m_1+m_2)^2]/2E}\times dW(W^2-\mathfrak{M}^2)^{\frac{1}{2}}\times\Gamma(W^2,W^2-\mathfrak{M}^2,m_3^2,m_4^2)\times\Gamma((E-W)^2,W^2-\mathfrak{M}^2,m_1^2,m_2^2)\right\}d\mathfrak{M}, \quad (8)$$

where

$$\Gamma(x^2,y^2,m_i^2,m_j^2)=\frac{1}{12}\left\{3x^2\left[1-\frac{(m_i^2-m_j^2)^2}{x^2-y^2}\right]\cdot\left[1-\frac{2(m_i^2+m_j^2)}{x^2-y^2}+\frac{(m_i^2-m_j^2)^2}{(x^2-y^2)^2}\right]^{\frac{1}{2}}-y^2\left[1-\frac{2(m_i^2+m_j^2)}{x^2-y^2}+\frac{(m_i^2-m_j^2)^2}{(x^2-y^2)^2}\right]^{\frac{3}{2}}\right\}. \quad (9)$$

We obtain the total phase space by integrating (8) over $\mathfrak{M}$ between the limits $\mathfrak{M}_{\max}=E-m_1-m_2$ and $\mathfrak{M}_{\min}=m_3+m_4$.

The momentum spectrum of particle 1 can be shown[7] to be

$$\frac{d\rho_4}{dp_1}=16\pi^3p_1\left\{\int_0^{p_2'}p_2[G((E-E_1-E_2)^2,(p_1+p_2)^2,4m^2)-G((E-E_1-E_2)^2,(p_1-p_2)^2,4m^2)]dp_2+\int_{p_2'}^{(p_2)_{\max}}p_2[G((E-E_1-E_2)^2,(E-E_1-E_2)^2-4m^2,4m^2)-G((E-E_1-E_2)^2,(p_1-p_2)^2,4m^2)]dp_2\right\}, \quad (10)$$

where we have restricted ourselves to the case that $m_3=m_4=m$. The function G is defined as

$$G(x^2,y^2,4m^2)=-\frac{1}{12}(x^2-y^2)^2\left(1-\frac{4m^2}{x^2-y^2}\right)^{\frac{3}{2}}-\frac{x^2}{12}(x^2-y^2)\left(1-\frac{4m^2}{x^2-y^2}\right)^{\frac{3}{2}}+\frac{(x^2-y^2)^2}{16}\times\left(1-\frac{4m^2}{x^2-y^2}\right)^{\frac{1}{2}}-\frac{4m^2}{32}(x^2-y^2)\left(1-\frac{4m^2}{x^2-y^2}\right)^{\frac{1}{2}}+\frac{(4m^2)^2}{32}\log[(x^2-y^2)^{\frac{1}{2}}-(x^2-y^2-4m^2)^{\frac{1}{2}}]. \quad (11)$$

Also,

$$(p_2)_{\max}=\frac{E-E_1}{2}\left(1-\frac{2(m_2^2+4m^2)}{E^2+m_1^2-2EE_1}+\frac{(m_2^2-4m^2)^2}{(E^2+m_1^2-2EE_1)^2}\right)^{\frac{1}{2}}+\frac{p_1}{2}\left(1+\frac{m_2^2-4m^2}{E^2+m_1^2-2EE_1}\right), \quad (12)$$

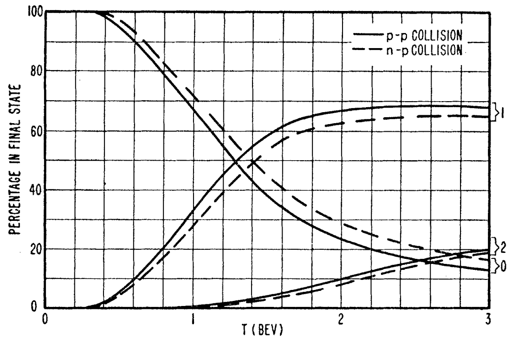

FIG. 1. The fraction of final states containing 0-, 1-, and 2-meson production for p-p and n-p collisions, as a function of the laboratory kinetic energy of the bombarding nucleon.

and

$$p_2'=\frac{E-E_1}{2}\left(1-\frac{2(m_2^2+4m^2)}{E^2+m_1^2-2EE_1}+\frac{(m_2^2-4m^2)^2}{(E^2+m_1^2-2EE_1)^2}\right)^{\frac{1}{2}}-\frac{p_1}{2}\left(1+\frac{m_2^2-4m^2}{E^2+m_1^2-2EE_1}\right). \quad (13)$$

To obtain the total phase space, (10) is integrated over p_1 between the limits.

$$(p_1)_{\max}=\frac{\{[E^2-(m_1+m_2+2m)^2][E^2-(m_1-m_2-2m)^2]\}^{\frac{1}{2}}}{2E}$$

and $(p_1)_{\min}=0$.

IV. FIVE-BODY SYSTEM

Because of the extreme complexity of the 5-body system, we restrict ourselves to the case of $m_1=m_2=M$, $m_4=m_5=m$, and only evaluate the momentum spectrum of particle 3, whose mass, however, can be different from m and M. The momentum spectrum can be shown[7] to be given by

$$\begin{aligned}(d\rho_5/dp_3)dp_3 &= 32\pi^4 p_3\\ &\times\Bigg\{\int_{2m}^{W'}[G(W^2,W^2-4m^2,4m^2)-G(W^2,0,4m^2)]\\ &\times[G((E-E_3-W)^2,(W+E_3)^2-(E_3+2m)^2,4M^2)\\ &-G((E-E_3-W)^2,0,4M^2)]dW\\ &+\int_{W'}^{W_0}[G(W^2,W^2-4m^2,4m^2)-G(W^2,0,4m^2)]\\ &\times[G((E-E_3-W)^2,(E-E_3-W)^2-4M^2,4M^2)\\ &-G((E-E_3-W)^2,0,4M^2)]dW\\ &+\int_{W_0}^{E-E_3-2M}[G(W^2,(E-W)^2-(E_3+2M)^2,4m^2)\\ &-G(W^2,0,4m^2)][G((E-E_3-W)^2,(E-E_3-W)^2\\ &-4M^2,4M^2)-G((E-E_3-W)^2,0,4M^2)]dW\Bigg\}dp_3, \quad (14)\end{aligned}$$

where G is defined in (11).

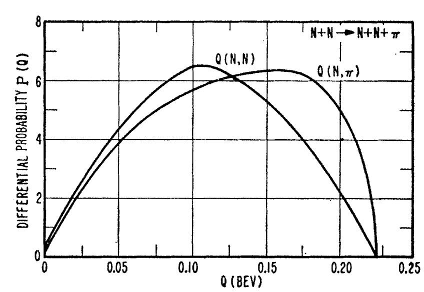

FIG. 3. Q distributions in single pion production, between the pion and either nucleon, and the two nucleons, at 0.8 Bev. The curves are normalized to unit area.

The integration limits over W are given by

$$W_0=\frac{E^2-E_3^2-4ME_3-4M^2+4m^2}{2E}, \quad (15)$$

$$W'=\frac{(E-E_3)^2+4mE_3-4M^2+4m^2}{2E}. \quad (16)$$

If we desire to obtain the total phase space from (14), we integrate over p_3 from

$$(p_3)_{\max}=\frac{\{[E^2-(m_3+2m+2M)^2][E^2-(m_3-2m-2M)^2]\}^{\frac{1}{2}}}{2E}$$

to $(p_3)_{\min}=0$.

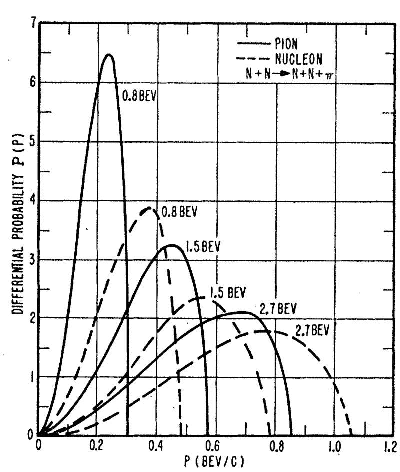

FIG. 2. The momentum spectra for nucleons and the pion in a 3-body final state (single pion production) at laboratory kinetic energies of 0.8, 1.5, and 2.7 Bev. The curves are normalized to unit area.

V. APPLICATION OF THE FERMI THEORY TO NUCLEON-NUCLEON COLLISIONS

In the preceding sections, we have developed formulas suitable for hand computation of ρ_n, the exact relativistic phase space factor for $n=2$-, 3-, 4- and 5-body systems. Using these results, we now obtain detailed numerical applications of the Fermi statistical theory of meson production for the case of nucleon-nucleon collisions. Following Fermi,[1,2] the probability for the production of s mesons in a final state containing n particles ($n=s+2$) is proportional to $\Omega^{n-1}\rho_n f_n/(2\pi\hbar)^{3(n-1)}$, where Ω is the spatial volume in which statistical equilibrium is supposed to take place. We choose Ω to be $(4/3)\pi r^3(2M/E)$, where M is the nucleon rest mass, E is the total center-of-mass energy and r is taken as the Compton wavelength of the pion, 1.2×10^{-13} cm. The factor $2M/E$ reflects the Lorentz con-

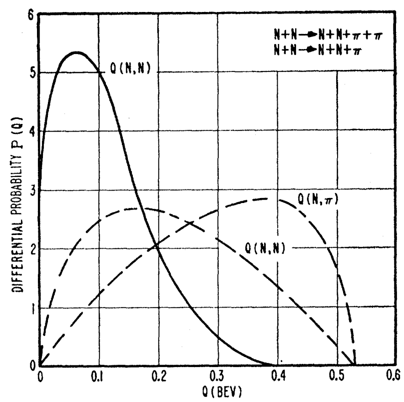

FIG. 4. Q distributions for single and double pion production at 1.5 Bev. The curves are normalized to unit area.

traction of the sphere. The term f_n arises from the assumption of charge independence and is proportional to number of final independent isotopic spin states that can be formed. If we denote R_s as the probability of producing $s=n-2$ mesons, then

$$\frac{R_s}{R_{s-1}}=\frac{\Omega}{(2\pi\hbar)^3}\cdot\frac{\rho_n}{\rho_{n-1}}\cdot\frac{f_n}{f_{n-1}}. \tag{17}$$

Using Milburn's[4] results on charge independence and neglecting triple and higher-order meson emission (very small at Cosmotron energies), the relative multiplicities were computed from (17), using (3) and numerically integrating (4) and (8) for 0-, 1-, and 2-meson production at laboratory bombarding kinetic energies up to 3 Bev. The results are shown in Fig. 1 for p-p and n-p collisions. The reason for the differing predictions for n-p

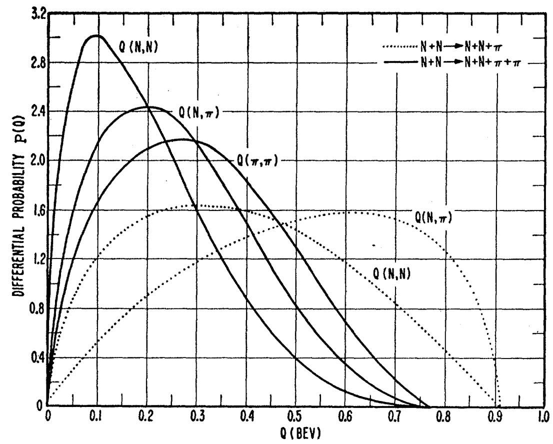

FIG. 5. Q distributions for single and double pion production at 2.7 Bev. The curves are normalized to unit area.

and p-p collisions is that the former are mixtures of isotopic spin states 0 and 1, whereas the latter are in a pure isotopic-spin-1 state.

Since experiments on p-p scattering are currently in progress at the Brookhaven Cosmotron at kinetic energies of 0.8, 1.5, and 2.7 Bev, we have evaluated momentum and Q distributions for these particular bombarding energies. The momentum spectra for the final state pion and either of the two nucleons, for the case of single meson production, have been calculated from (4) at the above beam energies and are presented in Fig. 2. The Q distributions between the two nucleons, and the pion and any one of the nucleons have been computed from (7) for the 0.8-Bev case and are plotted in Fig. 3. Figure 4 shows similar results for 1.5-Bev bombarding energy; in addition, the Q distribution between the two nucleons in a collision producing 2 mesons, calculated from (8), is shown for comparison. At 2.7 Bev, all possible Q combinations have been evaluated for both 1- and 2-meson production and these distributions compiled from (7) and (8) are shown in Fig. 5. It should perhaps be noted that both the momentum and Q distributions do not involve any knowledge of the parameter r, the radius of the sphere in which Fermi envisualized the statistical equilibrium taking place, and thus perhaps are simpler to compare directly with experiment than the predictions on multiplicity, whose results depend rather strongly on the radius.

ACKNOWLEDGMENTS

The author wishes to thank Dr. E. Harth, Dr. R. Sternheimer, and Dr. W. Wada for many stimulating discussions.