Application of the Bacher and Goudsmit Method to Nuclear Spectra*

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(Received September 22, 1955)

A method for the calculation of nuclear l^n matrix elements in terms of those of l^{n-1} is developed, assuming two-particle interactions. An example is the calculation of p^3 matrix elements as linear combinations of p^2 matrix elements. The coefficients in these linear combinations are tabulated. Application is made by calculating $_3Li^7$ energies from $_3Li^6$ energy data. A comparison of the calculated energies of $_3Li^7$ with experiment leads to some restrictions on the nuclear potential. To perform the calculation it is necessary to fit the $_3Li^6$ data; the results of the fit are presented. The existence of a $J=5/2^-$ energy level in $_3Li^7$ at or near 6.53 Mev is also discussed.

INTRODUCTION

BECAUSE of the growing success of the shell model we have attempted to apply the Bacher and Goudsmit[1–3] method of atomic spectroscopy to the calculation of nuclear energy levels. The initial purpose of our calculation was twofold: (1) to predict the energy levels of an n-particle nucleus from those of $n-1$ particles; (2) to try to pin down the type of nuclear force by requiring that the same force law fit energy data of successive isotopes.

In particular, owing to the relative ease of calculation, we have restricted ourselves to the beginning of the p shell. In a usual spectroscopic application of the Bacher and Goudsmit method we would calculate l^n *energies* as linear combinations of experimentally observed l^{n-1}, l^{n-2}, $\cdots l$ energies. In the nuclear case we calculate only the *matrix elements* of a nuclear l^n configuration as a linear combination of the matrix elements of a l^{n-1} configuration. A similar calculation was used to treat the atomic energy levels of vanadium.[4] The derivation of the matrix elements of l^n involves the assumption that the nuclear forces be two-particle ones. If we assume, in particular, central, spin-orbit, or tensor forces, our matrix elements reduce to the corrected elements of Elliott.[5] By central forces, we mean any linear combination of Wigner, Majorana, Bartlett, or Heisenberg forces. Throughout the calculation we have used LS coupling, although the calculation proceeds similarly for jj coupling. The calculation is discussed in detail in Sec. I.

As an application we consider the calculation of p^3 $_3Li^7$ matrix elements from those of p^2 $_3Li^6$. In Sec. II numerical Li^6 elements are derived from the experimental energies. The effect of different types of forces on parameters L, K, and a is discussed. L and K, the central force radial integrals, and a, the spin-orbit parameter which equals $6D'$ in Elliott's notation are defined in Elliott's paper.

In Sec. III, the actual calculation of Li^7 energies is described and the results are discussed. The evidence for the identification of an observed level of Li^7 at 6.53 Mev as a $J=5/2^-$ state is considered. Our results indicate that by introducing a spin-orbit force together with a central force intermediate to the Rosenfeld mixture ($V_M=0.8$, $V_B=0.2$) and an "almost Serber" mixture ($V_M=V_W=0.4$, $V_B=0.2$) we can fit both Li^6 and Li^7 fairly well. By intermediate we mean that the matrix element $^{33}P_0$ $^{33}P_0$ of Li^6 lies between the Rosenfeld mixture value $-0.6(L-3K)-a$ and the Serber-like mixture value of $0.2(L-3K)-a$; similar changes take place in the $^{11}P_1$ $^{11}P_1$, $^{33}P_1$ $^{33}P_1$, and $^{33}P_2$ $^{33}P_2$ elements.

I. DERIVATION OF p^3 MATRIX ELEMENTS

Elliott[5] has calculated the matrix elements of p^2, p^3 and p^6 configurations for central, spin-orbit and tensor forces in terms of various radial integrals. For our purposes however, this calculation does not suffice because it does not explicitly give the relative contributions to the p^3 elements arising from the various p^2 states. We have, therefore, calculated the matrix elements of the p^3 configuration as a linear combination of p^2 elements. We have used the coefficients of fractional parentage of Jahn and Van Wieringen[6] and tables of W coefficients by Biedenharn.[7] Elliott's paper has been used as a check because our elements should reduce to his. In the course of the calculation we found various errors in Elliott's paper which he has corroborated.[8]

* Some of this work was reported at the 1955 New York meeting of the American Physical Society [S. Meshkov and C. W. Ufford, Phys. Rev. **98**, 1204(A) (1955)].

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Our first step is to write a three-particle wave function in the $SLJM_J$ scheme as a linear combination of products of one- and two-particle wave functions which are also expressed in the above scheme. This involves a series of recouplings and transformations from the $SLJM_J$ scheme to the SLM_SM_L scheme and back again. In general for n particles

$$\Psi(l^n\gamma TSLJM_TM_J)=\sum_{\substack{M_SM_LM_{S'}M_{L'}m_sm_l\\ \gamma'T'S'L'jm_jJ'M_{J'}M_{T'}}}\langle l^n\gamma TSL|l^{n-1}\gamma'T'S'L'\rangle(SLM_SM_L|SLJM_J)(S'sM_{S'}m_s|S'sSM_S)$$
$$\times(L'lM_{L'}m_l|L'lLM_L)(S'L'M_{S'}M_{L'}|S'L'J'M_{J'})(slm_sm_l|sljm_j)\psi(l^{n-1}\gamma'T'S'L'J'M_{T'}M_{J'})\phi(sljm_j),\quad(1)$$

where $(SLM_SM_L|SLJM_J)$ is a Clebsch-Gordan coefficient, $\langle l^n\gamma TSL|l^{n-1}\gamma'T'S'L'\rangle$ are the fractional parentage coefficients for the reduction $\langle l^n|l^{n-1}\rangle$, and $\phi(sljm_j)$ is a one-electron wave function. More concisely,

$$\Psi(l^n\gamma TSLJM_TM_J)=\sum_{\gamma'T'S'L'jm_jJ'M_{J'}M_{T'}}(J'jM_{J'}m_j|J'jJM_J)[(2J'+1)(2j+1)(2S+1)(2L+1)]^{\frac{1}{2}}X\begin{Bmatrix}S' & L' & J'\\ s & l & j\\ S & L & J\end{Bmatrix}$$
$$\times\psi(l^{n-1}\gamma'T'S'L'J'M_{T'}M_{J'})\phi(sljm_j).\quad(2)$$

By letting $n=3$, the p^3 and p^2 configurations are treated.

The contributions of various l^{n-1} states to the l^n matrix elements are determined by the coefficients of the products of the allowed two particle matrix elements. Using (2) we determine the coefficient of the contribution of the l^{n-1} matrix element specified by the quantum numbers $(\gamma_1'T'S_1'L_1'J')$ and $(\gamma_2'T'S_2'L_2'J')$ to the l^n matrix element labeled by quantum numbers $(\gamma_1TS_1L_1J)$, $(\gamma_2TS_2L_2J)$. It is

$$d(1'2';12)=\frac{n}{n-2}\sum_{jm_jM_{J'}}(J'M_{J'}jm_j|J'M_{J'}JM_J)^2(2J'+1)(2j+1)[(2S_1+1)(2S_2+1)(2L_1+1)(2L_2+1)]^{\frac{1}{2}}$$
$$\times X\begin{Bmatrix}S_1' & L_1' & J'\\ s & l & j\\ S_1 & L_1 & J\end{Bmatrix}X\begin{Bmatrix}S_2' & L_2' & J'\\ s & l & j\\ S_2 & L_2 & J\end{Bmatrix}\langle l^n\gamma_1TS_1L_1|l^{n-1}\gamma_1'T'S_1'L_1'\rangle\langle l^n\gamma_2TS_2L_2|l^{n-1}\gamma_2'T'S_2'L_2'\rangle.\quad(3)$$

If $(\gamma_1'T'S_1'L_1'J')$ is not the same as $(\gamma_2'T'S_2'L_2'J')$, the above summation must also be carried out with $(\gamma_1'T'S_1'L_1'J')$ and $(\gamma_2'T'S_2'L_2'J')$ interchanged.

Each coefficient calculated must be multiplied by $n/(n-2)$ to account for the ratios of pairs of particles in the l^n and l^{n-1} configurations. The entries in Table I have not been multiplied by $n/(n-2)$. Orthogonality together with selection rules determines which matrix elements make any contributions. The coefficients which show the relative contributions to p^3 elements by the p^2 elements are listed in Table I. These coefficients may be used whatever the force is, so long as it be a two-particle force.

In order to obtain Elliott's results from this table we: (1) write the p^2 matrix elements in terms of the radial integrals appropriate for the force considered; (2) multiply the rewritten p^2 matrix elements by the coefficients of Table I; (3) add all of the products together; and (4) multiply each resulting p^3 matrix element by 3 when the force considered arises from the interaction of two p electrons, and by 3/2 when our two-particle interaction is a spin-orbit $(1s,2p)$ interaction.

The factor of 3 is found by letting $n=3$ in the coefficient $n/(n-2)$. As is pointed out in reference 2, the factor $n/(n-2)$ arises from taking the ratio of the number of ways of choosing pairs of n particles to the number of ways of choosing pairs of $n-1$ particles. In the case of the factor 3/2, if we consider the spin-orbit $(1s,2p)$ interactions, there are twelve ways to choose pairs of $1s$ and $2p$ particles out of the four $1s$ and three $2p$ particles of Li^7 and eight ways to choose pairs of $1s$ and $2p$ particles out of the four $1s$ and two $2p$ particles of Li^6. The proper multiplication factor is $12/8=1.5$.

Our method requires that we must know the matrix elements of Li^6 to start with. They are obtained from experiment as far as possible, although some radial integrals must be fitted to give the elements in those matrices that are not diagonal. Once we have obtained the Li^6 matrix elements, either from experimental or theoretical results, we multiply the central part by 3 and the spin-orbit contribution by 1.5. The sum of these two contributions is multiplied by the appropriate coefficient from Table I to yield the Li^7 matrix element. The resulting Li^7 matrices are then diagonalized on a desk computer to yield the eigenvalues. This procedure, of course, is identical with starting with the corrected Li^7 matrices of Elliott, expressed in terms of L, K, a, inserting the values of the parameters L, K, a derived from Li^6 data, and then diagonalizing the resulting matrix. The method that we employ, i.e., using the p^2 matrix elements, enables us to see from which Li^6 state energy contributions can arise. The use of this to fix a force law is described in Sec. II.

At this point let us note other types of calculations which might be performed with better results. The one we have just described errs since we do not account for

TABLE I. The elements of the energy matrices of the nuclear configuration p^3. The use of this table is illustrated by writing the $[3]^{22}P_{1/2}\,[3]^{22}P_{1/2}$ matrix element of p^3. $[3]^{22}P_{1/2}\,[3]^{22}P_{1/2} = 3[(5/18)^{31}S_0{}^{31}S_0 + (5/18)^{13}S_1{}^{13}S_1 + (1/9)^{13}D_1{}^{13}D_1 + (1/9)^{13}D_2{}^{13}D_2 + (2/9)^{31}D_2{}^{31}D_2]$. The coefficient 3 is replaced by 1.5 for a $(1s,2p)$ spin-orbit interaction.

$p^3 \backslash p^2$	$^{31}S_0\,^{31}S_0$	$^{33}P_0\,^{33}P_0$	$^{31}S_0\,^{33}P_0$	$^{13}S_1\,^{13}S_1$	$^{13}D_1\,^{13}D_1$	$^{11}P_1\,^{11}P_1$	$^{33}P_1\,^{33}P_1$	$^{13}S_1\,^{13}D_1$	$^{13}S_1\,^{11}P_1$	$^{13}D_1\,^{11}P_1$	$^{13}D_2\,^{13}D_2$	$^{33}P_2\,^{33}P_2$	$^{31}D_2\,^{31}D_2$	$^{33}P_2\,^{31}D_2$	$^{13}D_3\,^{13}D_3$
$[3]^{22}P_{1/2}\,[3]^{22}P_{1/2}$	5/18	0	0	5/18	1/9	0	0	0	0	0	1/9	0	2/9	0	0
$[21]^{22}P_{1/2}\,[21]^{22}P_{1/2}$	1/9	1/18	$(2)^{\frac{1}{2}}/9$	1/9	5/72	1/4	1/8	0	$(6)^{\frac{1}{2}}/9$	$(30)^{\frac{1}{2}}/36$	5/72	5/72	5/36	$5(2)^{\frac{1}{2}}/36$	0
$[21]^{24}P_{1/2}\,[21]^{24}P_{1/2}$	0	1/18	0	2/9	5/24	0	3/8	$-(5)^{\frac{1}{2}}/9$	0	0	5/72	5/72	0	0	0
$[21]^{24}D_{1/2}\,[21]^{24}D_{1/2}$	0	5/18	0	0	3/8	0	5/24	0	0	0	1/8	1/72	0	0	0
$[111]^{22}S_{1/2}\,[111]^{22}S_{1/2}$	0	1/18	0	0	0	1/2	1/6	0	0	0	0	5/18	0	0	0
$[3]^{22}P_{1/2}\,[21]^{22}P_{1/2}$	$-(10)^{\frac{1}{2}}/18$	0	$-(5)^{\frac{1}{2}}/18$	$(10)^{\frac{1}{2}}/18$	$-(10)^{\frac{1}{2}}/36$	0	0	0	$(15)^{\frac{1}{2}}/18$	$-(3)^{\frac{1}{2}}/18$	$-(10)^{\frac{1}{2}}/36$	0	$(10)^{\frac{1}{2}}/18$	$(5)^{\frac{1}{2}}/18$	0
$[3]^{22}P_{1/2}\,[21]^{24}P_{1/2}$	0	0	$-(5)^{\frac{1}{2}}/18$	0	$-(10)^{\frac{1}{2}}/36$	0	0	$-(2)^{\frac{1}{2}}/36$	0	0	$(10)^{\frac{1}{2}}/36$	0	0	$(5)^{\frac{1}{2}}/18$	0
$[3]^{22}P_{1/2}\,[21]^{24}D_{1/2}$	0	0	5/18	0	$(2)^{\frac{1}{2}}/12$	0	0	$-(10)^{\frac{1}{2}}/12$	0	0	$-(2)^{\frac{1}{2}}/12$	0	0	1/18	0
$[3]^{22}P_{1/2}\,[111]^{22}S_{1/2}$	0	0	$(5)^{\frac{1}{2}}/18$	0	0	0	0	0	$(15)^{\frac{1}{2}}/18$	$(3)^{\frac{1}{2}}/9$	0	0	0	$(5)^{\frac{1}{2}}/9$	0
$[21]^{22}P_{1/2}\,[21]^{24}P_{1/2}$	0	1/18	$(2)^{\frac{1}{2}}/18$	0	5/72	0	−1/8	$-(5)^{\frac{1}{2}}/9$	$-(6)^{\frac{1}{2}}/18$	$-(30)^{\frac{1}{2}}/72$	−5/72	5/72	0	$5(2)^{\frac{1}{2}}/72$	0
$[21]^{22}P_{1/2}\,[21]^{24}D_{1/2}$	0	$-(5)^{\frac{1}{2}}/18$	$-(10)^{\frac{1}{2}}/18$	0	$-(5)^{\frac{1}{2}}/24$	0	$(5)^{\frac{1}{2}}/24$	−1/6	0	$-(6)^{\frac{1}{2}}/8$	$(5)^{\frac{1}{2}}/24$	$(5)^{\frac{1}{2}}/72$	0	$(10)^{\frac{1}{2}}/72$	0
$[21]^{22}P_{1/2}\,[111]^{22}S_{1/2}$	0	−1/18	$-(2)^{\frac{1}{2}}/18$	0	0	0	−1/12	0	$(6)^{\frac{1}{2}}/18$	$-(30)^{\frac{1}{2}}/36$	0	5/36	0	$5(2)^{\frac{1}{2}}/36$	0
$[21]^{24}P_{1/2}\,[21]^{24}D_{1/2}$	0	$-(5)^{\frac{1}{2}}/18$	0	0	$(5)^{\frac{1}{2}}/24$	0	$(5)^{\frac{1}{2}}/24$	1/6	0	0	$-(5)^{\frac{1}{2}}/24$	$(5)^{\frac{1}{2}}/72$	0	0	0
$[21]^{24}P_{1/2}\,[111]^{22}S_{1/2}$	0	−1/18	0	0	0	0	−1/12	0	$(6)^{\frac{1}{2}}/9$	$-(30)^{\frac{1}{2}}/12$	0	5/36	0	0	0
$[21]^{24}D_{1/2}\,[111]^{22}S_{1/2}$	0	$(5)^{\frac{1}{2}}/18$	0	0	0	0	$-(5)^{\frac{1}{2}}/12$	0	0	0	0	$(5)^{\frac{1}{2}}/36$	0	0	0
$[21]^{42}P_{1/2}\,[21]^{42}P_{1/2}$	2/9	1/9	$-2(2)^{\frac{1}{2}}/9$	0	0	0	1/4	0	0	0	0	5/36	5/18	$-5(2)^{\frac{1}{2}}/18$	0
$[3]^{22}P_{3/2}\,[3]^{22}P_{3/2}$	5/18	0	0	5/18	1/90	0	0	0	0	0	1/18	0	2/9	0	7/45
$[21]^{22}P_{3/2}\,[21]^{22}P_{3/2}$	1/9	1/72	$-(2)^{\frac{1}{2}}/18$	1/9	1/144	1/4	1/16	0	$-(6)^{\frac{1}{2}}/18$	$-(30)^{\frac{1}{2}}/72$	5/144	25/144	5/36	$-5(2)^{\frac{1}{2}}/72$	7/72
$[21]^{24}P_{3/2}\,[21]^{24}P_{3/2}$	0	5/36	0	2/9	7/120	0	1/8	$4(5)^{\frac{1}{2}}/45$	0	0	13/72	17/72	0	0	7/180
$[21]^{22}D_{3/2}\,[21]^{22}D_{3/2}$	0	5/72	0	0	9/80	1/4	7/48	0	0	$-(30)^{\frac{1}{2}}/40$	17/144	5/144	1/4	$-(2)^{\frac{1}{2}}/8$	7/360
$[21]^{24}D_{3/2}\,[21]^{24}D_{3/2}$	0	5/36	0	0	9/40	0	7/24	0	0	0	17/72	5/72	0	0	7/180
$[3]^{22}P_{3/2}\,[21]^{22}P_{3/2}$	$-(10)^{\frac{1}{2}}/18$	0	$(5)^{\frac{1}{2}}/36$	$(10)^{\frac{1}{2}}/18$	$-(10)^{\frac{1}{2}}/360$	0	0	0	$-(15)^{\frac{1}{2}}/36$	$(3)^{\frac{1}{2}}/36$	$-(10)^{\frac{1}{2}}/72$	0	$(10)^{\frac{1}{2}}/18$	$-(5)^{\frac{1}{2}}/36$	$-7(10)^{\frac{1}{2}}/180$
$[3]^{22}P_{3/2}\,[21]^{24}P_{3/2}$	0	0	$-5(2)^{\frac{1}{2}}/36$	0	−1/45	0	0	$(5)^{\frac{1}{2}}/180$	0	0	−1/18	0	0	$5(2)^{\frac{1}{2}}/36$	7/90
$[3]^{22}P_{3/2}\,[21]^{22}D_{3/2}$	0	0	−5/36	0	$-(2)^{\frac{1}{2}}/40$	0	0	0	$5(3)^{\frac{1}{2}}/36$	$(15)^{\frac{1}{2}}/180$	$-(2)^{\frac{1}{2}}/72$	0	0	−1/36	$7(2)^{\frac{1}{2}}/180$
$[3]^{22}P_{3/2}\,[21]^{24}D_{3/2}$	0	0	$5(2)^{\frac{1}{2}}/36$	0	−1/30	0	0	$(5)^{\frac{1}{2}}/12$	0	0	1/9	0	0	$(2)^{\frac{1}{2}}/36$	−7/90
$[21]^{22}P_{3/2}\,[21]^{24}P_{3/2}$	0	$-(10)^{\frac{1}{2}}/72$	$(5)^{\frac{1}{2}}/18$	0	$(10)^{\frac{1}{2}}/180$	0	0	$(2)^{\frac{1}{2}}/18$	$-(15)^{\frac{1}{2}}/18$	$-5(3)^{\frac{1}{2}}/72$	$(10)^{\frac{1}{2}}/72$	$(10)^{\frac{1}{2}}/72$	0	$5(5)^{\frac{1}{2}}/72$	$-7(10)^{\frac{1}{2}}/360$
$[21]^{22}P_{3/2}\,[21]^{22}D_{3/2}$	0	$-(5)^{\frac{1}{2}}/72$	$(10)^{\frac{1}{2}}/36$	0	$(5)^{\frac{1}{2}}/80$	0	$-(5)^{\frac{1}{2}}/48$	0	$(30)^{\frac{1}{2}}/36$	$-5(6)^{\frac{1}{2}}/72$	$(5)^{\frac{1}{2}}/144$	$5(5)^{\frac{1}{2}}/144$	0	$-5(10)^{\frac{1}{2}}/72$	$-7(5)^{\frac{1}{2}}/360$
$[21]^{22}P_{3/2}\,[21]^{24}D_{3/2}$	0	$(10)^{\frac{1}{2}}/72$	$-(5)^{\frac{1}{2}}/18$	0	$(10)^{\frac{1}{2}}/120$	0	$-(10)^{\frac{1}{2}}/24$	$(2)^{\frac{1}{2}}/12$	0	$-(3)^{\frac{1}{2}}/8$	$-(10)^{\frac{1}{2}}/36$	$(10)^{\frac{1}{2}}/36$	0	$(5)^{\frac{1}{2}}/72$	$7(10)^{\frac{1}{2}}/360$
$[21]^{24}P_{3/2}\,[21]^{22}D_{3/2}$	0	$5(2)^{\frac{1}{2}}/72$	0	0	$(2)^{\frac{1}{2}}/20$	0	$-(2)^{\frac{1}{2}}/12$	$(10)^{\frac{1}{2}}/20$	$-(3)^{\frac{1}{2}}/18$	$(15)^{\frac{1}{2}}/360$	$-5(2)^{\frac{1}{2}}/72$	$(2)^{\frac{1}{2}}/72$	0	−1/8	$7(2)^{\frac{1}{2}}/360$
$[21]^{24}P_{3/2}\,[21]^{24}D_{3/2}$	0	−5/36	0	0	13/120	0	1/24	$(5)^{\frac{1}{2}}/15$	0	0	−5/72	7/72	0	0	−7/180
$[21]^{22}D_{3/2}\,[21]^{24}D_{3/2}$	0	$-5(2)^{\frac{1}{2}}/72$	0	0	$3(2)^{\frac{1}{2}}/40$	0	$(2)^{\frac{1}{2}}/24$	0	0	$(15)^{\frac{1}{2}}/40$	$-(2)^{\frac{1}{2}}/18$	$(2)^{\frac{1}{2}}/36$	0	−1/8	$-7(2)^{\frac{1}{2}}/360$
$[21]^{42}P_{3/2}\,[21]^{42}P_{3/2}$	2/9	1/36	$(2)^{\frac{1}{2}}/9$	0	0	0	1/8	0	0	0	0	25/72	5/18	$5(2)^{\frac{1}{2}}/36$	0
$[21]^{42}D_{3/2}\,[21]^{42}D_{3/2}$	0	5/36	0	0	0	0	7/24	0	0	0	0	5/72	1/2	$(2)^{\frac{1}{2}}/4$	0
$[111]^{44}S_{3/2}\,[111]^{44}S_{3/2}$	0	1/9	0	0	0	0	1/3	0	0	0	0	5/9	0	0	0
$[21]^{42}P_{3/2}\,[21]^{42}D_{3/2}$	0	$-(5)^{\frac{1}{2}}/36$	$-(10)^{\frac{1}{2}}/18$	0	0	0	$-(5)^{\frac{1}{2}}/24$	0	0	0	0	$5(5)^{\frac{1}{2}}/72$	0	$5(10)^{\frac{1}{2}}/36$	0
$[21]^{42}P_{3/2}\,[111]^{44}S_{3/2}$	0	1/18	$(2)^{\frac{1}{2}}/9$	0	0	0	1/12	0	0	0	0	−5/36	0	$-5(2)^{\frac{1}{2}}/18$	0
$[21]^{42}D_{3/2}\,[111]^{44}S_{3/2}$	0	$-(5)^{\frac{1}{2}}/18$	0	0	0	0	$(5)^{\frac{1}{2}}/12$	0	0	0	0	$-(5)^{\frac{1}{2}}/36$	0	0	0
$[3]^{22}F_{5/2}\,[3]^{22}F_{5/2}$	0	0	0	0	7/30	0	0	0	0	0	13/54	0	1/2	0	7/270
$[21]^{24}P_{5/2}\,[21]^{24}P_{5/2}$	0	0	0	2/9	1/360	0	1/8	$-(5)^{\frac{1}{2}}/45$	0	0	1/24	3/8	0	0	7/30
$[21]^{22}D_{5/2}\,[21]^{22}D_{5/2}$	0	0	0	0	1/120	1/4	1/24	0	0	$(30)^{\frac{1}{2}}/60$	13/216	5/24	1/4	$(2)^{\frac{1}{2}}/12$	49/270
$[21]^{24}D_{5/2}\,[21]^{24}D_{5/2}$	0	0	0	0	7/120	0	7/24	0	0	0	61/216	5/24	0	0	43/270
$[3]^{22}F_{5/2}\,[21]^{24}P_{5/2}$	0	0	0	0	$-(21)^{\frac{1}{2}}/180$	0	0	$(105)^{\frac{1}{2}}/45$	0	0	$(21)^{\frac{1}{2}}/108$	0	0	0	$-(21)^{\frac{1}{2}}/270$
$[3]^{22}F_{5/2}\,[21]^{22}D_{5/2}$	0	0	0	0	$-(7)^{\frac{1}{2}}/60$	0	0	0	0	$-(210)^{\frac{1}{2}}/60$	$-(7)^{\frac{1}{2}}/108$	0	0	$(14)^{\frac{1}{2}}/12$	$7(7)^{\frac{1}{2}}/270$
$[3]^{22}F_{5/2}\,[21]^{24}D_{5/2}$	0	0	0	0	−7/60	0	0	0	0	0	17/108	0	0	$(2)^{\frac{1}{2}}/6$	−11/270
$[21]^{24}P_{5/2}\,[21]^{22}D_{5/2}$	0	0	0	0	$(3)^{\frac{1}{2}}/360$	0	$(3)^{\frac{1}{2}}/24$	$-(15)^{\frac{1}{2}}/90$	$-(2)^{\frac{1}{2}}/6$	$(10)^{\frac{1}{2}}/120$	$5(3)^{\frac{1}{2}}/216$	$-(3)^{\frac{1}{2}}/24$	0	$-(6)^{\frac{1}{2}}/8$	$-7(3)^{\frac{1}{2}}/270$
$[21]^{24}P_{5/2}\,[21]^{24}D_{5/2}$	0	0	0	0	$(21)^{\frac{1}{2}}/360$	0	$-(21)^{\frac{1}{2}}/24$	$-(105)^{\frac{1}{2}}/90$	0	0	$5(21)^{\frac{1}{2}}/216$	$(21)^{\frac{1}{2}}/24$	0	0	$-7(21)^{\frac{1}{2}}/270$
$[21]^{22}D_{5/2}\,[21]^{24}D_{5/2}$	0	0	0	0	$(7)^{\frac{1}{2}}/120$	0	$-(7)^{\frac{1}{2}}/24$	0	0	$(210)^{\frac{1}{2}}/120$	$7(7)^{\frac{1}{2}}/216$	$(7)^{\frac{1}{2}}/24$	0	$-(14)^{\frac{1}{2}}/24$	$-11(7)^{\frac{1}{2}}/270$
$[21]^{42}D_{5/2}\,[21]^{42}D_{5/2}$	0	0	0	0	0	0	1/12	0	0	0	0	5/12	1/2	$-(2)^{\frac{1}{2}}/6$	0
$[3]^{22}F_{7/2}\,[3]^{22}F_{7/2}$	0	0	0	0	0	0	0	0	0	0	1/9	0	1/2	0	7/18
$[21]^{24}D_{7/2}\,[21]^{24}D_{7/2}$	0	0	0	0	0	0	0	0	0	0	1/18	1/2	0	0	4/9
$[3]^{22}F_{7/2}\,[21]^{24}D_{7/2}$	0	0	0	0	0	0	0	0	0	0	$-(2)^{\frac{1}{2}}/18$	0	0	1/2	$(2)^{\frac{1}{2}}/18$

any change of parameters as we go from one configuration to the next. We could overcome this difficulty by writing the matrix elements of Li^7 as a linear combination of the matrix elements of Li^6 plus a linear combination of changes in radial integrals as we go from Li^6 to Li^7. Such an approach has been taken for the atomic spectra of Ti[9] and V,[4] involves a knowledge of the magnitudes of the matrix elements of Li^6 and Li^7, and treats the *changes* in radial integrals as unknowns.

Another calculation would be a true Bacher and Goudsmit calculation in which we included the p energy contribution as well as the contribution from p^2.

An ideal application of the method we used would occur if all of the l^{n-1} matrices were diagonal so that the l^{n-1} matrix elements corresponded exactly to the l^{n-1} observed energies. Unfortunately, for light elements it appears that a large off-diagonal element due to spin-orbit interaction is necessary to explain the level structure. Therefore it becomes necessary to deviate from our ideal calculation and to find some way to get values for the diagonal and off-diagonal elements in Li^6.

II. FITTING Li^6 DATA

Another problem is the lack of knowledge of exactly what nuclear force to assume. Various combinations like the Rosenfeld or Serber forces are often used,[10,11] but these are still only rough approximations. If we had enough precise experimental information so that we could fit the energy levels of the two successive isotopes fairly well with the same force mixture, then it would be possible to pin down the relative amounts of the various forces involved. Here we would also make the reasonable assumption that the radial integrals involved vary slowly as we go from nucleus to nucleus. We consider only a few force mixtures, but believe we can indicate a reasonable array of forces to describe the Li isotopes by considering the magnitudes and sources of the various p^3 elements.

Let us consider what we know about Li^6 (Table II).[12,13] Of all the 10 levels of the p^2 configuration only the $^3D_2$ and $^3D_3$ are diagonal, unencumbered by any off-diagonal elements (see Elliott[5]). None of the P states have been observed. We tried to fit the Li^6 data by mixtures of central forces and tensor forces alone, but were unable to do so. We also briefly tried fitting a mixture of central, tensor, and spin-orbit forces, but the tensor force parameters gave too much trouble. We therefore restricted ourselves to the consideration of mixtures of spin-orbit and central forces.

It is tempting to identify the 6.63-Mev level of Li^6 as $^3P_2$. If this is done, however, there is no way to fit the observed experimental difference of 1.32 Mev between $^3P_2$ and $^1D_2$ with any combination of spin-orbit and central forces and still fit the observed $^3D_2$, $^3D_3$ splitting. This is seen from the $J=2$ matrix,

	$^3P_2$	$^1D_2$
$^3P_2$	A	$a/\sqrt{2}$,
$^1D_2$	$a/\sqrt{2}$	B

which, when solved for the difference between the eigenvalues, gives

$$\lambda_+-\lambda_-=1.32=[(A-B)^2+4(a/\sqrt{2})^2]^{\frac{1}{2}}. \quad (4)$$

The maximum value of a, i.e., if $(A-B)=0$, gives a value of $a=-0.94$ Mev, too small to fit the observed $^3D_2$, $^3D_3$ splitting of 2.33 Mev.[13] We assume then that the 6.63-Mev level is due to some higher configuration. Since we have only two diagonal levels, and no P states are identified, we must try to calculate the values for the various matrix elements from experimental energy level data. We proceed as follows in fitting the Li^6 energies.

TABLE II. Observed levels of Li^6. All energies are in units of Mev.

Energy	J	T	π
9.3[a]			
8.37		1	
7.40		0	
6.63		1	
5.6[a]	1	0	+
5.4			
5.35[a]	2	1	+
4.52	2	0	+
3.57	0	0	+
2.189	3	0	+
0	1	0	+

[a] See reference 12; the 5.6-Mev level is reported as 5.6±0.2 Mev. All other levels are taken from Ajzenberg and Lauritsen (reference 13).

TABLE III. Radial integrals and calculated Li^6 energy levels. All integrals and energies are in Mev. Results of three calculations are listed: (1) Rosenfeld force, with $^3D_1=5.6$ Mev; (2) Rosenfeld force, $^3D_1=5.4$ Mev; (3) Serber-like force, $^3D_1=5.6$ Mev.

Calc. No.	Rosenfeld (1)	Rosenfeld (2)	Serber-like (3)
	Radial integrals		
L	−6.72	−6.71	−7.15
K	−1.17	−1.16	−1.18
a	−1.41	−1.41	−1.53
L/K	5.74	5.76	6.07
a/K	1.21	1.21	1.30
	Energies observed		
...	12.94	12.92	11.12
...	12.72	12.71	10.93
...	11.80	11.78	10.07
...	10.60	10.58	8.63
5.6			
5.4	5.50	5.46	5.60
5.35	5.63	5.61	5.62
4.52	4.33	4.30	4.45
3.57	3.30	3.29	3.31
2.189	2.21	2.18	2.16
0	0	0	0

[9] S. Meshkov, Phys. Rev. **91**, 871 (1953).
[10] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).
[11] A. M. Lane, Proc. Phys. Soc. (London) **A68**, 197 (1955).
[12] Allen, Almqvist, and Bigham, Phys. Rev. **99**, 631(A) (1955).
[13] F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).

TABLE IV. Selected numerical matrix elements of the $J=5/2$, $T=1/2$ matrix. All matrix elements are in units of Mev. The first three Li^7 elements are calculated with the Rosenfeld force, the next three with the Serber-like force. The numerical Li^6 matrix elements listed, already have their central force contributions multiplied by 3 and their spin-orbit parts multiplied by 1.5. For example, the $^{13}D_1\ ^{13}D_1$ entry $=3\,[-5.55]+1.5[2.12]=-13.47$. In the listed matrix elements, the experimental $^{13}D_2$ and $^{13}D_3$ energies have been used.

Li^7 \ Li^6		$^{13}S_1\ ^{13}S_1$	$^{13}D_1\ ^{13}D_1$	$^{11}P_1\ ^{11}P_1$	$^{33}P_1\ ^{33}P_1$	$^{13}S_1\ ^{11}P_1$	$^{13}D_1\ ^{11}P_1$	$^{13}D_2\ ^{13}D_2$	$^{33}P_2\ ^{33}P_2$	$^{31}D_2\ ^{31}D_2$	$^{31}D_2\ ^{33}P_2$	$^{13}D_3\ ^{13}D_3$	Total
	a	−27.17	−13.47	9.62	6.83	−1.73	1.93	−14.96	4.72	−9.98	−1.50	−18.81	
$[3]^{22}F_{5/2}\ [3]^{22}F_{5/2}$		0	−3.14	0	0	0	0	−3.60	0	−4.99	0	−0.49	−12.22
$[21]^{24}P_{5/2}\ [21]^{24}P_{5/2}$		−6.04	−0.04	0	0.85	0	0	−0.62	1.77	0	0	−4.39	−8.47
$[3]^{22}F_{5/2}\ [21]^{24}P_{5/2}$		0	0.34	0	0	0	0	−0.63	0	0	0	0.32	0.03
	b	−28.53	−14.48	2.17	−1.02	−1.87	2.09	−16.57	−3.32	−10.76	−1.62	−20.14	
$[3]^{22}F_{5/2}\ [3]^{22}F_{5/2}$		0	−3.38	0	0	0	0	−3.99	0	−5.38	0	−0.52	−13.27
$[21]^{24}P_{5/2}\ [21]^{24}P_{5/2}$		−6.34	−0.04	0	−0.13	0	0	−0.69	−1.24	0	0	−4.70	−13.14
$[3]^{22}F_{5/2}\ [21]^{24}P_{5/2}$		0	0.37	0	0	0	0	−0.70	0	0	0	0.34	0.01

[a] Numerical values of Li^6 matrix elements with Rosenfeld force.
[b] Numerical values of Li^6 matrix elements with Serber-like force.

First we assume a force mixture for Li^6, and calculate the matrix elements in terms of radial integrals. We assume some reasonable numerical values for these integrals and obtain the numerical transformation matrices which diagonalize the Li^6 matrices expressed in terms of the radial integrals. This process yields a set of equations linear in the various radial integrals. The equations are then matched with the known experimental results, differences are taken and a least squares solution of these equations is undertaken for the best values of the radial integrals. With this new set of integrals we can evaluate the matrix elements numerically, diagonalize the matrices and compare our results with experiment. For better accuracy the approximation process may be repeated.

Two force mixtures were used in fitting the Li^6 data. The first mixture used was the Rosenfeld mixture with $V_M=0.8$, $V_B=0.2$ or $a_0=-0.1$, $a_\sigma=-0.1$, $a_\tau=-0.2$, $a_{\sigma\tau}=-0.2$. The a's are the coefficients used in Elliott[5] and are related to the V's by Eq. (5.32) of Blatt and Weisskopf.[14] The Li^6 levels used to fit the data were 0, 2.189, 3.57, 4.52, 5.35, and 5.6 Mev. The resulting parameters and calculated level positions are listed in Table III. A similar calculation was carried out replacing the 5.6-Mev level by 5.4 Mev, but almost no difference in parameters resulted. The 5.6-Mev value was taken from Allen,[12] the 5.4-Mev from Ajzenberg and Lauritsen.[13] In both these calculations the $^{13}D_1$ falls slightly below the $^{31}D_2$, a fact which is not too distressing in view of the inaccuracy of the $^{13}D_2$ and $^{31}S_0$ levels, both of which are too low by $\sim$0.3 Mev. As Inglis[10] points out, the P states are $\sim$5 Mev above the D states.

III. NUMERICAL CALCULATION OF p^3 MATRIX ELEMENTS OF Li^7 AND CHOICE OF FORCE

Using the p^2 matrix elements as computed from the parameters obtained in our Li^6 fitting, we form the Li^7 numerical matrices and diagonalize them. Listed in Table IV are the contributions of the various p^2 matrix elements to the p^3 $J=5/2$ matrix, which are relevant to the discussion. In the tables we have already multiplied the central force contribution by 3 and the spin orbit part by 1.5.

In an attempt to introduce as much experimental data as possible, we use the experimental $^{13}D_2$ and $^{13}D_3$ energies rather than the computed $^{13}D_2$ and $^{13}D_3$ matrix elements, inasmuch as they are single levels. In apportioning their central and spin-orbit contributions, we use the same relative amounts as those calculated from the radial integrals. The various Li^7 matrices were diagonalized by use of a desk computer and the results are shown in Table V. The calculations were also carried through with the computed $^{13}D_2$ and $^{13}D_3$ elements, but, as may be seen from Table V, this did not cause a significant change.

Table VI shows the experimentally observed states of Li^7. Also shown are some recent measurements by

TABLE V. Calculated energies of Li^7. All energies are in units of Mev. Results are listed for Rosenfeld and Serber-like potentials. Columns headed a are results obtained using observed $^{13}D_2$ and $^{13}D_3$ energies, those headed b used calculated $^{13}D_2$ and $^{13}D_3$ energies. It is not expected that the higher state energies are very significant due to lack of precision in potential and radial integrals, and also due to the omission of configuration interaction.

		Rosenfeld		Serber-like	
No.	Term	a	b	a	b
21	$[111]^{22}S_{1/2}$	26.52	26.54	20.77	20.82
20	$[111]^{44}S_{3/2}$	24.43	24.46	18.44	18.48
19	$[21]^{42}D_{3/2}$	17.28	17.31	14.68	14.72
18	$[21]^{24}D_{1/2}$	16.54	16.58	14.26	14.29
17	$[21]^{22}D_{3/2}$	16.53	16.60	14.12	14.10
16	$[21]^{42}D_{5/2}$	16.34	16.36	13.72	13.76
15	$[21]^{22}D_{5/2}$	15.84	15.79	13.22	13.32
14	$[21]^{24}D_{3/2}$	15.39	15.18	12.54	12.58
13	$[21]^{42}P_{1/2}$	14.93	14.96	12.31	12.35
12	$[21]^{42}P_{3/2}$	13.77	13.80	10.93	10.97
11	$[21]^{24}D_{5/2}$	13.22	13.12	10.30	10.28
10	$[21]^{22}P_{1/2}$	13.18	13.09	10.48	10.50
9	$[21]^{22}P_{3/2}$	12.82	12.80	10.03	10.06
8	$[21]^{24}D_{7/2}$	11.76	11.76	8.97	8.96
7	$[21]^{24}P_{1/2}$	11.72	11.71	8.82	8.84
6	$[21]^{24}P_{3/2}$	11.27	11.18	8.37	8.38
5	$[21]^{24}P_{5/2}$	9.71	9.73	6.97	6.96
4	$[3]^{22}F_{5/2}$	6.02	5.90	6.38	6.38
3	$[3]^{22}F_{7/2}$	4.32	4.31	4.82	4.82
2	$[3]^{22}P_{1/2}$	0.53	0.46	0.66	0.72
1	$[3]^{22}P_{3/2}$	0	0	0	0

[14] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952).

Chromchenko and Blinov.[15] Our calculation gives good agreement with the 0.477 Mev, $J=\frac{1}{2}$ level. The 4.61-Mev level seems to have $J=7/2$. According to our calculation the next two levels should have $J=5/2$. Experimentally, the 7.46-Mev level is identified as $5/2^-$, but the level at 6.56 Mev is $1/2^+$, $3/2^+$. The (d,p) experiments of Chromchenko and Blinov give a level at 6.53 Mev with undetermined parity and J. Until angular-distribution studies of this 6.53-Mev level are made, it cannot be ruled out as a possible candidate for being the $[3]^{22}F_{5/2}$ level. Such a choice gives a fairly reasonable value of $1.92/0.477=4.0$ for the ratio of the the ^2F separation to that of the ^2P separation. The ^2F separation in the LS limit, as Inglis[10] points out, should be 7/3 as great as the ^2P separation. Other possible candidates for the low $J=5/2^-$ state might be the states, mentioned by Stoll,[16] at 5.5 Mev or 6.8 Mev. The 5.5-Mev level would give 1.9 for the separation ratio. Inasmuch as it is difficult to separate the 6.8-Mev level in Stoll's experiment from the 7.46-Mev peak, it is doubtful whether his 6.8-Mev level could be the level sought.

Although the existence of the positive parity level at about 6.6 Mev is well established, it is still possible that a negative parity level may be masked by this rather broad positive parity level.

We shall make the assumption that the 7.46-Mev level is the $[21]^{24}P_{5/2}$ level and that there should exist, somewhere between 4.61 Mev and 7.46 Mev, a $J=5/2^-$ level which arises from the $[3]^{22}F_{5/2}$. We now ask the question, "How shall we modify our nuclear potential so as to bring the computed $[21]^{24}P_{5/2}$ energy down from the 9.71 Mev calculated with the Rosenfeld potential into agreement with the experimental value of 7.46 Mev?" Using Tables I and IV we consider the composition of the matrix elements $[3]^{22}F_{5/2}[3]^{22}F_{5/2}$ and $[21]^{24}P_{5/2}[21]^{24}P_{5/2}$ (called elements α and β hereafter). Element α arises only from the D states of Li^6, all of which are known fairly well. Element β, on the other hand, has 1/2 of its value arising from P states, whose magnitudes vary with the force considered. The off-diagonal element connecting states α and β depends only on S and D states and will be little affected by a new potential choice.

In order to change the spacing of the two $J=5/2^-$ levels we must, therefore, alter the potential so as to modify the P state contribution. To decrease the spacing, the P state contributions which are positive must be made less positive or perhaps even negative. Looking at the $^{33}P_1\,^{33}P_1$ matrix element[5] we see that inasmuch as $L-3K$ is negative, the $^{33}P_1\,^{33}P_1$ contribution can be made less positive by changing the coefficient of $L-3K$ to a less negative number than the Rosenfeld value $-3/5$, or perhaps to a positive number. A potential of the Serber type, which also includes spin exchange, satisfies this requirement. After modifying it to decrease the $J=5/2$ separation by the proper magnitude, we arrived at the mixture $V_M=V_W=0.4$, $V_B=0.2$, or $a_0=0.4$, $a_\sigma=0$, $a_\tau=-0.1$ and $a_{\sigma\tau}=-0.1$.

TABLE VI. Observed levels of Li^7. All energies are in units of Mev.

Energy	J	π
17.5		
14.0		
12.4		
10.8		
9.6		
7.46	5/2	−
6.6	1/2,3/2	+
6.53[a]		
5.5		
4.61		
4.454[a]		
0.477	1/2	−
0	3/2	−

[a] From Chromchenko and Blinov (reference 15).

Having chosen this potential we went through the same procedure as for the Rosenfeld case, fitting the Li^6 data anew, calculating the radial integrals and then evaluating the Li^7 matrices. The various values obtained in this calculation are tabulated in the same tables used in the Rosenfeld mixture calculations. Table V shows the first two lines of the $J=5/2$ matrix for the Serber-like calculation. We see that the contributions of the S and D states vary only slightly, whereas the P state contributions actually change sign.

The result of this calculation is to reduce the $J=5/2$ spacing too much, so that now the 7.46-Mev level is reduced to 6.97 Mev and the lower 5/2 level is raised to 6.38 Mev. In addition, the ^2P splitting is increased. A less extreme shift of potential would place the 6.97-Mev level closer to the experimental 7.46 Mev and would thus represent a better nuclear potential. When the levels of Li^7 are more definitely fixed experimentally, the potential may be chosen with more assurance.

ACKNOWLEDGMENTS

We are grateful to F. R. Innes and A. M. Feingold for many interesting discussions, and to N. Kalfe for translating the Russian paper. We also thank N. Kalfe and D. Pearlstein for help in some of the numerical calculations. One of us (S.M.) is grateful to the University of Delaware for granting him a Summer Research Fellowship to complete this work.

[15] L. M. Chromchenko and V. A. Blinov, J. Exptl. and Theoret. Phys. (U.S.S.R.) **28**, 741 (1955).

[16] P. Stoll, Helv. Phys. Acta. **27**, 395 (1954). *Note added in proof.*—A recent experiment [R. W. Gelinas and S. S. Hanna, Bull. Am. Phys. Soc. **30**, 7 (1955)] indicates that with a $Be^9(d,\alpha)Li^7$ reaction, no 5.5-Mev level of Li^7 was found. A $Li^6(d,p)Li^7$ reaction [S. H. Levine, R. S. Bender and J. N. McGreuer, Phys. Rev. **97**, 1249 (1955)] shows a level of Li^7 at 6.56 Mev, but no angular distributions were done.