a threshold of 10.1 Mev by neglecting the lower energy points, and a nuclear temperature of 1.17 Mev. Straggling due to the reduction in energy from 31 Mev to 10 Mev probably explains part of the difference.

The data in Table II were obtained from two separate bombardments. The absolute energy of the protons was not known for the first bombardment; so these data were fit into that of the second bombardment by matching threshold energies. The absolute amount of bismuth was known only for a few plates of Run I; so only the ratio $\sigma_{(p,2n)}/[\sigma_{(p,n)}+\sigma_{(p,2n)}]$ is given at many of the energies.

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Deformation Energy of a Charged Drop. I. Qualitative Features

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The qualitative features of the deformation energy of a charged drop are discussed with special reference to the fission process. It is shown that, under conditions more general than the model of an incompressible liquid drop with a sharp surface, the threshold energy for fission should be proportional to $[(Z^2/A)_0-(Z^2/A)]^3$ near the limit of stability $(Z^2/A)_0$. Similarly, if instability against asymmetry sets in below $(Z^2/A)_c$, the degree of asymmetry of the asymmetric saddle-point shapes which appear should be proportional to $[(Z^2/A)_c-(Z^2/A)]^{\frac{1}{2}}$ and the difference between the symmetric and asymmetric thresholds should be proportional to $[(Z^2/A)_c-(Z^2/A)]^2$. Some factors governing the stability against asymmetry of a strongly deformed drop are discussed qualitatively.

ACCORDING to the liquid-drop model, the process of fission is the result of a competition between the long-range electrostatic repulsion and the attractive short-range nuclear forces, idealized in the model as a surface tension.[1] Of importance for the theory of fission is the knowledge of the potential energy of such a system as a function of deformation. The present series of papers will be concerned with this problem. In the first part, we shall consider the qualitative features of the deformation energy.

A quantity of importance for the theory of fission is the ratio of electrostatic to surface energy which, for a nucleus, is approximately proportional to $(Z^2/A^{\frac{1}{3}})\div A^{\frac{2}{3}} = Z^2/A$. A charged drop for which this quantity is less than a certain critical value $[Z^2/A<(Z^2/A)_0]$ is stable against small deformations and the potential energy is an increasing function of the deformation. For larger distortions, a maximum will occur and the energy will decrease thereafter. The least energy necessary to divide the drop (corresponding to the height of the barrier along a suitable deformation path) is of importance in the discussion of fission thresholds. The shape of the drop when in the configuration of unstable equilibrium corresponding to the top of the barrier is also of interest, especially in connection with a discussion of fission asymmetry.

A general deformation of the drop may be specified by a number of deformation coordinates (in general infinite), which, in the case of an incompressible drop with a sharp boundary, could be taken as the coefficients in the expansion of the surface in spherical harmonics. Of special interest are configurations for which the potential energy is stationary with respect to all small distortions. It is convenient to restrict our attention from the beginning to configurations for which the energy is stationary with respect to all except a limited number of deformation parameters. In the case of axially symmetric configurations one may, for example, eliminate in this way all but two coordinates, one symmetric and one asymmetric, the latter specifying deviations from reflection symmetry. The deformation energy is then explicitly a function of two coordinates only and its properties can be discussed conveniently with reference to a deformation energy surface in three dimensions. The points where the energy is stationary with respect to the remaining two coordinates specify configurations for which the energy is stationary with respect to all deformations. The choice of the two coordinates is in principle arbitrary, but in practice it is advantageous to choose them so that they are capable of describing qualitatively the division of the drop into two equal or unequal fragments, the eliminated parameters being concerned with relatively less important features of the configuration. In the case in which the deformation parameters are the coefficients α_n in an expansion of the surface of the drop in Legendre polynomials, a convenient choice is

[1] N. Bohr and J. A. Wheeler, Phys. Rev. 56, 426 (1939).

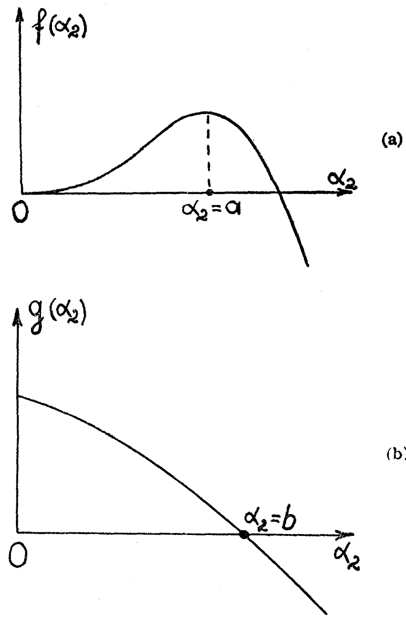

FIG. 1. (a) Qualitative appearance of the function $f(\alpha_2)$, specifying the deformation energy for symmetric distortions. The point $\alpha_2=a$ specifies the symmetric critical shape. (b) Qualitative appearance of the function $g(\alpha_2)$, specifying the stiffness against an asymmetric distortion. At $\alpha_2=b$, stability against the type of asymmetry specified by α_3 is lost.

α_2 and α_3, the symmetric and asymmetric distortions against which the spherical shape first becomes unstable with increasing Z^2/A [at $(Z^2/A)_0$ and $(7/4)(Z^2/A)_0$ respectively in the case of an incompressible drop].

Consider the deformation energy of a drop as function of two such coordinates, $E=E(\alpha_2,\alpha_3)$. In what follows we shall not restrict ourselves to the case of an incompressible drop with a sharp surface, but for the sake of definiteness we are using the notation α_2 and α_3 appropriate in that case. Most of the considerations which follow would hold for more general but qualitatively similar coordinates $\alpha_{\text{symmetric}}$ and $\alpha_{\text{asymmetric}}$.

For shapes whose deviation from reflection symmetry is small, we may write

$$E(\alpha_2,\alpha_3)=f(\alpha_2)+\alpha_3^2 g(\alpha_2)+\text{higher powers of } \alpha_3^2. \quad (1)$$

If terms beyond α_3^2 are neglected, the deformation energy surface is specified by the two functions f and g of the single variable α_2. The function $f(\alpha_2)$ gives the energy of purely symmetrical distortions and its qualitative behavior is shown in Fig. 1(a). The quantity $f(\alpha_2=a)$, where a corresponds to the maximum in f, gives the energy required to divide the drop under the restriction of symmetric distortions. We shall call it the symmetric threshold energy and the shape specified by $\alpha_2=a$ the symmetric critical shape.

The function $g(\alpha_2)$ specifies the stability against an asymmetric distortion of a symmetric shape described by α_2. The asymmetric distortion contemplated is that specified by α_3 with all other asymmetric coordinates chosen so as to make the energy a minimum. For a drop whose charge is below the critical value for instability against α_2, the initial shape $\alpha_2=0$ is stable also against all asymmetric deformations, so that $g(\alpha_2=0)$ is a finite positive quantity. With increasing symmetric deformation this stability may be lost. (This is, for example, the case for a drop with a sufficiently low charge. See below.) The qualitative appearance of g is then as shown in Fig. 1(b). The point $\alpha_2=b$, where g changes sign, specifies the configuration in the sequence of symmetric shapes where stability against the asymmetric distortion previously discussed is first lost.

The functions f and g, as well as the critical deformations a and b, are functions of Z^2/A. For $Z^2/A=(Z^2/A)_0$, f has a point of inflection at $\alpha_2=0$ and $a((Z^2/A)_0)=0$. With decreasing Z^2/A the threshold energy $f(a)$ will increase. We shall now show that the occurrence of a point of inflection in f for $x\equiv(Z^2/A)/(Z^2/A)_0=1$ leads to a $(1-x)^3$ dependence of the threshold on x for $(1-x)\ll 1$.

If, for $\alpha_2\ll 1$, we write

$$f(\alpha_2)=\tfrac{1}{2}f''(0)\alpha_2^2+\tfrac{1}{6}f'''(0)\alpha_2^3, \quad (2)$$

then

$$f'(\alpha_2)=f''(0)\alpha_2+\tfrac{1}{2}f'''(0)\alpha_2^2. \quad (3)$$

Using $f'(a)=0$, we may write the threshold $f(a)$ as

$$f(a)=-\tfrac{1}{12}f'''(0)a^3.$$

Expanding $a(x)$ in powers of $1-x$, [$a(1)=0$], we find

$$\begin{aligned} f(a)&=\tfrac{1}{12}f'''(0)[a'(1)]^3(1-x)^3\\ &=c_1[(Z^2/A)_0-(Z^2/A)]^3, \end{aligned} \quad (4)$$

where c_1 is a constant.

This is the well-known liquid-drop result[1] derived here in a more general way.

The critical deformation $\alpha_2=a$ is an increasing function of $(Z^2/A)_0-(Z^2/A)$ and, for some sufficiently small $Z^2/A=(Z^2/A)_c$, it becomes equal to b and then exceeds it. This follows from the fact that the symmetric critical shape for $Z^2/A=0$—the configuration of tangent spheres[1]—is unstable against asymmetry. For $Z^2/A=(Z^2/A)_c$, the symmetric critical shape is just unstable (or just stable) against asymmetry and for $Z^2/A<(Z^2/A)_c$ the instability occurs even before the maximum in $f(\alpha_2)$ has been reached. It is then possible, by making use of asymmetric distortions, to accomplish the division of the drop with an energy smaller than the symmetric threshold energy. The situation is illustrated in Fig. 2(c). The division of the drop is achieved with least energy by going over one of the two asymmetric saddle points which have become available. The positions of the saddle points in the α_2, α_3 plane are found by considering $\partial E/\partial\alpha_2$ evaluated on the line $\alpha_2=b$. This quantity, equal to $f'(b)+\alpha_3^2 g'(b)$, is positive for $\alpha_3=0$, but, since $g'(b)$ is negative, it will change sign

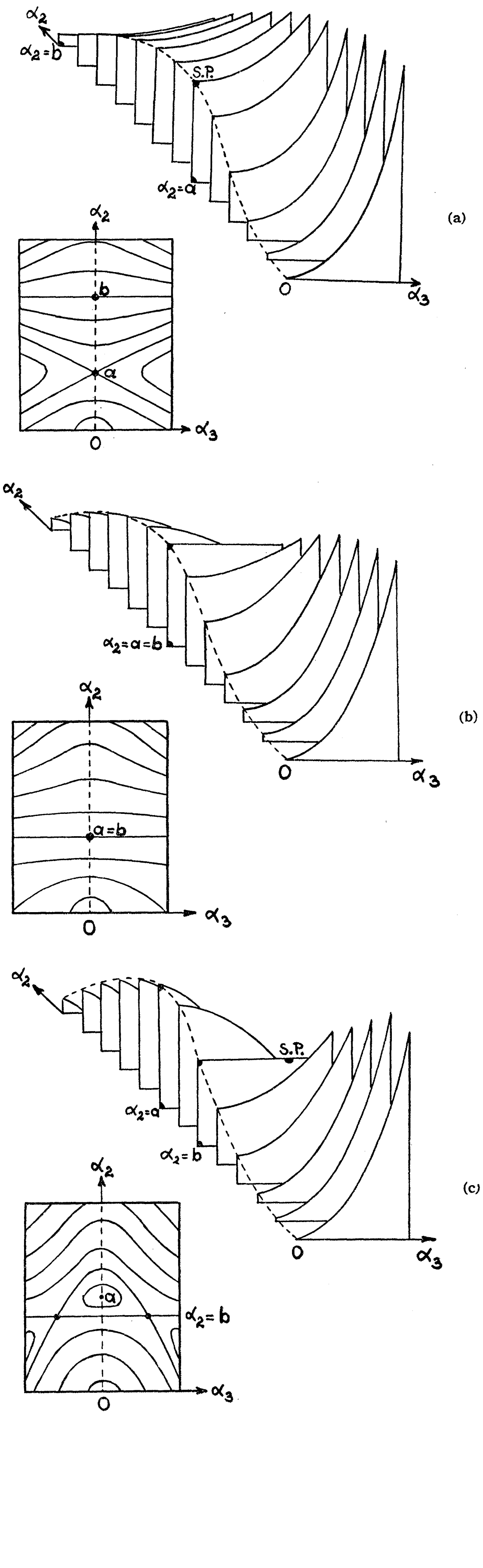

FIG. 2. The qualitative appearance of the deformation energy surface, considered as function of a symmetric coordinate α_2 and an asymmetric coordinate α_3. A map of the surface and a relief drawing is shown in each of the following three cases: (a) $(Z^2/A) > (Z^2/A)_c$. The threshold energy is determined by the symmetric saddle point, stable against asymmetry. (b) $(Z^2/A) = (Z^2/A)_c$. Stability of the symmetric critical shape is lost. (c) $(Z^2/A) < (Z^2/A)_c$. Two asymmetric saddle points have appeared, with energies below that of the symmetric critical shape.

at some α_3 given by

$$(\alpha_3)_{\text{S.P.}} = \pm[-f'(b)/g'(b)]^{\frac{1}{2}}. \tag{5}$$

This equation defines the positions of the saddle points.

The amount by which the asymmetric threshold energy lies below the symmetric threshold is $f(a)-f(b)$, which, for small $a-b$, is given by

$$f(a)-f(b) = -\tfrac{1}{2}f''(a)(a-b)^2. \tag{6}$$

Similarly, the degree of asymmetry of the asymmetric saddle points is given by

$$(\alpha_3)_{\text{S.P.}} = \pm[f''(a)/g'(a)]^{\frac{1}{2}}(a-b)^{\frac{1}{2}}. \tag{7}$$

If we expand $a(x)$ and $b(x)$ around

$$x = x_c = (Z^2/A)_c/(Z^2/A)_0,$$

we find that

$$a-b = [a'(x_c)-b'(x_c)](x-x_c), \quad \text{for} \quad x-x_c \ll 1.$$

Hence,

$$\begin{aligned} f(a)-f(b) &= -\tfrac{1}{2}f''(a)[b'(x_c)-a'(x_c)]^2(x_c-x)^2 \\ &= c_2[(Z^2/A)_c-(Z^2/A)]^2, \end{aligned} \tag{8}$$

and

$$\begin{aligned} (\alpha_3)_{\text{S.P.}} &= \pm[f''(a)/g'(a)]^{\frac{1}{2}}[b'(x_c)-a'(x_c)]^{\frac{1}{2}}(x_c-x)^{\frac{1}{2}} \\ &= \pm c_3[(Z^2/A)_c-(Z^2/A)]^{\frac{1}{2}}, \end{aligned} \tag{9}$$

where c_2 and c_3 are positive constants.

Similarly, the maximum in the deformation energy along a deformation path for which the degree of asymmetry in the neighborhood of the maximum is held fixed at some value α_3 (and which would be the threshold energy for a division constrained to proceed in this way) is found by expanding $E-E(\alpha_2=b)$ to second powers in α_2-b and determining the maximum. For the energy excess of such a threshold over the asymmetric threshold $E(b)$, expressed in units of the maximum excess $E(a)-E(b)$, we find the expression

$$\delta E/[E(a)-E(b)] = (1-s^2)^2, \tag{10}$$

where $s=\alpha_3/(\alpha_3)_{\text{S.P.}}$ is the prescribed degree of asymmetry in units of the asymmetry associated with the smallest threshold.

It will be noted that the forms of Eqs. (4), (8), (9), and (10) follow from the qualitative forms of f and g (the presence of a point of inflection in f and a zero in g) and have, therefore, more general validity than the model of an incompressible drop with a sharp surface. For example, the generalization to include a nonuniform density distribution, or even the inclusion of additional forces varying smoothly with Z and A, would affect only the numerical values of the constants in the above equations.

The relation of the trends with Z^2/A suggested by Eqs. (4), (8), and (9) to experimental asymmetries and thresholds has been considered by the author.[2,3] In connection with the semiempirical formulas discussed there, it may be remarked that the double-humped function $-(1-s^2)^2$ of Eq. (10) [or even a function of $(1-s^2)^2$] does not provide a good representation of the observed logarithmic fission yield curves, the decrease from the maximum on the asymmetric side being too rapid compared with that on the side towards symmetry. The observed asymmetry reflects the final result of tendencies during the whole division process up to the point of separation of the fragments, and the characteristics with respect to asymmetry of configurations other than those near the saddle-point shape will play a role. For example, if asymmetry could result in a sufficiently pronounced lowering of potential energy in the later stages of fission, the final division might be asymmetric even though the saddle-point shape was symmetric. (See, for example, Hill and Wheeler.[4])

The quantitative discussion of the stability against asymmetry of a charged drop and the estimation of the critical $(Z^2/A)_c$ will be undertaken in a later paper. Here we shall confine ourselves to some qualitative remarks about the factors which are at play in determining the stability or instability, especially in the case of shapes for which the two fragments have become discernible. (Configurations near the spherical shape are stable against asymmetry and instability would not occur in the early stages of the fission of a charged drop.) Such configurations will occur as the critical shapes for low x values (less than about 0.6) and, for higher x, in the later stages of fission after passage over the top of the potential barrier.

For completely separated fragments of fractional masses V and W $(V+W=1)$, the potential energy is the sum of the separate surface and electrostatic energies and is proportional to $V^{2/3}+W^{2/3}+2x(V^{5/3}+W^{5/3})$. In the same units the electrostatic interaction energy of the two fragments at a distance d is $(5/3)VWR/d$, where $R=$ radius of the undivided drop. The sum of these expressions plus a correction for the neck connecting the two fragments represents the main features of the potential energy of a strongly deformed drop. Considering asymmetric distortions which change the ratio $V:W$ without changing the effective separation between the fragments, the stability or instability against asymmetry will be related to the sign of $\partial^2E/\partial V^2$ evaluated at $V=\frac{1}{2}$. Neglecting the effect of the neck on the question of stability against asymmetry, we find

$$\left(\frac{\partial^2 E}{\partial V^2}\right)_{V=\frac{1}{2}} \propto -1+x\left(5-\frac{15}{4}\frac{1}{\lambda}\right), \tag{11}$$

where λ is the distance d, measured now in units of $2^{2/3}R$, the separation between the centers of two equal tangent spheres. The first term (-1) represents the tendency towards asymmetry associated with the fact that a symmetric division creates the largest amount of new surface energy; the second term $(5x)$ comes from the opposite tendency in the electrostatic energy of separated fragments and the last term expresses the modification of this caused by the interaction energy which is greatest for equal fragments. According to Eq. (11), a preference for asymmetry would be expected in the first place for low x, but will persist also for higher x values in configurations for which the interaction energy between the fragments is sufficiently large compared to their self energies. The relation between x and λ obtained by equating to zero Eq. (11) suggests that the critical value of x at which $\partial^2E/\partial V^2$ changes sign depends sensitively on the separation between the fragments for λ values around 1.

It may be remarked that, except for shapes whose energy is stationary with respect to all deformations, it is always possible to find an asymmetric distortion which decreases the energy [essentially by choosing the asymmetric distortion to contain a sufficient amount (in second order) of the deformation with respect to which there is no equilibrium]. The discussion of the stability against asymmetry of shapes other than equilibrium shapes can, therefore, be made only with reference to a more or less arbitrary restriction on the type of asymmetric distortion considered. The discussion may still be useful qualitatively, but the results are relative to the restriction imposed. In the example previously discussed, this restriction was in the form of the constancy of the effective separation d between the fragments. If d is made a function of the fragment ratio, the stability against asymmetry will depend on this function. The example of touching spheres, discussed for instance by Frankel and Metropolis,[5] defines one such function through the geometrical requirement of the tangency of the fragments, which implies that the separation between their centers is greatest in the symmetric configuration.

Further discussion in the light of quantitative estimates is reserved for later.

It is a pleasure to thank Professors I. Walker and T. Svedberg for the hospitality of their Institutes.

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