Theory of Multiple-Quantum Transitions in the Ground State of K^{30}

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The results of an earlier paper are applied to magnetic resonance transitions in the ground state of K^{39}. Formulas are obtained for the parameters characterizing single- and multiple-quantum transitions, and the values of these parameters are calculated for a Zeeman field corresponding to $x=0.21135$.

IN an earlier paper,[1] the author has calculated a general expression for the resonance transition probabilities in a rotating field which is applicable when the widths of "allowed" lines are small compared to their separation.[2] In the present paper, we shall evaluate the parameters involved for $\Delta F=0$ transitions in the ground state of potassium, in order to facilitate comparison with the results of Kusch.[3]

1. MATRIX ELEMENTS OF I_x AND F_x IN THE PRESENCE OF A CONSTANT MAGNETIC FIELD, H_z

We shall deal here with a system whose total angular momentum is given by

$$\mathbf{F}=\mathbf{I}+\mathbf{J}, \tag{1}$$

where $I^2=i(i+1)$, $J^2=j(j+1)$, and for the moment we shall take $j=\frac{1}{2}$, and i arbitrary.

If we consider only perturbations involving the various angular momentum substates of a given atomic state, we may write for the perturbed Hamiltonian:

$$\mathcal{H}_1=\mathcal{H}_0+(\pm)[\Delta W/(i+\tfrac{1}{2})]\mathbf{I}\cdot\mathbf{J}+(g_I\mu_0H_z/\hbar)F_z+[(g_J-g_I)\mu_0H_z/\hbar]J_z, \tag{2}$$

where $(\pm)\Delta W$ is the hyperfine separation (ΔW being taken as positive). The symbol $(\pm)$ will be used throughout this paper to denote the sign of the hyperfine separation.

The eigenstates of $\mathcal{H}_1$ may be found in a straightforward manner.[4] They are

$$\begin{aligned}|(\gamma,i+\tfrac{1}{2}),m\rangle&=a_m(x)\,|\gamma,i+\tfrac{1}{2},m\rangle+(\pm)b_m(x)\,|\gamma,i-\tfrac{1}{2},m\rangle,\\ |(\gamma,i-\tfrac{1}{2}),m\rangle&=-(\pm)b_m(x)\,|\gamma,i+\tfrac{1}{2},m\rangle+a_m(x)\,|\gamma,i-\tfrac{1}{2},m\rangle,\end{aligned} \tag{3}$$

where

$$\left.\begin{aligned}a_m&=1\\ b_m&=0\end{aligned}\right\}\quad\text{for}\quad |m|=i+\tfrac{1}{2}, \tag{4a}$$

$$\left.\begin{aligned}a_m&=\left\{\frac{[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{1/2}+[1+(\pm)mx/(i+\frac{1}{2})]}{2[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{1/2}}\right\}^{1/2}\\ b_m&=\left\{\frac{[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{1/2}-[1+(\pm)mx/(i+\frac{1}{2})]}{2[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{1/2}}\right\}^{1/2}\end{aligned}\right\}\ \text{for}\ |m|\neq i+\tfrac{1}{2}, \tag{4b}$$

and

$$x=(g_J-g_I)\mu_0H_z/\Delta W. \tag{5}$$

The corresponding energy values are

$$W_1(\gamma,i+\tfrac{1}{2},m)=E_0+(\pm)\frac{\Delta W}{2}\left\{-\frac{1}{2i+1}+(\pm)\frac{2g_I}{g_J-g_I}mx+\left(1+(\pm)\frac{mx}{i+\frac{1}{2}}\right)\right\}\quad\text{for } |m|=i+\tfrac{1}{2}, \tag{6a}$$

and

$$W_1(\gamma,i\pm\tfrac{1}{2},m)=E_0+(\pm)\frac{\Delta W}{2}\left\{-\frac{1}{2i+1}+(\pm)\frac{2g_I}{g_J-g_I}mx\pm(\pm)\left[1+(\pm)\frac{2mx}{i+\frac{1}{2}}+x^2\right]^{1/2}\right\}\quad\text{for } |m|\neq i+\tfrac{1}{2}, \tag{6b}$$

where, as mentioned before, the $(\pm)$ in parentheses refers to the sign of the hyperfine separation.

In calculating transition probabilities, we need the quantities

$$\begin{aligned}b\hbar\alpha(\gamma,f,m;\gamma,f',m')&\equiv(\mu_0H_0/\hbar)\langle(\gamma,f),m|g_II_x+g_JJ_x|(\gamma,f'),m'\rangle\\ &=(\mu_0H_0/\hbar)\langle(\gamma,f),m|g_IF_x+(g_J-g_I)J_x|(\gamma,f'),m'\rangle,\end{aligned} \tag{7}$$

[1] H. Salwen, Phys. Rev. **99**, 1274 (1955), which is hereafter referred to as I. When equations in I are quoted, the equation number is preceded by a Roman numeral I [e.g., Eq. (I.25) refers to Eq. (25) of I]. The notation of the present paper is in all cases based on that of I except that the total angular momentum is denoted by $\mathbf{F}=\mathbf{I}+\mathbf{J}$ rather than $\mathbf{J}=\mathbf{J}_1+\mathbf{J}_2+\cdots$.

[2] The results of I are applicable when the constant C-field is homogeneous and the rf amplitude may be considered to jump discontinually from the value zero outside the hairpin to a constant value inside the hairpin. Effects due to the continuous build-up of the rf amplitude near the ends of the hairpin will be considered in a later paper.

[3] P. Kusch, following paper [Phys. Rev. **101**, 627 (1956)].

[4] See Appendix. Also see Millman, Rabi, and Zacharias, Phys. Rev. **53**, 384 (1938).

where H_0 is the amplitude of the rotating magnetic field [see Eq. (I. 90)]. Thus, we need the matrix elements of J_x and F_x in the representation given by the states $|(\gamma,f),m\rangle$. We may use various results of Condon and Shortley[5] to obtain the matrix elements of J_x and F_x for the representation in which F^2 is diagonal. In this way, we find that the only nonzero matrix elements of F_x are [reference 5, Eq. $3^3(4)$]

$$\begin{aligned}\langle\gamma,f,m|F_x|\gamma,f,m+1\rangle&=\langle\gamma,f,m+1|F_x|\gamma,f,m\rangle\\&=\tfrac{1}{2}\hbar[(f-m)(f+m+1)]^{\frac{1}{2}},\end{aligned}\tag{8}$$

and the nonzero matrix elements of J_x are [reference 5, Eqs. $9^3(11)$ and $10^3(2a,b)$]

$$\begin{aligned}&\langle\gamma,i\pm\tfrac{1}{2},m|J_x|\gamma,i\pm\tfrac{1}{2},m+1\rangle\\&\quad=\langle\gamma,i\pm\tfrac{1}{2},m+1|J_x|\gamma,i\pm\tfrac{1}{2},m\rangle\\&\quad=\pm[\tfrac{1}{2}\hbar/(2i+1)][(i\pm\tfrac{1}{2}-m)(i\pm\tfrac{1}{2}+m+1)]^{\frac{1}{2}},\\&\langle\gamma,i\pm\tfrac{1}{2},m|J_x|\gamma,i\mp\tfrac{1}{2},m+1\rangle\\&\quad=\langle\gamma,i\mp\tfrac{1}{2},m+1|J_x|\gamma,i\pm\tfrac{1}{2},m\rangle\\&\quad=\pm[\tfrac{1}{2}\hbar/(2i+1)][(i+\tfrac{1}{2}\mp m)(i+\tfrac{1}{2}\mp(m+1))]^{\frac{1}{2}}.\end{aligned}\tag{9}$$

We may then combine Eqs. (8) and (9) with Eq. (3) to get

$$\begin{aligned}&\langle(\gamma,i\pm\tfrac{1}{2}),m|F_x|(\gamma,i\pm\tfrac{1}{2}),m+1\rangle\\&\quad=\langle(\gamma,i\pm\tfrac{1}{2}),m+1|F_x|(\gamma,i\pm\tfrac{1}{2}),m\rangle\\&\quad=\tfrac{1}{2}\hbar\{c_m^{\mp}(x)c_{m+1}^{\pm}(x)+d_m^{\pm}(x)d_{m+1}^{\mp}(x)\},\\&\langle(\gamma,i\pm\tfrac{1}{2}),m|F_x|(\gamma,i\mp\tfrac{1}{2}),m+1\rangle\\&\quad=\langle(\gamma,i\mp\tfrac{1}{2}),m+1|F_x|(\gamma,i\pm\tfrac{1}{2}),m\rangle\\&\quad=\mp(\pm)\tfrac{1}{2}\hbar\{c_m^{\mp}(x)d_{m+1}^{\pm}(x)-d_m^{\pm}(x)c_{m+1}^{\mp}(x)\},\end{aligned}\tag{10}$$

and

$$\begin{aligned}&\langle(\gamma,i\pm\tfrac{1}{2}),m|J_x|(\gamma,i\pm\tfrac{1}{2}),m+1\rangle\\&\quad=\langle(\gamma,i\pm\tfrac{1}{2}),m+1|J_x|(\gamma,i\pm\tfrac{1}{2}),m\rangle\\&\quad=\pm[\tfrac{1}{2}\hbar/(2i+1)]\{c_m^{\mp}(x)\mp(\pm)d_m^{\pm}(x)\}\\&\qquad\times\{c_{m+1}^{\pm}(x)\pm(\pm)d_{m+1}^{\mp}(x)\},\\&\langle(\gamma,i\pm\tfrac{1}{2}),m|J_x|(\gamma,i\mp\tfrac{1}{2}),m+1\rangle\\&\quad=\langle(\gamma,i\mp\tfrac{1}{2}),m+1|J_x|(\gamma,i\pm\tfrac{1}{2}),m\rangle\\&\quad=\pm[\tfrac{1}{2}\hbar/(2i+1)]\{c_m^{\mp}(x)\mp(\pm)d_m^{\pm}(x)\}\\&\qquad\times\{c_{m+1}^{\mp}(x)\mp(\pm)d_{m+1}^{\pm}(x)\},\end{aligned}\tag{11}$$

where

$$\begin{aligned}c_m^{\pm}(x)&=a_m[i+\tfrac{1}{2}\pm m]^{\frac{1}{2}},\\d_m^{\pm}(x)&=b_m[i+\tfrac{1}{2}\pm m]^{\frac{1}{2}}.\end{aligned}\tag{12}$$

Because of the factor $2(2i+1)$ in the denominator of the matrix element of J_x, it is convenient to define the b of Eq. (7) by

$$b\equiv\frac{(g_J-g_I)\mu_0H_0}{2(2i+1)\hbar},\tag{13}$$

so that

$$\begin{aligned}&\alpha(\gamma,i\pm\tfrac{1}{2},m;\gamma,i\pm\tfrac{1}{2},m+1)\\&\quad=\alpha(\gamma,i\pm\tfrac{1}{2},m+1;\gamma,i\pm\tfrac{1}{2},m)\\&\quad=\pm[c_m^{\mp}\mp(\pm)d_m^{\pm}][c_{m+1}^{\pm}\pm(\pm)d_{m+1}^{\mp}]\\&\quad+[(2i+1)g_I/(g_J-g_I)][c_m^{\mp}c_{m+1}^{\pm}+d_m^{\pm}d_{m+1}^{\mp}],\\&\alpha(\gamma,i\pm\tfrac{1}{2},m;\gamma,i\mp\tfrac{1}{2},m+1)\\&\quad=\alpha(\gamma,i\mp\tfrac{1}{2},m+1;\gamma,i\pm\tfrac{1}{2},m)\\&\quad=\pm[c_m^{\mp}\mp(\pm)d_m^{\pm}][c_{m+1}^{\mp}\mp(\pm)d_{m+1}^{\pm}]\\&\qquad\mp(\pm)[(2i+1)g_I/(g_J-g_I)]\\&\qquad\quad\times[c_m^{\mp}d_{m+1}^{\pm}-d_m^{\pm}c_{m+1}^{\mp}].\end{aligned}\tag{14}$$

2. APPLICATION TO TRANSITION PROBABILITIES

In I, we have shown that when the widths of the $\Delta m=1$ lines are small compared to their separations, the transition probability near any resonance may be given by a formula similar to the "Rabi flopping formula."[6] We shall summarize the results here.

For two states

$$|a\rangle=|(\gamma_a,f_a),m_a\rangle,\quad\text{and}\quad|b\rangle=|(\gamma_b,f_b),m_b\rangle,$$

there will be resonance near the frequency ν_{ab} given by[7]

$$(m_a-m_b)h\nu_{ab}=[W_1(\gamma_a,f_a,m_a)-W_1(\gamma_b,f_b,m_b)].\tag{15}$$

The probability that an atom or molecule initially in one of these states will undergo a transition after being in the rf field for a time τ will be [Eq. (I.85)]

$$P=\frac{b_{ab}^2}{(\nu_{ab}^*-\nu)^2+b_{ab}^2}\times\sin^2\{\pi|m_a-m_b|[(\nu_{ab}^*-\nu)^2+b_{ab}^2]^{\frac{1}{2}}\tau\},\tag{16}$$

where, for $m_b=m_a\pm k$, Eq. (I. 104) gives

$$\begin{aligned}\delta\nu_{ab}&=\nu_{ab}^*-\nu_{ab}\\&=\mp(1/k)\left\{\sum_{m'=m_a-1}\frac{|\alpha(n',a)|^2}{\nu_{an'}-\nu_{ab}}+\sum_{m'=m_a+1}\frac{|\alpha(n',a)|^2}{\nu_{ab}-\nu_{an'}}-\sum_{m'=m_b-1}\frac{|\alpha(n';b)|^2}{\nu_{bn'}-\nu_{ab}}-\sum_{m'=m_b+1}\frac{|\alpha(n';b)|^2}{\nu_{ab}-\nu_{bn'}}\right\},\end{aligned}\tag{17}$$

and Eq. (I. 105) gives

$$b_{ab}=\begin{cases}2(b/2\pi)\alpha(a;b) & \text{for } k=1\\[2ex](2/k!)(b/2\pi)^k\left|\displaystyle\sum_{\substack{m'=m_a\pm1\\m''=m_b\pm2\\\cdots\\m^{(k-1)}=m_a\pm(k-1)}}\frac{\alpha(a;n')\alpha(n';n'')\cdots\alpha(n^{(k-1)};b)}{(\nu_{ab}-\nu_{bn'})(\nu_{ab}-\nu_{bn''})\cdots(\nu_{ab}-\nu_{bn^{(k-1)}})}\right| & \text{for } k>1.\end{cases}\tag{18}$$

[5] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, London, 1951).
[6] I. I. Rabi, Phys. Rev. **51**, 652 (1937).
[7] ν_{ab} is the frequency of the rotating field and may be positive or negative.

TABLE I. Eigenvalues and coefficients of the eigenstates ($x=0.21135$).

	$m=2$	$m=1$	$m=0$	$m=-1$	$m=-2$
$W(2,m)-E_0$	$0.4806451\Delta W$	$0.4353463\Delta W$	$0.3860452\Delta W$	$0.3314465\Delta W$	$0.2693549\Delta W$
$W(1,m)-E_0$		$-0.6853762\Delta W$	$-0.6360452\Delta W$	$-0.5814166\Delta W$	
$a_m(x)$	1	0.9966377	0.9945821	0.9949101	1
$b_m(x)$	0	0.0819345	0.1039542	0.1007659	0

In these equations, we have used notations of the sort,

$$|n'\rangle=|(\gamma',f'),m'\rangle,$$

$$(m'-m'')h\nu_{n';n''}=[W_1(\gamma',f',m')-W_1(\gamma'',f'',m'')]$$
$$\equiv[W_1(n')-W_1(n'')], \quad (19)$$

$$\alpha_{n';n''}=\alpha(\gamma',f',m';\gamma'',f'',m'').$$

The α's are defined by Eqs. (7) and (13) above.

We are thus in a position to apply the results of Sec. 1 to a particular problem.

3. GROUND-STATE TRANSITIONS IN K^{39}

Experiments on the ground state of K^{39} give[8]

$$i=\tfrac{3}{2},\quad j=\tfrac{1}{2},$$
$$g_J/g_I=-14\,130, \quad (20)$$
$$(\pm)\Delta W/h=+461.723\ \text{Mc/sec}.$$

The possible substates (f,m) are then $f=2$, $m=0$, ±1, ±2 and $f=1$, $m=0$, ±1.

Equations (15), (17), and (18) now enable us to write down expressions for $\nu_{fm;f'm'}$, $\delta\nu_{fm;f'm'}$, and $b_{fm;f'm'}$. In the case of the transition $(2,2)\leftrightarrow(2,0)$, for example, these are

$$\nu_{2,2;2,0}=(1/2h)|W_1(2,2)-W_1(2,0)|, \quad (21)$$

$$\delta\nu_{2,2;2,0}=-\tfrac{1}{2}(b/2\pi)^2\left\{\frac{|\alpha(2,1;2,2)|^2}{\nu_{2,2;2,1}-\nu_{2,2;2,0}}+\frac{|\alpha(1,1;2,2)|^2}{\nu_{2,2;1,1}-\nu_{2,2;2,0}}-\frac{|\alpha(2,-1;2,0)|^2}{\nu_{2,0;2,-1}-\nu_{2,2;2,0}}-\frac{|\alpha(1,-1;2,0)|^2}{\nu_{2,0;1,-1}-\nu_{2,2;2,0}}-\frac{|\alpha(2,1;2,0)|^2}{\nu_{2,2;2,0}-\nu_{2,0;2,1}}-\frac{|\alpha(1,1;2,0)|^2}{\nu_{2,2;2,0}-\nu_{2,0;1,1}}\right\}, \quad (22)$$

$$b_{2,2;2,0}=(b/2\pi)^2\left|\frac{\alpha(2,2;1,1)\alpha(2,1;2,0)}{\nu_{2,2;2,0}-\nu_{2,0;2,1}}+\frac{\alpha(2,2;1,1)\alpha(1,1;2,0)}{\nu_{2,2;2,0}-\nu_{2,0;1,1}}\right|. \quad (23)$$

In order to facilitate a comparison with the results of Kusch,[3] it is useful to evaluate the various parameters at a field corresponding to $x=0.21135$. This is done in Tables I and II.

Table I contains $W_1(f,m)-E_0$, $a_m(x)$, and $b_m(x)$ for $x=0.21135$. Table II contains $\nu_{fm;f'm'}$, $\alpha(f,m;\,f',m')$, $\delta\nu_{fm;f'm'}$, and $b_{fm;f'm'}$. The quantities $\delta\nu_{fm;f'm'}$ and $b_{fm;f'm'}$ depend on the rf amplitude through the quantity $(b/2\pi)=[(g_J-g_I)\mu_0H_0/2(2i+1)h]$ defined in Eq. (13) above, where H_0 is the amplitude of the rotating field. If H_x is the amplitude of an applied oscillating field, $H_0=H_x/2$.

4. DISCUSSION

The succeeding paper by P. Kusch[3] contains a comparison of the above results with experiment. It will be seen from that paper that, while good agreement is obtained for low rf amplitudes, there is a substantial discrepancy at high rf amplitudes. We attribute this disagreement to the failure of the theory to take into account the continuous buildup of the r.f. amplitude at the ends of the hairpin (see reference 2). Discussion of effects resulting from the continuity of the oscillating field is deferred to a later paper.

TABLE II. Parameters characterizing magnetic resonance transitions in the ground state of K^{39} at a field corresponding to $x=0.21135$. The quantities $\delta\nu$ and b are given only for the $\Delta F=0$ transitions. All frequencies are given in megacycles per second.

$(f,m;\,f',m')$	$\nu_{fm;f'm'}$	$\alpha(f,m;\,f',m')$	$\delta\nu_{fm;f'm'}$	$b_{fm;f'm'}$
(2,2; 2,1)	20.9155	1.708882	$-2.796\ (b/2\pi)^2$	$3.418\ (b/2\pi)$
(2,1; 2,0)	22.7635	2.276473	$-1.009\ (b/2\pi)^2$	$4.553\ (b/2\pi)$
(2,0; 2,−1)	25.2095	2.519920	$0.540\ (b/2\pi)^2$	$5.040\ (b/2\pi)$
(2,−1; 2,−2)	28.6691	2.338321	$1.819\ (b/2\pi)^2$	$4.677\ (b/2\pi)$
(2,2; 2,0)	21.8395	0	$0.290\ (b/2\pi)^2$	$4.217\ (b/2\pi)^2$
(2,1; 2,−1)	23.9865	0	$0.373\ (b/2\pi)^2$	$4.703\ (b/2\pi)^2$
(2,0; 2,−2)	26.9393	0	$0.358\ (b/2\pi)^2$	$3.275\ (b/2\pi)^2$
(2,2; 2,−1)	22.9628	0	$0.153\ (b/2\pi)^2$	$1.433\ (b/2\pi)^3$
(2,1; 2,−2)	25.5474	0	$0.171\ (b/2\pi)^2$	$1.043\ (b/2\pi)^3$
(2,2; 2,−2)	24.3894	0	$0.109\ (b/2\pi)^2$	$0.1553(b/2\pi)^4$
(1,1; 1,0)	−22.7777	−1.328271	$0.887\ (b/2\pi)^2$	$2.657\ (b/2\pi)$
(1,0; 1,−1)	−25.2233	−1.473360	$-0.711\ (b/2\pi)^2$	$2.947\ (b/2\pi)$
(1,1; 1,−1)	−24.0005	0	$-0.161\ (b/2\pi)^2$	$1.552\ (b/2\pi)^2$
(2,2; 1,1)	538.379	−3.616370		
(2,1; 1,0)	494.686	−2.809135		
(2,0; 1,−1)	446.699	−1.816832		
(2,2; 1,0)	257.801	0		
(2,1; 1,−1)	234.731	0		
(2,2; 1,−1)	163.459	0		
(1,1; 2,0)	−494.700	1.076541		
(1,0; 2,−1)	−446.713	2.043529		
(1,−1; 2,−2)	−392.821	3.244367		
(1,1; 2,−1)	−234.745	0		
(1,0; 2,−2)	−209.122	0		
(1,1; 2,−2)	−146.941	0		

[8] P. Kusch and H. Taub, Phys. Rev. **75**, 1477 (1949).

APPENDIX. EIGENVECTORS IN A CONSTANT MAGNETIC FIELD. (i ARBITRARY, $j=\frac{1}{2}$)[9]

Taking the field in the z-direction, we may write

$$\mathcal{H}_1=\mathcal{H}_0+A\mathbf{I}\cdot\mathbf{J}+\frac{g_J\mu_0H_z}{\hbar}J_z+\frac{g_I\mu_0H_z}{\hbar}I_z. \quad (24)$$

Let $\mathbf{F}=\mathbf{I}+\mathbf{J}$. The eigenvalues of F^2 are $f(f+1)\hbar^2$, where $f=i\pm\frac{1}{2}$.

For $H_z=0$, the eigenstates are $|\gamma,f,m\rangle$ and the energies are given by

$$W_1(\gamma,f,m)=E_0(\gamma)+(A/2)\{f(f+1)-i(i+1)-j(j+1)\}\hbar^2. \quad (25)$$

Then [letting $(\pm)$ be the sign of energy difference] we have

$$(\pm)\Delta W\equiv W_1(\gamma,\,i+\tfrac{1}{2},\,m)-W_1(\gamma,\,i-\tfrac{1}{2},\,m)=(i+\tfrac{1}{2})A\hbar^2, \quad (26)$$

and

$$A=(\pm)[\Delta W/(i+\tfrac{1}{2})\hbar^2]. \quad (27)$$

We may thus rewrite $\mathcal{H}_1$ as

$$\mathcal{H}_1=\mathcal{H}_0+(\pm)[\Delta W/(i+\tfrac{1}{2})\hbar^2]\mathbf{I}\cdot\mathbf{J}+(g_I\mu_0H_z/\hbar)F_z+[(g_J-g_I)\mu_0H_z/\hbar]J_z. \quad (28)$$

We shall use the results of Condon and Shortley[5] [Eqs. 9³(11) and 10³(2,a,b)]. The only nonzero matrix elements of J_z are

$$\langle\gamma,\,i\pm\tfrac{1}{2},\,m|J_z|\gamma,\,i\pm\tfrac{1}{2},\,m\rangle=\pm\tfrac{1}{2}m\hbar/(i+\tfrac{1}{2}),$$

$$\langle\gamma,\,i\pm\tfrac{1}{2},\,m|J_z|\gamma,\,i\mp\tfrac{1}{2},\,m\rangle=\tfrac{1}{2}\hbar\{1-[m/(i+\tfrac{1}{2})]^2\}^{\frac{1}{2}}.$$

The nonzero matrix elements of $\mathcal{H}_1$ are then

$$\begin{aligned}
&\langle\gamma,\,i+\tfrac{1}{2},\,m|\mathcal{H}_1|\gamma,\,i+\tfrac{1}{2},\,m\rangle\\
&\quad=E_0+(\pm)\frac{\Delta W}{2}\left\{-\frac{1}{2i+1}+(\pm)\frac{2g_I}{g_J-g_I}mx\right\}+(\pm)\frac{\Delta W}{2}\left\{1+(\pm)\frac{mx}{i+\frac{1}{2}}\right\},\\
&\langle\gamma,\,i-\tfrac{1}{2},\,m|\mathcal{H}_1|\gamma,\,i-\tfrac{1}{2},\,m\rangle\\
&\quad=E_0+(\pm)\frac{\Delta W}{2}\left\{-\frac{1}{2i+1}+(\pm)\frac{2g_I}{g_J-g_I}mx\right\}-(\pm)\frac{\Delta W}{2}\left\{1+(\pm)\frac{mx}{i+\frac{1}{2}}\right\},\\
&\langle\gamma,\,i+\tfrac{1}{2},\,m|\mathcal{H}_1|\gamma,\,i-\tfrac{1}{2},\,m\rangle\\
&\quad=\langle\gamma,\,i-\tfrac{1}{2},\,m|\mathcal{H}_1|\gamma,\,i+\tfrac{1}{2},\,m\rangle\\
&\quad=(x\Delta W/2)\{1-[m/(i+\tfrac{1}{2})]^2\}^{\frac{1}{2}},
\end{aligned} \quad (29)$$

where

$$x\equiv(g_J-g_I)\mu_0H_z/\Delta W. \quad (30)$$

For $|m|=i+\frac{1}{2}$ we have only one state:

$$|(\gamma,\,i+\tfrac{1}{2}),\,\pm(i+\tfrac{1}{2})\rangle=|\gamma,\,i+\tfrac{1}{2},\,\pm(i+\tfrac{1}{2})\rangle, \quad (31a)$$

$$W_1(\gamma,\,i+\tfrac{1}{2},\,m=\pm(i+\tfrac{1}{2}))=E_0+(\pm)\frac{\Delta W}{2}\left\{-\frac{1}{2i+1}+(\pm)\frac{2g_I}{g_J-g_I}mx+\left(1+(\pm)\frac{mx}{i+\frac{1}{2}}\right)\right\}. \quad (32a)$$

For $|m|\neq i+\frac{1}{2}$, the matrix elements (29) cause the mixing of two states. The new eigenstates are

$$\begin{aligned}
|(\gamma,\,i+\tfrac{1}{2}),\,m\rangle&=a_m\,|\gamma,\,i+\tfrac{1}{2},\,m\rangle+(\pm)b_m\,|\gamma,\,i-\tfrac{1}{2},\,m\rangle,\\
|(\gamma,\,i-\tfrac{1}{2}),\,m\rangle&=-(\pm)b_m\,|\gamma,\,i+\tfrac{1}{2},\,m\rangle+a_m\,|\gamma,\,i-\tfrac{1}{2},\,m\rangle,
\end{aligned} \quad (31b)$$

where

$$\begin{aligned}
a_m&=\left\{\frac{[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{\frac{1}{2}}+[1+(\pm)mx/(i+\frac{1}{2})]}{2[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}},\\
b_m&=\left\{\frac{[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{\frac{1}{2}}-[1+(\pm)mx/(i+\frac{1}{2})]}{2[1+(\pm)2mx/(i+\frac{1}{2})+x^2]^{\frac{1}{2}}}\right\}^{\frac{1}{2}}.
\end{aligned} \quad (33b)$$

The corresponding energies (for $|m|\neq i+\frac{1}{2}$) are

$$W_1(\gamma,\,i\pm\tfrac{1}{2},\,m)=E_0+(\pm)\frac{\Delta W}{2}\left\{-\frac{1}{2i+1}+(\pm)\frac{2g_I}{g_J-g_I}mx\pm(\pm)\left[1+(\pm)\frac{2mx}{i+\frac{1}{2}}+x^2\right]^{\frac{1}{2}}\right\}. \quad (32b)$$

If we write

$$a_m=1,\quad b_m=0\quad\text{for}\quad |m|=i+\tfrac{1}{2}, \quad (33a)$$

we may put (31a) in the same form as (31b).

[9] This treatment is substantially the same as that given in the appendix to a paper by Millman, Rabi, and Zacharias [Phys. Rev. **53**, 384 (1938)]. Those authors, however, do not bother to obtain the eigenstates.