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Tensor Operator Methods and the Tensor Force

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Tensor operator methods are applied to find the matrix elements of the two-nucleon tensor force between states of two inequivalent nucleons in LS coupling. The results are used to obtain the direct and exchange terms arising from a tensor-force interaction between states of a shell closed except for a single vacancy and external inequivalent nucleon.

INTRODUCTION

THE subject matter of this paper originated in an investigation into the ordering of energy levels of the lowest excited state of O^{16} and the ground state of the Na^{22} nucleus, based on central and noncentral forces. Using the harmonic oscillator model, we assumed the former state to be a mixture of the configurations $(1s)^4(2p)^{11}(2s)$ and $(1s)^4(2p)^{11}(3d)$, and the latter to have the configuration $(1s)^4(2p)^{12}(2s)^4(3d)^2$. In approaching the problem, there was an early appreciation of the need for evaluating the energy of interaction between a shell closed except for a single nucleon, and a group of external nucleons—also with interconfigurational mixing in this case—and the matrix elements of the two-particle noncentral force operators between states of two inequivalent nucleons.

As is well known, these results are particularly simple in the case of a central force alone, a fact which led us to enquire whether or not some similar simplification occurred with noncentral forces. It was found that, while a general expression for a two-particle matrix element remained unwieldy,[1] the tensor force interaction energy between an almost closed shell[2] and single external nucleon reduced to a relatively compact expression.[3]

We here present a reasonably comprehensive derivation of a general expression for the two-nucleon matrix elements and apply the result to a study of an almost closed shell problem.

To this end we propose to use much of the algebra of tensor operators introduced by Racah[4] in the theory of complex spectra together with an extension to the tensor product of tensor operators discussed by several authors.[5,6] The notation employed is largely that of Racah.

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[1] L. W. Longdon, Phys. Rev. **90**, 1125 (1953).

[2] The phrase "almost closed shell" used throughout this paper implies a shell closed except for a single vacancy.

[3] J. Hope, Phys. Rev. **89**, 884 (1953).

[4] G. Racah, Phys. Rev. **62**, 438 (1942). Hereinafter referred to as I.

[5] J. Schwinger, Nuclear Development Associates, Inc., Report NYO-3071, White Plains, New York, 1952 (unpublished).

[6] I. Talmi, Phys. Rev. **89**, 1065 (1953).

1. TENSOR OPERATOR THEORY

We begin by recalling the relevant aspects of the algebra of tensor operators as developed in I.

In this paper an irreducible tensor operator T^k of degree k is defined as a set of $2k+1$ quantities $T_q^k(-k\leqslant q\leqslant k)$ which, under rotations in a three-dimensional Euclidean space, transform like the $2k+1$ components of a spherical harmonic of degree k. In addition they satisfy the same commutation rules with respect to the angular momentum vector components J_z, $J_x\pm iJ_y$, as these functions.

On representing the components T_q^k in the scheme αjm, and writing these commutation rules in the form of matrices, one is led to the result

$$(\alpha jm|T_q^k|\alpha' j'm') = (-1)^{j+j'+k}(\alpha j\|T^k\|\alpha' j')(kqj'm'|jm)(2j+1)^{-\frac{1}{2}}. \quad (1)$$

This equation separates the physical properties of the tensor, which are described by the amplitude matrix

$$(\alpha j\|T^k\|\alpha' j'),$$

from its geometrical properties as exhibited by the Wigner coefficient.

It will be remembered that an irreducible tensor operator T^k is said to be Hermitian [I, Eq. (25)] if its components and those of the adjoint operator $T^{k\dagger}$ are connected by the relation

$$T_q^{k\dagger}=(-1)^q T_{-q}^k.$$

As in vector algebra, one may define many kinds of products of tensor operators. By analogy with the vector addition law in quantum mechanics, the tensor product of order t of two irreducible tensor operators R^r, S^s may be defined by the equation:

$$T_\tau^t=(R^r\odot_\tau^t S^s)=(r\rho s\sigma|t\tau)R_\rho^r S_\sigma^s,$$

in which we have used a dummy suffix notation, i.e., Greek letters occurring more than once imply summation over all of their allowed values. It may be verified that T^t is an irreducible tensor operator. In keeping with the tradition of the theory of atomic spectra the scalar product of two tensor operators of degree k is given by the relation:

$$(R^k\cdot S^k)=(-1)^\mu R_\mu^k S_{-\mu}^k=(-1)^k(2k+1)^{\frac{1}{2}}T_0^0,$$

an example of which is the addition theorem for spherical harmonics of degree k.

The law of combination of irreducible tensor operators given above has very wide applications. T^k is a tensor operator not only when R^r and S^s operate on different systems but also when they operate on different parts of the same system; they are not even required to commute.

To examine the Hermitian character of the tensor product, consider the adjoint operator

$$T_\tau{}^{t\dagger}=(r\rho s\sigma|t\tau)R_\rho{}^{r\dagger}S_\sigma{}^{s\dagger}.$$

Upon assuming that R^r, S^s are Hermitian and commute, it follows at once from Eqs. (16′), (19), and (25) of I that

$$T_\tau{}^{t\dagger}=(R^r\odot_\tau{}^t S^s)^\dagger=(-1)^{r+s-t+\tau}(R^r\odot_{-\tau}{}^t S^s).$$

Thus the tensor product is Hermitian or skew-Hermitian as $r+s-t$ is even or odd. It will, however, be recalled that a tensor operator is necessarily Hermitian if it possesses a real nonzero diagonal element.

2. MATRIX ELEMENTS OF TENSOR PRODUCTS

In order to discuss the matrix element of the tensor product

$$T_\tau{}^t=(r\rho s\sigma|t\tau)R_\rho{}^rS_\sigma{}^s,$$

when R^r, S^s operate on different quantum-mechanical systems, let $\psi(l,\lambda)$, $\psi(k,\kappa)$ be orthonormal wave functions describing these systems, and let $\Psi(\alpha J\mu)$ be a wave function of the system consisting of both. Then extending our summation convention to primed Greek letters as well as unprimed, we have

$$(\alpha J\mu|T_\tau{}^t|\alpha'J'\mu')=(l\lambda k\kappa|J\mu)(r\rho s\sigma|t\tau)(l'\lambda'k'\kappa'|J'\mu')\times(\alpha l\lambda|R_\rho{}^r|\alpha''l'\lambda')(\alpha''k\kappa|S_\sigma{}^s|\alpha'k'\kappa').$$

Using (1), we may write the matrix elements occurring on each side of this equation in terms of the amplitude matrices of their corresponding tensors; multiplying each side of the resulting equation by

$$(-1)^{J+J'+t}(t\tau J'\mu'|J\mu)(2J+1)^{-\frac{1}{2}}$$

and summing over μ, μ', one finds with the help of an orthogonality relation of the Wigner coefficients, that

$$\begin{aligned}(\alpha J\|T^t\|\alpha'J')=&(-1)^{J+J'+t+l+l'+k+k'+r+s}(2t+1)\\&\times[(2l+1)(2k+1)(2J+1)]^{-\frac{1}{2}}\\&\times(\alpha l\|R^r\|\alpha''l')(\alpha''k\|S^s\|\alpha'k')\\&\times(l\lambda k\kappa|J\mu)(r\rho s\sigma|t\tau)(l'\lambda'k'\kappa'|J'\mu')\\&\times(r\rho l'\lambda'|l\lambda)(s\sigma k'\kappa'|k\kappa)(t\tau J'\mu'|J\mu).\end{aligned}$$

The summation over the six Wigner coefficients occurring in this expression is of some significance in a study of the matrix elements of noncentral force operators and various functions have been introduced in the literature to represent it.[5,6] We write

$$\begin{aligned}\chi(lkl'k';JJ';rs;t)&\\=&(-1)^{J+J'+l+l'+k+k'+r+s+t}\{[r]\cdot[s]\cdot[t]\}^{\frac{1}{2}}\\&\times\{[k]\cdot[l]\cdot[J]\}^{-\frac{1}{2}}(r\rho s\sigma|t\tau)(t\tau J'\mu'|J\mu)\\&\times(l\lambda k\kappa|J\mu)(r\rho l'\lambda'|l\lambda)(l'\lambda'k'\kappa'|J'\mu')(s\sigma k'\kappa'|k\kappa),\quad(2)\end{aligned}$$

in which $[r]=(2r+1)$, $\cdots$, etc., so that the equation giving the amplitude factor of the tensor product becomes

$$(\alpha J\|T^t\|\alpha'J')=\{[t]\}^{\frac{1}{2}}\{[r]\cdot[s]\}^{-\frac{1}{2}}\chi(lkl'k';JJ';rs;t)\times(\alpha l\|R^r\|\alpha''l')(\alpha''k\|S^s\|\alpha'k').\quad(3)$$

This is an extension of Eq. (38) of I to tensor products.

The χ function will be recognized as a matrix element in the orthogonal transformation between vector coupled states of four nucleons; its symmetry properties and orthogonality relations have been discussed by Hope and Jahn.[7]

In what follows we shall be concerned with the product of a scalar product of tensor operators with a tensor product, in particular when the scalar product is a product of spherical harmonics C^k of degree k, i.e., we consider:

$$(C^k{}_{(1)}\cdot C^k{}_{(2)})(R^r{}_{(1)}\odot_\tau{}^t S^s{}_{(2)}).$$

With the definitions already given this operator may be written

$$(-1)^\mu(r\rho s\sigma|t\tau)C_\mu{}^k(1)R_\rho{}^r(1)C_{-\mu}{}^k(2)S_\sigma{}^s(2),$$

where the bracketed numbers refer to the quantum-mechanical systems upon which the respective operators act: the unitarity of the Wigner coefficients permits one to write:

$$C_\mu{}^kR_\rho{}^r=\textstyle\sum_{w,m}(k\mu r\rho|wm)L_m{}^w,$$
$$C_{-\mu}{}^kS_\sigma{}^s=\textstyle\sum_{w',m'}(k-\mu s\sigma|w'm')M_{m'}{}^{w'}.$$

Hence this tensor operator may be expressed in terms of operators acting on the separate systems:

$$\sum_{w,w',m,m'}(-1)^\mu(r\rho s\sigma|t\tau)(k\mu r\rho|wm)\times(k-\mu s\sigma|w'm')L_m{}^w(1)M_{m'}{}^{w'}(2).$$

With the aid of a standard result for the sum over a product of three Wigner coefficients, *viz.*,

$$(a\alpha b\beta|c\gamma)(c\gamma d\delta|s\sigma)(b\beta d\delta|r\rho)=\{[c]\cdot[r]\}^{\frac{1}{2}}(a\alpha r\rho|s\sigma)W(absd;cr),\quad(4)$$

this last result becomes:

$$\textstyle\sum_{w,w'}(-1)^k\{[w]\cdot[w']\}^{\frac{1}{2}}W(wkts;rw')(L^w{}_{(1)}\odot_\tau{}^tM^{w'}{}_{(2)}).$$

Thus, in general, one may write

$$(C^k{}_{(1)}\cdot C^k{}_{(2)})(R^r{}_{(1)}\odot^t S^s{}_{(2)})=\textstyle\sum_{w,w'}(-1)^k\{[w]\cdot[w']\}^{\frac{1}{2}}\times W(wkts;rw')(L^w{}_{(1)}\odot^tM^{w'}{}_{(2)}).$$

It is then evident that by using Eq. (3) the amplitude factor of this operator is given by

$$\begin{aligned}(\alpha J\|(C^k{}_{(1)}\cdot C^k{}_{(2)})(R^r{}_{(1)}\odot^t S^s{}_{(2)})\|\alpha'J')&\\=\textstyle\sum_{w,w'}(-1)^k\{[t]\}^{\frac{1}{2}}\chi(lkl'k';JJ';ww';t)&\\\times W(wkts;rw')(\alpha l\|L^w\|\alpha''l')(\alpha''k\|M^{w'}\|\alpha'k').&\end{aligned}$$

We conclude this section by establishing the analog for the χ function, or any of its variants, of Eq. (75)

[7] J. Hope and H. A. Jahn, Phys. Rev. **93**, 318 (1954).

of I. Since this function is a matrix element in the transformation of states of four particles, it is evident that

$$\chi(abcd;ef;gh;k)=(-1)^{2d-f-h+\beta}\chi(abdc;ef;\alpha\beta;k)\times\chi(adcb;\alpha\beta;gh;k). \quad (5a)$$

If we now consider the expansion obtained by multiplying each term of the summation in this relation by $(-1)^\alpha$, and then putting $a=d$, it may be verified from an orthogonality property of the functions and their symmetry relations that:

$$(-1)^{2a-f-h+\alpha+\beta}\chi(abac;ef;\alpha\beta;k)\chi(aacb;\alpha\beta;gh;k)=(-1)^{c+b-k}\delta(e,h)\delta(f,g). \quad (5b)$$

This relation has been found valuable in the consideration of noncentral force interactions in low excited states of closed shell nuclei, as well as in the noncentral force almost closed shell problem.

3. STATIC TENSOR FORCE OPERATOR

We specify each of the quantum-mechanical systems already mentioned to consist of two inequivalent nucleons. The isotopic spin, intrinsic spin, and orbital quantum numbers of the individual nucleons are vector-coupled to form resultants T, S, and L respectively and the two-nucleon state is characterized by the totally antisymmetric wave function

$$\Psi(\gamma n_al_a,n_bl_b,TSLM_TM_SM_L),$$

where γ is some additional quantum number or numbers. This wave function may be separated into mutually independent isotopic spin, intrinsic spin, and orbital wave functions. We specify the orbital wave function to be of the form

$$\psi(\gamma n_al_a,n_bl_b;LM_L)=u_1(n_al_a)u_2(n_bl_b)(l_a\lambda l_b\mu|LM_L)\times Y_\lambda^{l_a}(\theta_1,\varphi_1)Y_\mu^{l_b}(\theta_2,\varphi_2), \quad (6)$$

where the u_i are arbitrary, normalized, single-particle radial wave functions of the argument r_i/b_i (r_i is the radius vector of the ith nucleon and b_i is a parameter), the Y's are spherical harmonics, and the indices denote the particles to which they belong. The two-nucleon static tensor force operator is:

$$H_{12}=T_{12}J(r_{12})\left\{\frac{(\boldsymbol{\sigma}_1\cdot\mathbf{r}_{12})(\boldsymbol{\sigma}_2\cdot\mathbf{r}_{12})}{|\mathbf{r}_{12}|^2}-\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\right\},$$

in which the indices 1, 2 refer to the separate nucleons, $\mathbf{r}_{12}$ is the relative position vector, $J(r_{12})$ is a function describing the nucleon separation and T_{12} may be any one of the following isotopic spin operators:

Neutral: 1; Symmetric: $(\boldsymbol{\tau}_1\cdot\boldsymbol{\tau}_2)$;

Charged: $(\tau_{(1)x}\tau_{(2)x}+\tau_{(1)y}\tau_{(2)y})$.

These may be expressed as tensor products of irreducible tensor operators of degrees 0 and 1. For either nucleon the components of the tensor operator of degree 1 are the three orthonormal functions[8]

$$\tau_{\pm1}{}^1=2^{-\frac{1}{2}}(\mp\tau_x-i\tau_y),\quad \tau_0{}^1=\tau_z$$

formed from the Cartesian components τ_x, τ_y, τ_z of the isotopic spin vector, and which transform according to the representation $D^{(1)}$ of the three-dimensional rotation group. One finds that these operators become:

Neutral: $(\tau^0{}_{(1)}\odot_0{}^0\tau^0{}_{(2)})$,

Symmetric: $-3^{\frac{1}{2}}(\tau^1{}_{(1)}\odot_0{}^0\tau^1{}_{(2)})$,

Charged: $-2[3^{-\frac{1}{2}}(\tau^1{}_{(1)}\odot_0{}^0\tau^1{}_{(2)})+6^{-\frac{1}{2}}(\tau^1{}_{(1)}\odot_0{}^2\tau^1{}_{(2)})]$.

As is well known, the spin-orbital part of the tensor force operator may be expressed as a scalar product of an irreducible tensor $S^{(2)}$ operating in the combined spin spaces of the two nucleons, and a tensor $L^{(2)}$ acting in their combined orbital spaces.

The components of $S^{(2)}$ are constructed from the Cartesian components of the spin vectors $\boldsymbol{\sigma}_k$ ($k=1, 2$) by taking combinations of the orthonormal functions:

$$2^{-\frac{1}{2}}(\mp\sigma_{(k)x}-i\sigma_{(k)y}),\quad \sigma_{(k)z};\quad (k=1,2),$$

which transform according to the representation $D^{(2)}$ of the rotation group. It is not difficult by using Eq. (1) and the properties of the Pauli spin matrices to show that the amplitude matrix of this operator is nonzero only for triplet spin states, i.e.,

$$(S\|S^{(2)}\|S')=2\times5^{\frac{1}{2}}\delta(S,S')\delta(S,1).$$

One may construct in a similar manner the five components of the two nucleon orbital space tensor operator $L^{(2)}$. Using the choice of phase factors in I and the axis of z as axis of quantization one may form from the Cartesian components X, Y, and Z of the unit vector $\mathbf{r}_{12}/|\mathbf{r}_{12}|$ three orthonormal functions which transform according to the $D^{(1)}$ representation:

$$2^{-1}(\mp X-iY),\quad Z.$$

Combining these with themselves to form a set of quantities transforming according to the representation $D^{(2)}$, one obtains the five components of the tensor $L^{(2)}$:

$$L_{\pm2}{}^2=2^{-1}(X\pm iY)^2,\quad L_{\pm1}{}^2=Z(\mp X-iY),$$
$$L_0{}^2=-6^{-\frac{1}{2}}(3Z^2-1).$$

From the spherical polar coordinates (r_k,θ_k,φ_k) of the kth nucleon ($k=1, 2$) one derives expressions for the components of the vector $\mathbf{r}_{12}$ which, after substitution in the components $L_q{}^{(2)}$ and rearrangement in terms of spherical harmonics yields the result:

$$|\mathbf{r}_{12}|^2L_q{}^2=(2/3)^{\frac{1}{2}}\{(C^2{}_{(1)}\odot_q{}^2C^0{}_{(2)})r_1{}^2+(C^0{}_{(1)}\odot_q{}^2C^2{}_{(2)})r_2{}^2\}-2(C^1{}_{(1)}\odot_q{}^2C^1{}_{(2)})r_1r_2,$$

in which the suffices of the spherical harmonic operators refer to the separate nucleons.

[8] The choice of phase factors is that of I.

Thus, upon writing

$$J(r_{12})/|\mathbf{r}_{12}|^2=\sum_{k=0}^{\infty}(C^k{}_{(1)}\cdot C^k{}_{(2)})J_k(r_1,r_2),$$

one arrives at the following expression for the two particle orbital tensor operator

$$J(r_{12})L^{(2)}=[(2/3)^{\frac{1}{2}}\{(C^2{}_{(1)}\odot^2 C^0{}_{(2)})r_1{}^2+(C^0{}_{(1)}\odot^2 C^2{}_{(2)})r_2{}^2\}-2(C^1{}_{(1)}\odot^2 C^1{}_{(2)})r_1r_2]\times\sum_{k=0}^{\infty}(C^k{}_{(1)}\cdot C^k{}_{(2)})J_k(r_1,r_2). \quad (7)$$

The products of spherical harmonics are of a type already discussed and may be written as

$$(C^k{}_{(1)}\cdot C^k{}_{(2)})(C^2{}_{(1)}\odot^2 C^0{}_{(2)})=5^{-\frac{1}{2}}\sum_x(-1)^x\{[x]\}^{\frac{1}{2}}(k020|x0)(C^x{}_{(1)}\odot^2 C^k{}_{(2)}),$$

$$(C^k{}_{(1)}\cdot C^k{}_{(2)})(C^0{}_{(1)}\odot^2 C^2{}_{(2)})=5^{-\frac{1}{2}}\sum_x(-1)^x\{[x]\}^{\frac{1}{2}}(k020|x0)(C^k{}_{(1)}\odot^2 C^x{}_{(2)}),$$

$$(C^k{}_{(1)}\cdot C^k{}_{(2)})(C^1{}_{(1)}\odot^2 C^1{}_{(2)})=\sum_{xy}(-1)^{1+y}\{[x]\cdot[y]\}^{\frac{1}{2}}(k010|x0)(k010|y0)\times W(11xy;2k)(C^x{}_{(1)}\odot^2 C^y{}_{(2)}).$$

These results enable the tensor force operator to be written in terms of single nucleon operators: it is felt that this is the most convenient form for dealing with the almost closed shell problem.

Let us next examine the amplitude matrix of the operator $J(r_{12})$. $L^{(2)}$ in the general two-nucleon states (6). For convenience denote the integration over the radial wave functions by

$$A_1{}^k=(n_al_a,n_bl_b|J_k(r_1,r_2)r_1{}^2|n_cl_c,n_dl_d),$$

$$A_2{}^k=(n_al_a,n_bl_b|J_k(r_1,r_2)r_2{}^2|n_cl_c,n_dl_d),$$

$$B^k=(n_al_a,n_bl_b|J_k(r_1,r_2)r_1r_2|n_cl_c,n_dl_d).$$

Upon substituting in (7) the expressions for the products of the spherical harmonic operators and applying (3), it is apparent that the amplitude matrix of this operator is readily written in terms of these radial integrals and the single particle amplitude matrices of the relevant spherical harmonic operators. The value of the latter is obtained from the general result given in I, *viz.*,

$$(j\|C^q\|j')=(-1)^{g-j}[\{[j][j']C(j,q,j')\}/2]^{\frac{1}{2}} \quad (2g=j+q+j'), \quad (8)$$

in which $C(a,b,c)=0$ if $a+b+c$ is odd, and

$$C(a,b,c)=\frac{2(a+b-c)!(b+c-a)!(c+a-b)!g!^2}{(a+b+c+1)!(g-a)!^2(g-b)!^2(g-c)!^2}$$

when $a+b+c=2g$, where g is integral. Certain values of these coefficients have been tabulated by Shortley and Fried.[9] The explicit connection with the Wigner coefficients may be argued from paragraph 4 of I, and is

$$(c0b0|a0)=(-1)^{\frac{1}{2}(a+b+c)-a}[\{[a]C(a,b,c)\}/2]^{\frac{1}{2}}. \quad (9)$$

[9] G. H. Shortley and B. Fried, Phys. Rev. **54**, 739 (1938).

We do not present a detailed account of the manipulation involved, but it is not too tedious to verify that the amplitude matrix under consideration is given by

$$\Delta\sum_{k=0}^{\infty}[6^{-\frac{1}{2}}\{A_1{}^k\sum_x\Theta(l_al_bl_cl_d;LL';xk;k202)+A_2{}^k\sum_x\Theta(l_al_bl_cl_d;LL';kx;k022)\}-B^k\sum_{xy}\Theta(l_al_bl_cl_d;LL';xy;k112)], \quad (10)$$

where

$$\Delta=(-1)^{\frac{1}{2}(l_c+l_d-l_a-l_b)-1}2^{-1}\{5[l_a]\cdot[l_b]\cdot[l_c]\cdot[l_d]\}^{\frac{1}{2}},$$

and

$$\Theta(abcd;ef;xy;krst)=\{[x][y]C(a,x,c)C(b,y,d)C(k,r,x)C(k,s,y)\}^{\frac{1}{2}}\times W(rsxy;tk)\chi(abcd;ef;xy;t).$$

To conclude this aspect of the development we remind the reader that (1) and Eq. (38) of I may be employed to furnish an expression for the matrix element of the tensor force operator in the γTM_TLSJM scheme. Explicitly:

$$(\gamma n_al_a,n_bl_b;TM_TLSJM|H_{12}|\gamma'n_cl_c,n_dl_d;TM_TL'SJM)=(-1)^{1+L-J}\delta(s,1)\Phi(T)2\times 5^{\frac{1}{2}}\,W(L1L'1;J2)\times(\gamma n_al_a,n_bl_b;L\|J(r_{12})L^{(2)}\|\gamma'n_cl_c,n_dl_d;L'). \quad (11)$$

Here $\Phi(T)$ is a factor whose value is determined by the type of charge dependence assumed, and is tabulated:

	Neutral	Symmetric	Charged	
$\Phi(T)$	$(\delta(T,0)+\delta(T,1))$	$(-3\delta(T,0)+\delta(T,1))$	$(-2\delta(T,0)+2\delta(T,1)\delta(M_T,0))$.	(12)

The calculation of this matrix element for special states can be very tedious, especially if large values of the quantum numbers $l_a, \ldots$ are involved. However, labor may be saved in several cases by evaluating the relevant functions from tables of Racah's W function[10] and the Shortley and Fried coefficients.

In the course of s, p, and d shell calculations employing (10), (11), and oscillator wave functions it was found, in company with Elliott,[11] that the separate radial integrals contain a divergent part which is proportional to $(2k+1)$. This singularity disappeared on taking the correct linear combination of these integrals. In an appendix we demonstrate that if the integrals $A_i{}^k$ $(i=1, 2)$, B^k are each equal to $\alpha(2k+1)$ where α is a constant, then they do not occur in the evaluated matrix element.

The special case of the result at (10) when all the nucleons are equivalent is of some interest in problems involving a group of equivalent nucleons. Reverting to the radial integrals $A_i{}^k$ $(i=1, 2)$, B^k, it is easy to show by interchanging coordinates, that when

$$n_a=n_b=n_c=n_d=n, \quad l_a=l_b=l_c=l_d=l,$$

[10] L. C. Biedenharn, Oak Ridge National Laboratory Report, No. 1098, 1952 (unpublished); Biedenharn, Blatt, and Rose, Revs. Modern Phys. **24**, 249 (1952).

[11] J. P. Elliott, Proc. Roy. Soc. (London) **A218**, 345 (1953).

we have $A_1{}^k = A_2{}^k = A^k$. The connection between the coefficients of these radial integrals arises from the symmetry property

$$\chi(llll; LL'; xk; 2) = \chi(llll; LL'; kx; 2)$$

of the χ function; consequently:

$$((nl)^2L\|J(r_{12})L^{(2)}\|(nl)^2L')$$
$$= \Delta_1 \sum_{k=0}^{\infty} [(2/3)^{\frac{1}{2}}\{A^k \sum_x \Theta(llll; LL'; xk; k202)\} - B^k \sum_{xy} \Theta(llll; LL'; xy; k112)],$$

where $\Delta_1 = -5^{\frac{1}{2}}(2l+1)^2/2$.

Finally the expression at (10) may be adapted to yield the direct and exchange terms of the tensor force interaction between the two nucleon states (6). The specialization which gives rise to the exchange term requires multiplication by an exchange phase factor,

$$(-1)^{l_c+l_d-L'}.$$

4. APPLICATION OF TENSOR OPERATOR METHODS TO ALMOST CLOSED AND CLOSED SHELLS

In this section it is assumed that the quantum-mechanical systems referred to earlier consist of a group of equivalent nucleons forming an almost closed or closed shell, and an isolated nucleon not of the same shell. To discuss the tensor force interaction energy of such a configuration we follow the method of I, paragraph 6, for the atomic central force: that is, we write down a general element in the energy matrix for a two nucleon configuration, and then replace the amplitude matrix element of a single nucleon tensor operator by that appropriate to the group. Clearly the operator whose amplitude matrix is to be replaced must act entirely within a single shell.

We have seen that the amplitude matrix of the tensor force may be written as a product of amplitude matrices of tensor products in the isotopic spin, intrinsic spin and orbital spaces, typical direct and exchange terms of which are:

$$d_j = (j_a j_b j\|(R^r \odot^t S^s)\|j_c j_d j'), \quad (13)$$

$$e_j = (-1)^{j_c+j_d-j'}(j_a j_b j\|(R^r \odot^t S^s)\|j_d j_c j'). \quad (14)$$

An application of Eq. (3) to the direct term d_j shows that if either $j_a = j_c$ or $j_b = j_d$, or both, then one or the other of the tensor operators acts within a shell. This, however, is not true of the exchange term e_j, for on applying Eq. (3)

$$e_j = (-1)^{j_c+j_d-j'}\{[t]\}^{\frac{1}{2}}\{[r]\cdot[s]\}^{-\frac{1}{2}} \times\chi(j_a j_b j_d j_c; jj'; rs; t)(j_a\|R^r\|j_d)(j_b\|S^s\|j_c). \quad (15)$$

Now the exchange term arises from the antisymmetric nature of the wave function characterizing the two nucleon system, as may readily be seen by constructing the relevant Slater determinant from the single particle wave functions of the constituent nuclei.

The immediate aim is to show that the tensor product whose amplitude matrix appears at (14) can be replaced by one formed from single particle tensors which operate in the same shell, when either of the above specializations of quantum numbers is made.

From Eq. (5a) one may put

$$\chi(j_a j_b j_d j_c; jj'; rs; t) = (-1)^{2j_c-j'-s+\beta} \times\chi(j_a j_b j_c j_d; jj'; \alpha\beta; t)\chi(j_a j_c j_d j_b; \alpha\beta; rs; t),$$

in which α and β play the role of summation indices.

Upon introducing tensor operators u^p, v^q defined by the relations

$$(j_a\|u^p\|j_c) = (2p+1)^{\frac{1}{2}}, \quad (j_b\|v^q\|j_d) = (2q+1)^{\frac{1}{2}} \quad (16)$$

for all p and q, one finds, after some algebraic manipulation, that

$$e_j = (-1)^{3j_c-2j'+j_d-s+\beta}\{[r]\cdot[s]\}^{-\frac{1}{2}} \times(j_a\|R^r\|j_d)(j_b\|S^s\|j_c)\chi(j_a j_c j_d j_b; \alpha\beta; rs; t) \times(j_a j_b j\|(u^\alpha \odot^t v^\beta)\|j_c j_d j'). \quad (17)$$

Consequently the desired result will be obtained by replacing

$$(R^r \odot^t S^s)$$

in Eq. (14) by

$$\{[r]\cdot[s]\}^{-\frac{1}{2}}(j_a\|R^r\|j_d)(j_b\|S^s\|j_c)(-1)^{2j_c-j'-s+\beta} \times\chi(j_a j_c j_d j_b; \alpha\beta; rs; t)(u^\alpha{}_{(1)} \odot^t v^\beta{}_{(2)}). \quad (18)$$

To provide some measure of uniformity the operator occurring in the direct term may be written in the form

$$\{[r]\cdot[s]\}^{-\frac{1}{2}}(j_a\|R^r\|j_c)(j_b\|S^s\|j_d)(u^r{}_{(1)} \odot^t v^s{}_{(2)}). \quad (19)$$

With the interaction written in terms of operators of this form, the method outlined at the beginning of the section can be applied.

Some digression is indicated at this stage to make trivial extensions of the atomic almost-closed-shell theory of Racah to cover the nuclear case.

Following I, paragraph 5, we define a triple tensor of degree (tsk) to be a quantity which behaves as an irreducible tensor of degree t in isotopic spin space, an irreducible tensor of degree s in intrinsic spin space, and an irreducible tensor of degree k in orbital space.

Let $S^{(tsk)}(\mathfrak{L})$ be a triple tensor operator acting with a group $\mathfrak{L}$ of ϵ equivalent nucleons and defined as the sum

$$S^{(tsk)}(\mathfrak{L}) = \sum_{r=1}^{\epsilon} t^{(tsk)}{}_{(r)}$$

of single-nucleon triple tensors. Also let $S^{(tsk)}(\mathfrak{N})$ be a triple tensor acting on the group $\mathfrak{N}$ of $(m-\epsilon)$ equivalent nucleons which go to make up the closed shell. The relation between the amplitude matrices of $S^{(tsk)}(\mathfrak{N})$ and the adjoint of $S^{(tsk)}(\mathfrak{L})$ is the strict analogy of Eq. (74) of I, i.e.,

$$(\gamma''T''S''L''\|S^{(tsk)}(\mathfrak{N})\|\gamma'T'S'L')$$
$$= (-1)^{1+t+s+k}(\gamma''T''S''L''\|S^{(tsk)}(\mathfrak{L})^\dagger\|\gamma'T'S'L'),[12] \quad (20)$$

[12] One of us (J.H.) wishes to point out an error in his Ph.D. (London) thesis. The footnote on page 49 is incorrect and the

and may be established by a similar argument. The result may still be said to be valid for a triple scalar operator, except for a constant diagonal term, for if

$$(\gamma'T'S'L'\|S^{(000)}(\mathfrak{L})\|\gamma'T'S'L')=a\epsilon,$$

where

$$a=(\tfrac{1}{2}\tfrac{1}{2}l\|t^{(000)}\|\tfrac{1}{2}\tfrac{1}{2}l),$$

then

$$(\gamma'T'S'L'\|S^{(000)}(\mathfrak{R})\|\gamma'T'S'L')=(m-\epsilon)a.$$

The representation of the two nucleon tensor force in terms of irreducible single nucleon tensor operators enables one to introduce a triple tensor $s^{(tsk)}$ operating in the combined isotopic spin, intrinsic spin and orbital spaces of any one nucleon, whose amplitude matrix satisfies the relation

$$(\tfrac{1}{2}\tfrac{1}{2}nl\|s^{(tsk)}\|\tfrac{1}{2}\tfrac{1}{2}nl)=(\tfrac{1}{2}\|\tau^t\|\tfrac{1}{2})(\tfrac{1}{2}\|\sigma^s\|\tfrac{1}{2})(nl\|T^k\|nl). \quad (21)$$

It will be noted that each of the single-particle operators τ^t, σ^s, and T^k in respectively isotopic spin, intrinsic spin, and orbital space possesses nonzero, real diagonal elements, and is consequently Hermitian.

The result of replacing each single-nucleon amplitude matrix in (21) by that appropriate to the almost closed shell may, from (20), be written as

$$\begin{aligned}(\tfrac{1}{2}\tfrac{1}{2}nl\|S^{(tsk)}(m-1)\|\tfrac{1}{2}\tfrac{1}{2}nl)\\ =(-1)^{1+t+s+k}(\tfrac{1}{2}\tfrac{1}{2}nl\|s^{(tsk)}\|\tfrac{1}{2}\tfrac{1}{2}nl)\\ +m(\tfrac{1}{2}\tfrac{1}{2}nl\|s^{(tsk)}\|\tfrac{1}{2}\tfrac{1}{2}nl)\delta(t,0)\delta(s,0)\delta(k,0). \quad (22)\end{aligned}$$

For convenience in reference, the terms on the right-hand side of this last equation are called the "normal" and "null" terms respectively.

We now return to the main theme and apply these extensions to find the direct and exchange terms for the tensor force in quantum-mechanical systems symbolized by the wave functions:

$$\Psi(\gamma(nl)^{m-1},n_al_a,TM_TLSJM),$$
$$\Psi(\gamma'(nl)^{m-1},n_bl_b,T'M_{T'}L'S'J'M'),$$

where $m=4(2l+1)$.

It will be recalled that the tensor force operator is:

$$\begin{aligned}H_{12}=(\tau^m{}_{(1)}\odot^n\tau^m{}_{(2)})(\sigma^1{}_{(1)}\odot^2\sigma^1{}_{(2)})\sum_{k=0}^{\infty}J_k(r_1,r_2)\\ \times[(2/15)^{\frac{1}{2}}\textstyle\sum_x(-1)^x\{[x]\}^{\frac{1}{2}}(k020|x0)\\ \times\{r_1^2(C^x{}_{(1)}\odot^2C^k{}_{(2)})+r_2^2(C^k{}_{(1)}\odot^2C^x{}_{(2)})\}\\ +2r_1r_2\textstyle\sum_{xy}(-1)^y\{[x]\cdot[y]\}^{\frac{1}{2}}(k010|x0)\\ \times(k010|y0)W(11xy:2k)(C^x{}_{(1)}\odot^2C^y{}_{(2)})].\end{aligned}$$

From this form it is immediate that the two nucleon direct term consists of the product of

$$(T\|(\tau^m{}_{(1)}\odot^n\tau^m{}_{(2)})\|T')(S\|(\sigma^1{}_{(1)}\odot^2\sigma^1{}_{(2)})\|S') \quad (23)$$

content of Chapter II, paragraph 6 is therefore misleading. The effect of this error on the numerical work of Chapter IX, paragraph 6 has not been completely explored, but it is felt that the general trend of results will not be greatly changed.

with a linear combination of

$$\begin{aligned}(ll_aL\|(C^x{}_{(1)}\odot^2C^k{}_{(2)})\|ll_bL'),\\ (ll_aL\|(C^k{}_{(1)}\odot^2C^x{}_{(2)})\|ll_bL'),\\ (ll_aL\|(C^x{}_{(1)}\odot^2C^y{}_{(2)})\|ll_bL').\end{aligned} \quad (24)$$

Hence the transition to the almost closed (n,l) shell is obtained by replacing the amplitude matrices at (23) and (24) by the appropriate "normal" terms, there being no other contribution owing to the presence of the Kronecker delta symbol $\delta(1,0)$ in the intrinsic spin "null" term. The factors

$$(-1)^{x+1},\quad(-1)^{k+1},\quad(-1)^{x+1},$$

by which the amplitude matrices at (24) are respectively modified in the transition may be easily evaluated by recalling from Eqs. (3) and (8) that these amplitude matrices give rise respectively to the Shortley and Fried coefficients:

$$C(l,x,l),\quad C(l,k,l),\quad C(l,x,l).$$

Consequently both x and k are even, and the factors are each equal to -1. Noticing that the "normal" term at (22) may be written as

$$(-1)^{1+t}(\tfrac{1}{2}\|\tau^t\|\tfrac{1}{2})(-1)^{1+s}(\tfrac{1}{2}\|\sigma^s\|\tfrac{1}{2})(-1)^{1+k}(l\|T^k\|l),$$

it is apparent that the transition to the almost closed shell case induces a phase factor $(-1)^{1+m}$ in the two-nucleon amplitude factors at (23). Thus, taking into account the above result for the orbital part of the tensor force operator, the matrix elements D_{m-1}, D of the m nucleon and two nucleon operators are connected by:

$D_{m-1}=D$ for neutral isotopic spin,

$D_{m-1}=-D$ for symmetric and charged isotopic spin.

While the reduction of the almost closed shell exchange term requires considerably more manipulation, the final result is surprisingly simple in contrast with the corresponding two-nucleon term, as the following analysis shows.

As before, apart from exchange phase factors, the two-nucleon amplitude matrix consists of the product of (23) with a linear combination of

$$\begin{aligned}(ll_aL\|(C^x{}_{(1)}\odot^2C^k{}_{(2)})\|l_blL'),\\ (ll_aL\|(C^k{}_{(1)}\odot^2C^x{}_{(2)})\|l_blL'),\\ (ll_aL\|(C^x{}_{(1)}\odot^2C^y{}_{(2)})\|l_blL'),\end{aligned} \quad (25)$$

we accordingly restrict ourselves to the consideration of a typical amplitude matrix:

$$(jj_aJ\|(X^x{}_{(1)}\odot^zY^y{}_{(2)})\|j_bjJ'). \quad (26)$$

Successive applications of Eqs. (17) and (3) show that this becomes:

$$\begin{aligned}(-1)^{\beta-J'-y}(2z+1)^{\frac{1}{2}}\{[x]\cdot[y]\cdot[\alpha]\cdot[\beta]\}^{-\frac{1}{2}}\\ \times(j\|X^x\|j_b)(j_a\|Y^y\|j)\chi(jj_ajj_b;JJ';\alpha\beta;z)\\ \times\chi(jjj_bj_a;\alpha\beta;xy;z)(j\|u^\alpha\|j)(j_a\|v^\beta\|j_b).\end{aligned}$$

For a reason already stated, the transition to the almost closed shell case can only give rise to a "normal" term which may be rearranged as

$$(-1)^{\alpha+\beta-2j-J'-y+1}\{[x]\cdot[y]\}^{-\frac{1}{2}}(j\|X^x\|j_b)(j_a\|Y^y\|j)\times\chi(jj_ajj_b;JJ';\alpha\beta;z)(2z+1)^{\frac{1}{2}}\chi(jjj_bj_a;\alpha\beta;xy;z).$$

Recalling that α and β play the role of dummy suffices and applying the symmetry relation of the χ function given at (5b) one finds that this "normal" term may be written as

$$(-1)^{1+j_a+j_b-z}(2z+1)^{\frac{1}{2}}\{[x]\cdot[y]\}^{-\frac{1}{2}}\times(j\|X^x\|j_b)(j_a\|Y^y\|j)\delta(J,y)\delta(J',x).$$

This result may be repeatedly applied to the almost closed shell amplitude matrices corresponding to those listed at (23) and (25): using properties of the Pauli spin matrices and Eqs. (9), (9a) one evaluates the resulting single-nucleon amplitude matrices and thus obtains expressions for those of the almost closed shell case. After multiplying by appropriate exchange phase factors and using Eqs. (1), it may be shown that the exchange matrix element for the tensor force operator in the almost closed shell case reduces to:

$$\begin{aligned}E_{m-1}=&(-1)^{1+l-J+\frac{1}{2}(l_a+l_b)}\delta(S,1)\delta(S,S')\delta(J,J')\\&\times\delta(M_T,M_{T'})\delta(M,M')(2l+1)\Phi(T)\\&\times\{5[l_a]\cdot[l_b]\cdot C(l,l_a,L)\cdot C(l,l_b,L')\}^{\frac{1}{2}}\\&\times W(L1L'1;J2)[\{C(L,L',2)\}^{\frac{1}{2}}\cdot\{3[L]\cdot[L']\}^{-\frac{1}{2}}\\&\times\{(2L'+1)E_1^L+(2L+1)E_2^{L'}\}\\&-\sum_{k=0}^{\infty}E^kW(11L'L;2k)\\&\times\{5[L]\cdot[L']\cdot C(k,1,L')\cdot C(k,1,L)\}^{\frac{1}{2}}].\end{aligned}$$

The radial integrals E_1^L, $E_2^{L'}$, E^k are the exchange counterparts of the integrals A_1^L, $A_2^{L'}$, B^k already introduced. The function $\Phi(T)$ is specified by the assumed isotopic spin dependence and takes on the values:

	Neutral	Symmetric	Charged
$\Phi(T)$	$-2\delta(T,0)\times\delta(T',0)$	$-2\delta(T,1)\times\delta(T',1)$	$-2\{\delta(M_T,1)+\delta(M_T,-1)\}\times\delta(T,1)\delta(T',1)$. (27)

Finally elements in the energy matrix of the interconfigurational mixing of the configurations $(n,l)^{m-1}(n_a,l_a)$; $(n,l)^{m-1}(n_b,l_b)$ are given by $D_{m-1}-E_{m-1}$.

We conclude by recalling that the value of the amplitude matrix of the two-nucleon spin operator S^2 implies the vanishing of tensor force interactions between a closed shell and external nucleon, also between a closed shell and an almost closed shell.

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APPENDIX

We indicate how to show that

$$\sum_{k=0}^{\infty}(2k+1)[6^{-\frac{1}{2}}\sum_x\{\Theta(l_al_bl_cl_d;LL';xk;k202)+\Theta(l_al_bl_cl_d;LL';kx;k022)\}-\sum_{xy}\Theta(l_al_bl_cl_d;LL';xy;k112)]=0.$$

First we rewrite the above in terms of Wigner coefficients and Racah's W function by using the definition of Θ and an expression for the χ function in terms of three W functions[7]: one chooses the latter so that, in each case, only one of the W functions contains the parameter k.

The summation over k is performed by replacing a Wigner coefficient and W function by the sum over the product of three Wigner coefficients, e.g., for the first term of the above by putting

$$(k0l_d0|l_b0)W(kxl_by;2l_d)=\{5[l_d]\}^{-\frac{1}{2}}(k0x\xi|2\xi)(2\xi y-\xi|l_b0)(x\xi y-\xi|l_d0).$$

The summation over k then embraces two Wigner coefficients and may be identified with an orthogonality relation of these symbols.

An identical technique is successful for summing over x and the summation parameter arising from the χ function for the first two terms, resulting in:

$$(-1)^{1-\frac{1}{2}(l_a+l_b+l_c+l_d)+l_b+l_c}4\{2[L]\}^{\frac{1}{2}}\{3\times5\cdot[l_a]\cdot[l_c]\}^{-\frac{1}{2}}\times(l_d0L'0|l_c0)(l_b0L0|l_a0)(20L0|L'0).$$

Repeatedly summing over k, x, y and the parameter occurring in the representation of χ function as a product of three W functions, shows that the last term reduces to an expression identical to but of opposite sign to that already quoted. The result follows.

[7] H. A. Jahn and J. Hope, Phys. Rev. 93, 318 (1954).