The recoiling thallium nuclides formed *directly* have forward to backward ratios almost equal to unity. Inasmuch as a forward velocity component would be expected from the transfer of momentum to the target nucleus, similar in magnitude to that observed for lead and bismuth recoils, the apparent cancellation of this in the production of thallium nuclides must be the result of another process, isotropic in nature, and capable of transferring appreciable momentum. It is proposed that the thallium nuclides studied are formed by the evaporation of neutrons and an alpha particle following the passage of the bombarding proton through the nucleus. The recoil momentum of the thallium nucleus resulting from an alpha particle emitted with an energy of the order of 30 Mev would be about 2.5 times the momentum of the struck nucleus and would lead to a more isotropic distribution of resultant recoils. The range of the thallium recoils can then be taken to be about four times the "effective range," which would be about 0.4 mg/cm^2 Bi, or about 5 to 6 times the range of the lead and bismuth recoils. The range-velocity dependence for heavy fragments at these low kinetic energies, where the range is varying with the second to third power of the velocity,[32] yields an expected value of about 8 for the ratio of the range of the thallium recoils relative to those of the bismuth and lead recoils, from their velocity ratio of about 2.5. The agreement between the expected and observed values for the ratio of the ranges lends support to the model advanced for the production of the thallium nuclides of mass number ~200, in which an alpha particle and neutrons are emitted. In earlier work on the photoactivation of bismuth with 86 Mev bremsstrahlung,[33] the yield of *directly* formed Tl^{201} was higher than that of *directly* formed Pb^{201}. The formation of Pb^{201} by photons involves the emission of one proton and neutrons, and that of Tl^{201} two protons, or an alpha particle, and neutrons. The fact that the yield of Tl^{201} is higher than the yield of Pb^{201} probably means that in this case, too, the Tl^{201} nuclide formation involves alpha-particle emission.

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[32] J. Knipp and E. Teller, Phys. Rev. 59, 659 (1941).

[33] N. Sugarman and R. Peters, Phys. Rev. 81, 951 (1951).

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Meson Production in Nucleon-Nucleon Collisions at High Energies*

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Calculations based on a statistical model have yielded results concerning the relative probabilities for the different multiplicities and charge distribution of mesons produced in nucleon-nucleon collisions. In deviating from a pure statistical model the important effects of final state interactions and various selection rules have been included, using results of meson-nucleon scattering experiments. Notably in the results the suppression of some one-meson final states by consideration of the Pauli principle and conservation of angular momentum and parity, along with the enhancement of two-meson states due to resonance effects, have brought about results which are in closer agreement with experiment than predictions of a pure statistical nature. Account was taken of the final state interactions by considering separately the nucleon-nucleon and meson-nucleon interactions, a separation made plausible by consideration of the small amount of kinetic energy taken away by the more massive particles. Meson-meson interactions were neglected. The nucleon-nucleon interaction was taken care of by introducing in the statistical weight a factor which is the square of the wave function for the scattering of two nucleons evaluated at the origin of their interaction. The meson-nucleon final state scattering was treated by the method discussed by Chew, modified for the case of a meson scattering off two stationary and superposed nucleons. Multiplicities up to two mesons were considered. On comparing with experimental results, at 1.7-Bev bombarding energy of neutrons on protons the ratio of the probability of occurrence of the final states $(np+-)$:$(pp-0)$:$(pp-)$ is calculated to be 3.0:1.0:0.9, while experiment gives 3.3:1:0.8. The ratio of the total probability for double meson production to that for single meson production at this energy is 1.2, while a modified result of observations gives 1.4. Results on proton-proton collisions do not yield good agreement with present observations.

I. INTRODUCTION

In Fermi's[1,2] statistical theory of multiple meson production, the relative probabilities for alternative processes initiated by a nucleon-nucleon collision depend primarily upon the volume in phase space available to each final state when the energy of the colliding nucleons is high. The dependence upon the dynamics involved is argued to be diminishingly small under the assumption that all possible final states are equally excited due to the strong interactions involved. Thus,

* Supported in part by the Office of Naval Research and a grant from the National Science Foundation.

[1] E. Fermi, Progr. Theoret. Phys. (Japan) 5, 570 (1950).

[2] E. Fermi, Phys. Rev. 92, 452 (1953); 93, 1434 (1954).

with the inclusion of necessary restrictions and refinements (conservation of energy, momentum and charge; charge independence[2]; physical indistinguishability of particles[2,3]; etc.), the relative probabilities of the competing final states are obtained by calculating the fraction of phase space accessible to them in the collision region when equilibrium is established. However, even if such an equilibrium is achieved, it alone cannot be expected to determine the weighting of the final states. One considers that in the collision region very strong interactions rapidly bring about equilibrium, but as the particles leave this volume the less strong but longer range interactions become predominant. The final state particles emerge not into plane wave states of free particles, but into the perturbed states of their mutual interaction. Familiar in low-energy phenomena, (beta decay, near-threshold meson production, etc.), these final-state interaction effects are known to be appreciable. For example,[4–6] the final-state nucleon-nucleon interaction increases single-meson production by 2 or 3 orders of magnitude over what is predicted neglecting this effect.

Recent experimental investigations with the Brookhaven[7] Cosmotron have made it possible to compare the statistical theory with experiment. At Cosmotron energies single and double meson production were observed in collisions between Cosmotron-produced neutrons and protons in a hydrogen-filled diffusion cloud chamber. The observed ratio of the probability for double meson production to that for single meson production is more than 20 times greater than that predicted by the statistical model. In view of this seemingly poor agreement of the statistical theory with what is observed, it has been thought worth while to investigate the above-mentioned refinement due to the effects of final-state interactions. Of the final state interactions in this type of process which can be expected to be appreciable, namely, the nucleon-nucleon, meson-nucleon, and meson-meson interactions, the first two, at least, can easily be seen to bring about corrections to the statistical theory in the desired direction for agreement with experiment. That is, the nucleon-nucleon interaction favors the higher multiplicities of mesons produced, since for higher multiplicities the nucleons are left with less kinetic energy. Moreover, the meson-nucleon interaction with its resonance can presumably enhance double meson production at the expense of triple as well as single production in the proper energy region (around an energy such that the two mesons can both rescatter at resonance off the nucleons, while if one or three mesons are produced the most probable energy each will take off is beyond or below the resonance region). This latter effect should occur roughly around 1.5 Bev of laboratory energy, near the mean energy of the Brookhaven experiments.[7]

In what follows, the production of mesons at Cosmotron energies will be treated by a statistical theory with the added refinement of the final-state effects of nucleon-nucleon and meson-nucleon interactions. The further refinement due to the requirements of the Pauli principle and of angular momentum and parity considerations will be included in obtaining the relative probabilities of occurrence of the various possible charge states of nucleons and mesons in nucleon-nucleon encounters.

II. NUCLEON-NUCLEON FINAL STATE INTERACTION

As the first step, the more familiar interaction of the nucleons in the final state will be considered, neglecting the effect of the mesons. This separation and independent treatment of the involved interactions of the final-state particles into specific interactions between the different types of particles is only approximate. However, it finds some support in the relative duration of, say, the nucleon-nucleon and meson-nucleon interactions in the final state. Even after the mesons have left the region of their production, and their interactions have ceased to be effective, the more slowly moving nucleons will still be moving well within the range of their mutual interaction. This then may be considered as the case at hand.

From the assumption of statistical equilibrium in the region of primary interaction, the matrix element connecting the initial and final states (according to the statistical theory) when N particles appear in the final state is just[1,8]

$$(f|\mathcal{H}|i)=\text{const.}(\Omega/V)^{N/2}\equiv\mathcal{H}_F, \tag{1}$$

where the constant is independent of the final state, V is the normalization volume for the final state free particle wave functions, and Ω is the volume of collision which initially contains the N-particle virtual states. From the transition rate

$$\omega\sim|(f|\mathcal{H}|i)|^2\rho_F, \tag{2}$$

the relative probability that n pions will emerge from a nucleon-nucleon high-energy interaction is

$$S_n{}^0=\left(\frac{\Omega}{(2\pi)^3}\right)^{n+1}\int\cdots\int d\mathbf{P}d\mathbf{k}_1\cdots \times d\mathbf{k}_n\delta\left(E-\frac{P^2}{M}-\omega_1-\cdots-\omega_n\right), \tag{3}$$

where the nucleons are treated nonrelativistically and $\mathbf{P}$ is their relative momentum, ω_i is the energy of the ith meson,[9] E is the total available energy, and where

[3] C. N. Yang and R. Christian, Brookhaven National Laboratory Report (unpublished). See also footnote 22, reference 7.
[4] K. Brueckner, Phys. Rev. 82, 598 (1951).
[5] K. M. Watson and K. A. Brueckner, Phys. Rev. 83, 1 (1951).
[6] Kenneth M. Watson, Phys. Rev. 88, 1163 (1952).
[7] Fowler, Shutt, Thorndike, and Whittemore, Phys. Rev. 95, 1026 (1954).
[8] Richard H. Milburn, Revs. Modern Phys. 27, 1 (1955).
[9] We use $\hbar=c=\mu$ (the pion mass) $=1$.

the number of dynamically independent particles is $(n+1)$ instead of $(n+2)$ due to the conservation of linear momentum.

If instead of a Born approximation we use the actual wave function of the final nucleons, $(\mathbf{r}|\psi)$, where $\mathbf{r}$ is the relative nucleon coordinate, then the transition matrix element (using a momentum representation) is[10]

$$(\psi|\mathcal{H}|\mathbf{P}')=\int(\psi|\mathbf{P})(\mathbf{P}|\mathcal{H}|\mathbf{P}')d\mathbf{P}/(2\pi)^3, \qquad (4)$$

where only the relative nucleon momentum labels the states, the meson coordinates being omitted. From the Fermi assumption, Eq. (1), $(\mathbf{P}|\mathcal{H}|\mathbf{P}')$ is independent of the final state variables and can be removed from the integral. The remaining integration yields just the nucleon wave function evaluated at the origin of the relative coordinate, $(\mathbf{r}=0|\psi)\equiv\psi_{(0)}$ so that

$$(\psi|\mathcal{H}|\mathbf{P})=\mathcal{H}_F\psi_{(0)}. \qquad (5)$$

Hence, the weighting of a final state containing n pions becomes

$$S_n=\left(\frac{\Omega}{(2\pi)^3}\right)^{n+1}\int\cdots\int|\psi_{(0)}|^2d\mathbf{P}d\mathbf{k}_1\cdots \times d\mathbf{k}_n\delta\left(E-\frac{P^2}{M}-\omega_1-\cdots-\omega_n\right). \qquad (6)$$

It will suffice for our calculations to use square-well wave functions derived from a potential of depth V_0 and range r_0 (neglecting spin dependence). In the low energy approximation[11] (with only S-waves contributing) this can easily be seen to give

$$(\mathbf{r}|\psi)=\left[1+\frac{4V_0}{(a_0+b_0E_N)(E_N+\epsilon)}\right]^{\frac{1}{2}}\frac{\sin Kr}{Kr}, \quad r<r_0 \qquad (7)$$

where $a_0=[2+(M\epsilon)^{\frac{1}{2}}r_0]^2$, $b_0=Mr_0^2$, and the normalization is to unit amplitude in the asymptotic region. Here ϵ is the deuteron binding energy, M is the nucleon mass, and $K=[M(V_0+E_N)]^{\frac{1}{2}}$. Admitting the bound state solution as well, for the case of a neutron-proton final state, the deuteron wave function must also be considered.[12] For a square well this is[11]

$$(\mathbf{r}|\psi_D)=\left[\frac{\beta^2\alpha}{2\pi(1+\alpha r_0)}\right]^{\frac{1}{2}}\frac{\sin\beta r}{\beta r}, \quad r<r_0, \qquad (8)$$

where $\beta=[M(V_0-\epsilon)]^{\frac{1}{2}}$ and $\alpha=(M\epsilon)^{\frac{1}{2}}$.

Since such a nonsingular potential may underestimate the magnitude of the wave functions at small separations, the depth of the well was somewhat arbitrarily adjusted. Instead of using those square well parameters which give a best fit to experiment, a depth V_0 was used (for $r_0=1.09$) which matched the square-well wave function for the deuteron, Eq. (8), with the Hulthén wave function[13] at $\mathbf{r}=0$. Such a depth is $V_0=0.78$.

The computation of the weights, Eq. (6), was carried out with appropriate approximations for the low- and high-energy regions, while retaining the relativistic inclusion of the meson rest mass in all cases. That is, in the near-threshold region in terms of the meson kinetic energies, $T_i=\omega_i-1$, the condition that the total energy ω_i will be slowly varying was used in removing some average value for it from the integrations. Thus,

$$S_{n\,T_i\ll1}=\left(\frac{4\pi\Omega}{(2\pi)^3}\right)^{n+1}\frac{M^{\frac{3}{2}}}{2}[\bar{\omega}(\bar{\omega}+1)^{\frac{1}{2}}]^n\int\cdots\int|\psi_{(0)}|^2 \times(T_0-T_1-\cdots-T_n)^{\frac{1}{2}}(T_1\cdots T_n)^{\frac{1}{2}}dT_1\cdots dT_n, \qquad (9)$$

where T_0 is the total available kinetic energy, and the assumption has been made that momentum conservation need be applied strictly only to the nucleons. At higher energies, where k/ω is slowly varying, the integrals were again simplified to

$$S_n^{\text{rel}}=\left(\frac{4\pi\Omega}{(2\pi)^3}\right)^{n+1}\frac{M^{\frac{3}{2}}}{2}\left\langle\frac{k}{\omega}\right\rangle_{\text{Av}}^n\int\cdots\int|\psi_{(0)}|^2 \times(E-\omega_1-\cdots-\omega_n)^{\frac{1}{2}}\omega_1^2d\omega_1\cdots\omega_n^2d\omega_n. \qquad (10)$$

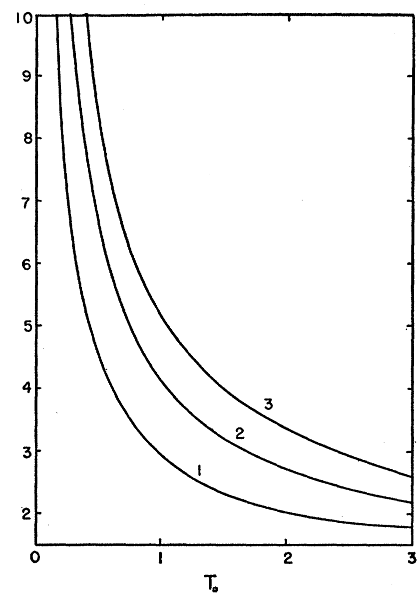

FIG. 1. Probability near threshold of multiple meson production with nucleon-nucleon interaction in the final state, relative to the statistical result. The curves are labeled by the multiplicity of the mesons produced.

[10] See appendix of reference 4.

[11] H. A. Bethe and R. F. Bacher, Revs. Modern Phys. 8, 82 (1936); see page 119.

[12] Since the formation of the bound state places another restriction on the momenta, the weighting factor in this case is

$$S_n(D)=(2\pi)^3\left(\frac{\Omega}{(2\pi)^3}\right)^{n+1}|\psi_D(0)|^2\int\cdots\int d\mathbf{k}_1\cdots \times d\mathbf{k}_n\delta(E-\omega_1-\cdots-\omega_n).$$

[13] See, for example, G. F. Chew, Phys. Rev. 84, 710 (1951).

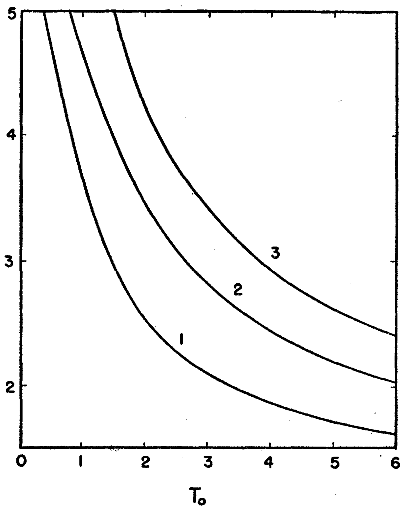

FIG. 2. Probability at energies well above threshold of multiple meson production with nucleon-nucleon interaction in the final state, relative to the statistical result. The curves are labeled by the multiplicity of mesons produced.

The weights for deuteron formation were similarly handled.

As an indication of the effect of the use of the actual nucleon wave functions, these weights were calculated relative to the statistical result (i.e., where $|\psi_{(0)}|^2=1$) and are shown in Figs. 1 and 2 for the two energy regions, Eqs. (10) and (11), as a function of the kinetic energy available to the mesons and their multiplicity. The corresponding ratios for the case of deuteron formation are shown in Figs. 3 and 4. In more restrictive energy regions these effects can be seen from general formulas for the production of n pions. For instance, near threshold but at energies high enough that ϵ, the deuteron binding energy, can be neglected compared to the available kinetic energy, we have

$$S^n \underset{\substack{\epsilon\to 0\\ T_i\ll 1}}{=} \left(\frac{4\pi\Omega}{(2\pi)^3}\right)^{n+1} \frac{M^{\frac{3}{2}}}{2} \frac{[\bar{\omega}(\bar{\omega}+1)^{\frac{1}{2}}]^n \pi^{(n+1)/2} T_0^{(3n+1)/2}}{2^n(3n+1)\Gamma\left(\frac{3n+1}{2}\right)} \times\left[1+\frac{(3n+1)\gamma V_0}{T_0}\right], \quad (11)$$

where $\gamma\approx 4/a_0=0.73$ at low nucleon energies [see Eq. (7)]. The last factor compared to unity indicates the enhancement of the cross section near threshold due to the nucleon-nucleon interaction as a function of the meson multiplicity and available kinetic energy. From this it is seen that at, say, 80 Mev above threshold for two-meson production in the center-of-mass system (which is 260 Mev above threshold for the production of a single meson), the cross section is eight times that obtained from the pure phase-space result, while single production is enhanced by a factor of 2.2. However, as indicated by Fig. 2 for the high-energy region and by

$$S_n^{\text{rel}} \underset{\mu\to 0}{=} \left(\frac{4\pi\Omega}{(2\pi)^3}\right)^{n+1} \frac{M^{\frac{3}{2}}}{2} \frac{\pi^{\frac{1}{2}} 2^{n-1} T_0^{(6n+1)/2}}{\Gamma\left(\frac{6n+1}{2}\right)} \times\left[1+\frac{(6n+1)\gamma V_0}{T_0}\right] \quad (12)$$

for the extreme relativistic region, the weights for the lower multiplicities and high meson energies are enhanced by about the same amount by the nucleon-nucleon final state interaction, and the enhancement becomes negligible in the limit where the meson mass can be neglected. Hence, in the region of Cosmotron energies, although the enhancement is still considerable, the relative probability of single to double meson production is not much changed from the pure phase space result by the effect of the unbound final state interaction of the nucleons. Therefore, this effect alone cannot be expected to bring the statistical result into better agreement with the experimental observations.

The effect of the formation of deuterons is even more inappreciable at Cosmotron energies. As shown in Figs. 3 and 4, although it is quite important near threshold,[14] this effect falls off even more rapidly than the effect of the unbound final state interaction. Hence, in what follows deuteron formation will not be considered.

III. MESON-NUCLEON FINAL STATE INTERACTION

A. Single Meson Production

In order to take account of the meson-nucleon final state interaction in the spirit of this calculation, a solution to the problem will be found in terms of the

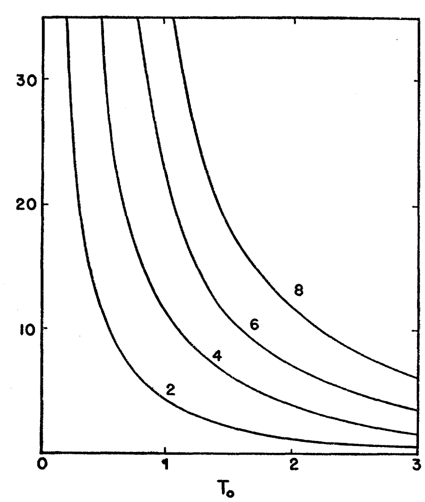

FIG. 3. Probability near threshold of multiple meson production with deuteron formation, relative to the statistical result neglecting neutron-proton interaction. The curves are labeled by multiplicity of mesons produced.

[14] K. A. Brueckner and J. S. Kovacs, Phys. Rev. 94, 726 (1954).

Møller[15] wave matrix into which are built the features of the Fermi statistical model for the actual production of the mesons. For example, that part of the wave matrix which corresponds to the production of a single meson in a nucleon-nucleon collision is[16]

$$\Omega=\Omega_s\frac{1}{a}H', \tag{13}$$

where H' is the interaction which produces the meson, $a\equiv E-H_0+i\eta$, η being an infinitesimal parameter specifying the contour of integration for outgoing waves, H_0 is the Hamiltonian for the noninteracting fields, and Ω_s is the wave matrix which describes the scattering of the final state particles and satisfies the Lippmann-Schwinger[17] equation

$$\Omega_s=1+\frac{1}{a}\mathcal{V}\Omega_s. \tag{14}$$

Identifying H' with the phenomenological Fermi matrix element corrected by using the actual wave functions for the final state nucleons, and with a factor which ensures meson production in P-states, we have

$$H'=V_F(\mathbf{A}\cdot\mathbf{k}/\omega_k), \tag{15}$$

where V_F is the matrix element given by Eq. (5). Here $\mathbf{A}\cdot\mathbf{k}/\omega_k$ will be of order unity at high energies if we make $\mathbf{A}$ an unspecified unit vector which may depend on spins, etc., but which we shall leave arbitrary and average over classically when necessary, in keeping with the statistical method. From Eqs. (13), (14), and (15) we get in momentum representation

$$(\mathbf{q},\mathbf{P}|\Omega|\mathbf{P}')=\frac{1}{E-\omega_q+i\eta}\left\{V_F\frac{\mathbf{A}\cdot\mathbf{q}}{\omega_q}+\int\frac{d\mathbf{k}}{(2\pi)^3}(\mathbf{q}|t|\mathbf{k})\frac{1}{E-\omega_k+i\eta}V_F\frac{\mathbf{A}\cdot\mathbf{k}}{\omega_k}\right\}, \tag{16}$$

where $\mathbf{P}'$ and $\mathbf{P}$ are the initial and final relative nucleon momenta, t is defined by

$$t\equiv\mathcal{V}\Omega_s, \tag{17}$$

and satisfies the integral equation[18]

$$t=\mathcal{V}+\mathcal{V}\frac{1}{a}t. \tag{18}$$

Thus, when t is determined, the transition matrix T for the whole process (which can be seen to be the quantity enclosed in the { } in Eq. (16)) is also determined.

To the lowest order for linear coupling, the potential constructed for the scattering[16] includes terms which express the nuclear force interaction between the nucleons, the absorption of the created meson by one nucleon and emission by the other, as well as the scattering of the meson by the individual nucleons. We shall deal with just the last of these[19] and use as the potential for the scattering of the produced meson by the two fixed and superposed nucleons

$$(\mathbf{k}|\mathcal{V}|\mathbf{k}')=\frac{f^2}{2(\omega_k\omega_{k'})^{\frac{1}{2}}}\left\{\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\tau}_1\cdot\mathbf{U}_k^*\frac{1}{E}\boldsymbol{\sigma}_1\cdot\mathbf{k}'\boldsymbol{\tau}_1\cdot\mathbf{U}_{k'}+\boldsymbol{\sigma}_2\cdot\mathbf{k}\boldsymbol{\tau}_2\cdot\mathbf{U}_k^*\frac{1}{E}\boldsymbol{\sigma}_2\cdot\mathbf{k}'\boldsymbol{\tau}_2\cdot\mathbf{U}_{k'}+\boldsymbol{\sigma}_1\cdot\mathbf{k}'\boldsymbol{\tau}_1\cdot\mathbf{U}_{k'}\frac{1}{E-\omega_k-\omega_{k'}}\boldsymbol{\sigma}_1\cdot\mathbf{k}\boldsymbol{\tau}_1\cdot\mathbf{U}_k^*+\boldsymbol{\sigma}_2\cdot\mathbf{k}'\boldsymbol{\tau}_2\cdot\mathbf{U}_{k'}\frac{1}{E-\omega_k-\omega_{k'}}\boldsymbol{\sigma}_2\cdot\mathbf{k}\boldsymbol{\tau}_2\cdot\mathbf{U}_k^*\right\}, \tag{19}$$

where the interaction used is the pseudovector coupling term for the charge-independent pseudoscalar theory. Here the nucleons are both assumed to be at the origin of our coordinate system, f is the pseudovector coupling constant, $\boldsymbol{\sigma}_i$ and $\boldsymbol{\tau}_i$ are the spin and isotopic spin operators corresponding to the ith nucleon, and $\mathbf{U}_k$ and $\mathbf{U}_k^*$ are vectors in charge space which serve as annihilation and creation operators for the meson.[5]

In order to simplify the calculations, we shall express the problem in terms of eigenstates of angular momentum and isotopic spin. If we assume that the nucleons come off in S-states while the meson is produced in a P-state relative to either of these, then the possible states of angular momentum are labeled by $J=0$, 1, and 2 when the nucleons are in the triplet state, and

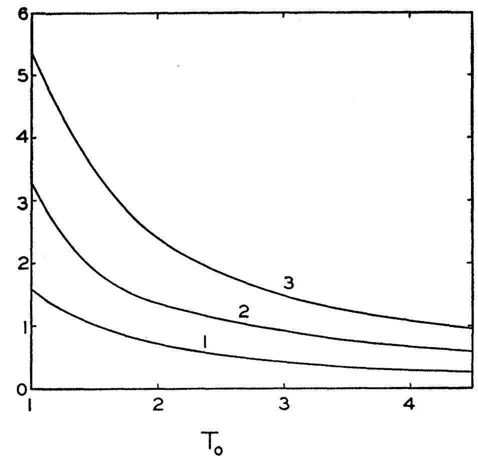

FIG. 4. Probability well above threshold of multiple meson production with deuteron formation, relative to the statistical result neglecting neutron-proton interaction. The curves are labeled by multiplicity of mesons produced.

[15] C. Møller, Kgl. Danske. Videnskab. Selskab, Mat.-fys. Medd. 23, No. 1 (1945).
[16] K. A. Brueckner and K. M. Watson, Phys. Rev. 90, 699 (1953).
[17] B. Lippmann and J. Schwinger, Phys. Rev. 79, 469 (1950).
[18] G. F. Chew and M. L. Goldberger, Phys. Rev. 77, 470 (1950).
[19] Compare, for example, with Aitken et al., Phys. Rev. 93, 1349 (1954).

$J=1$ for the singlet state. Similarly, the total isotopic spin can be $I=0$, 1, and 2 for the nucleons in the isotopic spin triplet state, and $I=1$ for the singlet. In this representation the potential can be written in the form

$$\mathcal{V}=2(B-A)\{E_3-2E_1-E_2-2F_1-F_2+F_3\}+2(A+B)\{(E_3-E_2-2E_1)(F_3-F_2-2F_1)+2E_5F_5+1\} \quad (20)$$

where

$$A=\frac{4\pi f^2kk'}{6(\omega_k\omega_{k'})^{\frac{1}{2}}}\frac{1}{E}, \quad (21a)$$

$$B=\frac{4\pi f^2kk'}{6(\omega_k\omega_{k'})^{\frac{1}{2}}}\frac{1}{E-\omega_k-\omega_{k'}}, \quad (21b)$$

E_1, E_2, E_3, and E_4 are projection operators onto the four states of angular momentum, and

$$E_5=(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{l}/2\sqrt{2}, \quad (22)$$

with the F_j the analogs of these for the isotopic spin states. The projection operators,[20]

$$(J=0,S=1):\ E_1=\tfrac{1}{12}\{-3-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l}]^2\}, \quad (23a)$$

$$(J=1,S=1):\ E_2=\tfrac{1}{8}\{6+2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-2(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l}-[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l}]^2\}, \quad (23b)$$

$$(J=2,S=1):\ E_3=(1/24)\{6+2\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2+6(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l}+[(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l}]^2\}, \quad (23c)$$

$$(J=1,S=0):\ E_4=\tfrac{1}{4}\{1-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\}, \quad (23d)$$

can be easily seen to have the usual properties, while E_5 operating on the states $\psi_{J,S}{}^M$ has the properties

$$E_5\psi_{2,1}{}^M=E_5\psi_{0,1}{}^0=0,$$
$$E_5\psi_{1,1}{}^M=-\psi_{1,0}{}^M,$$
$$E_5\psi_{1,0}{}^M=-\psi_{1,1}{}^M.$$

Here

$$(\mathbf{k}|\mathbf{l}|\mathbf{k}_0)=-\frac{3i}{4\pi}\frac{\mathbf{k}}{k}\times\frac{\mathbf{k}_0}{k_0} \quad\text{and}\quad (\mathbf{k}|1|\mathbf{k}_0)=\frac{3}{4\pi}\frac{\mathbf{k}}{k}\cdot\frac{\mathbf{k}_0}{k_0}.$$

Thus $\mathcal{V}$ is not diagonal in this representation, although the off-diagonal elements arise merely from the mixing of the two $J=1$ states as well as from the mixing of the corresponding isotopic spin states. Diagonalization of these submatrices diagonalizes the whole potential and reduces the problem to one of nonmixing channels for which the integral equations, Eq. (18), depend only upon the magnitude of the momentum. In this form the equations readily lend themselves to an approximate solution by one of Schwinger's variational principles[21] according to which

[20] With corresponding relations for the isotopic spin states.

[21] J. Schwinger, "Lectures on Nuclear Physics," Harvard University, 1947 (unpublished). See also, for example, G. F. Chew, Phys. Rev. **93**, 341 (1954).

TABLE I. Elements of diagonalized potential $\mathcal{V}$. States are labeled by the total angular momentum J, total spin of the nucleons S (for cases where $J=1$, a and b indicate symmetric and antisymmetric combinations, respectively, of states of $S=1$ and 0), and similarly defined labels for isotopic spin quantities. A and B are defined in Eq. (21).

J	S	I	T	$(i\|\mathcal{V}\|i)$
2	1	2	1	$8B$
2	1	1	a	$2B$
2	1	1	b	$2B$
2	1	0	1	$-4B$
1	a	2	1	$2B$
1	a	1	a	$(17/2)A+(9/2)B$
1	a	1	b	$(1/2)A-(7/2)B$
1	a	0	1	$9A-B$
1	b	2	1	$2B$
1	b	1	a	$(1/2)A-(7/2)B$
1	b	1	b	$(17/2)A+(9/2)B$
1	b	0	1	$9A-B$
0	1	2	1	$-4B$
0	1	1	a	$9A-B$
0	1	1	b	$9A-B$
0	1	0	1	$18A+2B$

$$(k|t_\alpha|k_0)=\frac{(k|\mathcal{V}_\alpha|k_0)}{1-(k|\mathcal{V}_\alpha|k_0)^{-1}\frac{1}{(2\pi)^3}\int\frac{(k|\mathcal{V}_\alpha|k')(k'|\mathcal{V}_\alpha|k_0)k'^2dk'}{E-\omega_{k'}}}, \quad (24)$$

where the $\mathcal{V}_\alpha$ are listed in Table I.[22] With the same arguments by which the scattering in only the $\frac{3}{2}$, $\frac{3}{2}$ state is seen to be enhanced in meson-nucleon scattering,[21,23] it can be seen upon examination of the entries in Table I that only for those potentials which are equal to $8B$ or $2B$ can there be enhancement in the scattering of a meson off two nucleons. Of these, all but two are ruled out by conservation of isotopic spin (I can be only 1 or 0 for an initial state consisting of two nucleons). For the remaining two channels, with the labels $\alpha'\equiv(J,S;I,\text{“}T\text{”})=(2,1;1,b)$ and $\alpha'=(2,1;1,a)$, we have

$$(k_E|t_{\alpha'}|k_0)=-\frac{4\pi f^2k_Ek_0}{3\omega_{k0}(E\omega_{k0})^{\frac{1}{2}}[1-I(E,\omega_{k0})]}, \quad (25)$$

where

$$I(E,\omega_{k0})=\frac{f^2}{6\pi^2}\int_1^{\omega_{\max}}\frac{\omega_{k0}(\omega_{k'}{}^2-1)^{\frac{3}{2}}d\omega_{k'}}{\omega_{k'}(E-\omega_{k'})(E-\omega_{k'}-\omega_{k0})}, \quad (26)$$

using a cutoff, and where the final nucleon energies have been neglected. This differs from the meson-single nucleon scattering operator mainly in that off-the-energy-shell scatterings are included. The meson can be produced at any energy ω_{k0} and scatter back on the energy shell.

[22] The complete set of projection operators for the new states after diagonalization can easily be seen to be E_1, E_3, $E_a=\frac{1}{2}(E_2+E_4-E_5)$, and $E_b=\frac{1}{2}(E_2+E_4+E_5)$ where the new states brought about by the transformation are $\psi_a=1/\sqrt{2}(\psi_2+\psi_4)$, $\psi_b=1/\sqrt{2}(\psi_4-\psi_2)$ and where ψ_2 and ψ_4 correspond to $(J,S)=(1,1)$ and $(1,0)$ respectively.

[23] Geoffrey F. Chew, Phys. Rev. **89**, 591 (1953).

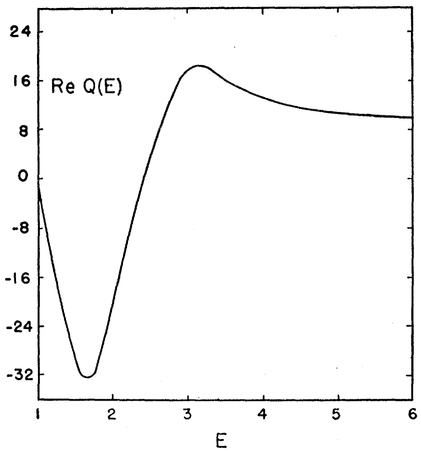

FIG. 5. The real part of the function $Q(E)$ defined in Eq. (31).

A careful examination of $t_{\alpha'}$ shows that there can indeed be enhancement in the scattering in these states in the energy range under consideration. That is, when the coupling constant and cut-off energy are chosen to agree with meson-single nucleon scattering results,[24] the principal value of the integral, $I(E,\omega_{k0})$, is close to unity near $E\approx 2$ for the off-the-energy-shell scatterings as well as for those on. This makes the scattering in this energy range important for only these two states, justifying the neglect of the contributions of the other states in which the scattering is suppressed. The results obtained in this manner are not expected to be too bad considering the approximations used. Particularly in the region near resonance, where the fit was made to meson-nucleon scattering data, this semiphenomenological treatment should reproduce the main features of the final state scattering.

The transition matrix which for the complete process is

$$(\mathbf{q},\mathbf{P}|T|\mathbf{P}') = \left\{\frac{\mathbf{A}\cdot\mathbf{q}}{\omega_q}+\int\frac{d\mathbf{k}}{(2\pi)^3}\frac{(\mathbf{q}|t|\mathbf{k})}{(E-\omega_k+i\eta)}\frac{\mathbf{A}\cdot\mathbf{k}}{\omega_k}\right\}V_F \quad (27)$$

can now be calculated if t is written in the form

$$(\mathbf{q}|t|\mathbf{k})=\mathbf{\Lambda}\cdot\mathbf{k},$$

where $\mathbf{\Lambda}$ is determined by

$$(\mathbf{q}|t|\mathbf{k})=\sum_\alpha(q|t_\alpha|k)(EF)_\alpha.$$

Then carrying out the angular part of the integral, T is

$$(\mathbf{q},\mathbf{P}|T|\mathbf{P}')=\left\{\frac{\mathbf{A}\cdot\mathbf{q}}{\omega_q}+\frac{1}{6\pi^2}\int\frac{k^4dk\mathbf{A}\cdot\mathbf{\Lambda}}{\omega_k(E-\omega_k+i\eta)}\right\}V_F. \quad (28)$$

With only the resonant terms considered so that t is[22]

$$(\mathbf{q}|t|\mathbf{k})=-\frac{4\pi f^2qk(F_2+F_4)E_3}{3\omega_k(\omega_k\omega_q)^{\frac{1}{2}}[1-I(\omega_k,E)]} \quad (29)$$

Eq. (28) becomes

$$(\mathbf{q},\mathbf{P}|T|\mathbf{P}')=\left\{\frac{\mathbf{A}\cdot\mathbf{q}}{\omega_q}-\frac{2f^2qQ(E)\,'E_3(F_2+F_4)}{9\pi\omega_q^{\frac{1}{2}}}\right\}V_F, \quad (30)$$

where

$$Q(E)=\int_1^{\omega_{\max}}\frac{(\omega_k^2-1)^{\frac{3}{2}}d\omega_k}{\omega_k^{\frac{3}{2}}(E-\omega_k)(1-I)} \quad (31)$$

and $'E_3$ is E_3 with $\mathbf{k}/k$ replaced by $\mathbf{A}$. The real and imaginary parts of the function $Q(E)$ are plotted in Figs. 5 and 6 with $\omega_{\max}=6$. From these it is seen that in the region up $E\approx 3$ the second term gives its greatest contribution to T.

In order to evaluate the weights for the different charge distributions of the final state particles, some assumption will have to be made about the way the creation operators appear in the phenomenological matrix element V_F. Let us simply assume that there is some operator $\mathcal{O}$ appended to V_F which creates the meson while keeping constant the total isotopic spin I and its z-component I_3. For example, to create mesons into the isotopic $L=1$ state $\mathcal{O}$ should be of the form $\mathbf{B}\cdot\mathbf{U}_k^*$, where $\mathbf{B}$ is some vector in charge space which, except for the above requirements, would be left unspecified in accordance with the Fermi statistical method. We can then write the transition matrix as

$$(\mathbf{q},\mathbf{P}|T|\mathbf{P}')=[\mathcal{X}-\mathcal{Y}F]\mathcal{O}, \quad (32)$$

where for brevity we have set

$$\mathcal{X}=V_F\mathbf{A}\cdot\mathbf{q}/\omega_q,\quad F=F_2+F_4, \quad (33a)$$

and

$$\mathcal{Y}=\frac{2f^2}{9\pi}\frac{qQ(E)}{\omega_q^{\frac{1}{2}}}\,'E_3V_F. \quad (33b)$$

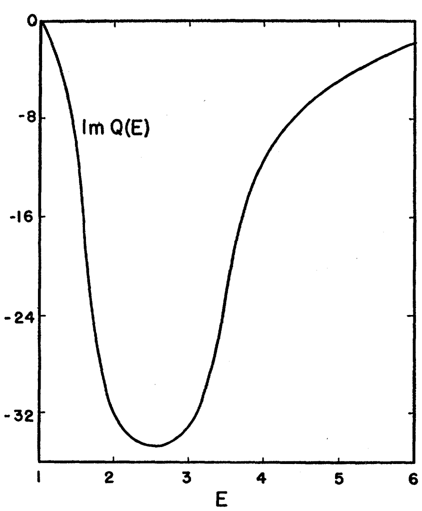

FIG. 6. The imaginary part of the function $Q(E)$ defined in Eq. (31).

[24] A rough estimate of the parameters was obtained by requiring the phase shift for the scattering in $(\frac{3}{2},\frac{3}{2})$ state to pass through 90° at $E=2$ (see also reference 23). For a chosen cutoff $\omega_{\max}=6$, $f^2/4\pi=0.16$ was found.

TABLE II. Matrix elements of T for one-meson production allowed by hypothesis of charge independence including final state meson-nucleon scattering contribution. Initial and final states are labeled according to isotopic spin notation where t^+1^0 corresponds to a state containing 2 nucleons in the isotopic triplet state and a neutral meson, t^+ being the state consisting of two protons, etc. $\mathcal{O}_{10}$, $\mathcal{O}_{11}$, $\mathcal{O}_{01}$ are the three different matrix elements of the production operator allowed by charge independence.[5] The first subscript refers to the isotopic spin of the initial state while the second refers to the isotopic spin of the final nucleons. $\mathfrak{X}$ and $\mathfrak{Y}$ are defined in Eq. (33).

i	f	$(f\|T\|i)$
t^+	t^+1^0	$(\mathfrak{X}-\mathfrak{Y})\mathcal{O}_{11}/\sqrt{2}$
t^+	t^01^+	$-(\mathfrak{X}-\mathfrak{Y})\mathcal{O}_{11}/\sqrt{2}$
t^+	$s1^+$	$(\mathfrak{X}-\mathfrak{Y})\mathcal{O}_{10}$
s	t^+1^-	$\mathfrak{X}\mathcal{O}_{01}/\sqrt{3}$
s	t^-1^+	$\mathfrak{X}\mathcal{O}_{01}/\sqrt{3}$
s	$s1^0$	0
s	t^01^0	$-\mathfrak{X}\mathcal{O}_{01}/\sqrt{3}$
t^0	t^+1^-	$(\mathfrak{X}-\mathfrak{Y})\mathcal{O}_{11}/\sqrt{2}$
t^0	t^-1^+	$-(\mathfrak{X}-\mathfrak{Y})\mathcal{O}_{11}/\sqrt{2}$
t^0	$s1^0$	$(\mathfrak{X}-\mathfrak{Y})\mathcal{O}_{10}$
t^0	t^01^0	0

The matrix elements of T between the various initial and final charge states are then given by

$$(f|T|i)=\mathfrak{X}(f|\mathcal{O}|i)-\mathfrak{Y}\sum_{\text{II}}(f|F|\text{II})(\text{II}|\mathcal{O}|i). \quad (34)$$

The elements $(f|F|\text{II})$ are most naturally evaluated by making use of the projection operator properties of F and expressing $|f)$ and $|\text{II})$ in terms of angular momentum eigenfunctions in charge space for a meson and two nucleons, $|I,I_3,T)$. The evaluation of these matrix elements of T is then straightforward and the results are shown in Table II. These are in agreement with the general relations obtained by Brueckner and Watson in their analysis of pion production using the hypothesis of charge independence. [See Eq. (17) of reference 5.]

With no meson-nucleon interaction in the final state, $\mathfrak{Y}=0$, these matrix elements should correspond to those of Fermi.[2] That is (including the factor giving the nucleon-nucleon final state interaction), $|T|^2$ should equal $|\mathfrak{X}|^2$ times the weights listed in his Tables II and III of reference 2. This will be so if the square of all of the matrix elements of $\mathcal{O}$ between eigenstates of isotopic spin are assumed to be equal. Then, since it is relative weights we are after, we can disregard these matrix elements of $\mathcal{O}$. Thus fitting the matrix elements of our unspecified operator $\mathcal{O}$ to give the Fermi result in the limit of no meson-nucleon final state interaction, we see that the matrix elements of our transition operator are given by the Fermi matrix element plus the correction term for those final states in which we get resonance scattering—all multiplied by the factor obtained from the assumption of charge independence.

On the basis of the Pauli principle and from angular momentum and parity considerations,[5] all of these transitions are not allowed if we assume the final nucleons to be in S-states and the meson in P-states. For instance, for the transition $t^+\to t^01^+$ conservation of parity allows only those transitions in which the initial state is symmetric in space, and, since the initial state consists of two protons, antisymmetric in spin. Since the final state is symmetric in isotopic spin, it must be antisymmetric in spin. This cannot give even J-values, hence this transition is forbidden. Table III lists for those transitions allowed by charge independence the transitions allowed by these latter considerations. Column 3 lists the matrix elements which result from all of these requirements combined.

B. Production of Two Mesons

Considering two-meson production, we shall again treat the primary interaction by means of the statistical model, and assume that the final state interaction with the nucleons can be considered independently for each meson. That is, we shall neglect the meson-meson interaction. Then we shall again take as the wave matrix

$$\Omega=\Omega_s\frac{1}{a}H', \quad (35)$$

where H' creates the two mesons

$$H'=\left[\frac{\mathbf{A}_1\cdot\mathbf{k}_1}{\omega_{k1}}+\frac{\mathbf{A}_2\cdot\mathbf{k}_2}{\omega_{k2}}\right]V_F' \quad (36)$$

and Ω_s accounts for the final state scattering of the mesons by the nucleons

$$\Omega_s=1+\frac{1}{a}t_1+\frac{1}{a}t_2, \quad (37)$$

where

$$(\mathbf{q}_1,\mathbf{q}_2|t_1|\mathbf{k}_1,\mathbf{k}_2)=(2\pi)^3\delta(\mathbf{q}_2-\mathbf{k}_2)(\mathbf{q}_1|t|\mathbf{k}_1), \quad (38a)$$

$$(\mathbf{q}_1,\mathbf{q}_2|t_2|\mathbf{k}_1,\mathbf{k}_2)=(2\pi)^3\delta(\mathbf{q}_1-\mathbf{k}_1)(\mathbf{q}_2|t|\mathbf{k}_2), \quad (38b)$$

and

$$(\mathbf{q}_1,\mathbf{q}_2|1|\mathbf{k}_1,\mathbf{k}_2)=(2\pi)^6\delta(\mathbf{q}_1-\mathbf{k}_1)\delta(\mathbf{q}_2-\mathbf{k}_2). \quad (38c)$$

Then in a manner similar to the treatment of one-meson

TABLE III. Matrix elements of T for single meson production allowed by charge independence, Pauli principle, conservation of parity, and conservation of angular momentum assuming final nucleons in S-states and meson in P-states.

Transitions allowed by charge independence hypothesis	Transitions of nucleon states allowed by Pauli principle, conservation of spin and angular momentum	Matrix elements of T
$t^+\to t^+1^0$	Forbidden	$\cdots$
$t^+\to t^01^+$	Forbidden	$\cdots$
$t^+\to s1^+$	${}^1D_2\to{}^3S_1$	$\mathfrak{X}-\mathfrak{Y}$
	${}^1S_0\to{}^3S_1$	$\mathfrak{X}$
$s\to t^+1^-$	${}^3S_1$, ${}^3D_1\to{}^1S_0$	$1/\sqrt{3}\,\mathfrak{X}$
$s\to t^-1^+$	${}^3S_1$, ${}^3D_1\to{}^1S_0$	$1/\sqrt{3}\,\mathfrak{X}$
$s\to t^01^0$	${}^3S_1$, ${}^3D_1\to{}^1S_0$	$1/\sqrt{3}\,\mathfrak{X}$
$t^0\to t^+1^-$	Forbidden	$\cdots$
$t^0\to t^-1^+$	Forbidden	$\cdots$
$t^0\to s1^0$	${}^1D_2\to{}^3S_1$	$\mathfrak{X}-\mathfrak{Y}$
	${}^1S_0\to{}^3S_1$	$\mathfrak{X}$

TABLE IV. Elements of transition matrix T for the production of two mesons when initial state is t^+. For the various final states the coefficients of the matrix elements of the unspecified production operator, $(T,I'|\mathcal{O}'|t^+)$, are listed. T and I' are the isotopic spins of the final state nucleons and of the combination of the two nucleons and the first meson, respectively. I and I_3 for the final state are suppressed in denoting the states since they are determined by the initial state.

f	$(f\|\mathcal{T}\|1,2)$	$(f\|\mathcal{T}\|1,1)$	$(f\|\mathcal{T}\|1,0)$	$(f\|\mathcal{T}\|0,1)$
$s1_1{}^01_2{}^+$	0	0	0	$-\frac{1}{\sqrt{2}}(\mathcal{X}'-\mathcal{Y}_1'-\mathcal{Y}_2')$
$s1_1{}^+1_2{}^0$	0	0	0	$\frac{1}{\sqrt{2}}(\mathcal{X}'-\mathcal{Y}_1'-\mathcal{Y}_2')$
$t^+1_1{}^+1_2{}^-$	$\frac{1}{\sqrt{60}}[6\mathcal{X}'-(5/2)\mathcal{Y}_2']$	$\frac{1}{4}\mathcal{Y}_2'$	$\frac{1}{2\sqrt{3}}\mathcal{Y}_2'$	0
$t^+1_1{}^-1_2{}^+$	$\frac{1}{\sqrt{60}}\mathcal{X}'$	$-\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1')$	$\frac{1}{\sqrt{3}}\mathcal{X}'$	0
$t^+1_1{}^01_2{}^0$	$-\frac{1}{\sqrt{60}}[3\mathcal{X}'-(5/2)\mathcal{Y}_2']$	$\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1'-\frac{1}{2}\mathcal{Y}_2')$	$-\frac{1}{2\sqrt{3}}\mathcal{Y}_2'$	0
$t^-1_1{}^+1_2{}^+$	$\frac{1}{\sqrt{60}}[\mathcal{X}'+(5/2)\mathcal{Y}_2']$	$\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1'-\frac{1}{2}\mathcal{Y}_2')$	$\frac{1}{\sqrt{3}}(\mathcal{X}'-\frac{1}{2}\mathcal{Y}_2')$	0
$t^01_1{}^+1_2{}^0$	$-\frac{3}{\sqrt{60}}\mathcal{X}'$	$-\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1')$	0	0
$t^01_1{}^01_2{}^+$	$\frac{1}{\sqrt{60}}[2\mathcal{X}'-(5/2)\mathcal{Y}_2']$	$\frac{1}{4}\mathcal{Y}_2'$	$-\frac{1}{\sqrt{3}}(\mathcal{X}'-\frac{1}{2}\mathcal{Y}_2')$	0

production we get

$$(\mathbf{q}_1,\mathbf{q}_2,\mathbf{P}|T|\mathbf{P}')=\{\mathcal{X}'-\mathcal{Y}_1'F_1'-\mathcal{Y}_2'F_2'\}\mathcal{O}', \quad (39)$$

where

$$\mathcal{X}'=\left(\frac{\Omega}{V}\right)^{\frac{3}{2}}\psi_{(0)}\left[\frac{\mathbf{A}_1\cdot\mathbf{q}_1}{\omega_{q1}}+\frac{\mathbf{A}_2\cdot\mathbf{q}_2}{\omega_{q2}}\right], \quad (40a)$$

$$\mathcal{Y}_1'=\left(\frac{\Omega}{V}\right)^{\frac{3}{2}}\psi_{(0)}\frac{2f^2q_1Q(E-\omega_{q2})}{9\pi\omega_{q1}{}^{\frac{1}{2}}}{}'E_3(q_1), \quad (40b)$$

and where $\mathcal{Y}_1'$ refers to the final state scattering of the first meson only. $\mathcal{O}'$ is the operator which creates two mesons and satisfies the requirements of charge independence.

The matrix elements of T between the appropriate initial and final charge states for the production of two mesons are obtained in a manner similar to Eq. (34) for single production,

$$(f|T|i)=\sum_{I'T}(f|\mathcal{T}|I,I_3,T,I')(I,I_3,T,I'|\mathcal{O}'|i), \quad (41)$$

where

$$(f|\mathcal{T}|I,I_3,T,I')=\mathcal{X}'(f|1|I,I_3,T,I') -\mathcal{Y}_1'(f|F_1'|I,I_3,T,I')-\mathcal{Y}_2'(f|F_2'|I,I_3,T,I'), \quad (42)$$

and where, in our approximation, the matrix elements of F_1' and F_2' are the same as those of F. The states $|I,I_3,T,I')$ are the isotopic spin eigenfunctions corresponding to the four states[2,25] $I=1$ and the two states $I=0$, where I and I_3 are the isotopic spin and its third component for the final state and T and I' are the isotopic spins of the final state nucleons, and of the combination of the final nucleons and the meson labeled 1 respectively. For the different initial states, t^+, t^0, and s, the matrix elements of $\mathcal{T}$ which enter in are tabulated in Tables IV, V, and VI respectively. Expressing the transition matrix elements in this manner allows us again to take care of the unspecified creation operator by means of the Fermi assumption of statistical equilibrium in the volume of collision. This means that we can set the squares of all of the matrix elements of $\mathcal{O}'$ between the different eigenstates of isotopic spin equal to unity. Thus, entering into the determination of the cross sections will be[26]

$$|(f|T|i)|^2=\sum_{T,I'}|(f|\mathcal{T}|I,I_3,T,I')|^2. \quad (43)$$

Again in the limit of no final state interaction and zero mass for the meson, this leads to the Fermi result.[2]

[25] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (Cambridge University Press, Cambridge, 1953), p. 76.

[26] All transitions listed in Tables IV, V, and VI are allowed by angular momentum and parity considerations.

TABLE V. Elements of transition matrix T for the production of two mesons when initial state is t^0. For the various final states the coefficients of the matrix elements of the unspecified production operator, $(T,I'|\mathcal{O}'|t^0)$, are listed. T and I' are the isotopic spins of the final state nucleons and of the combination of the two nucleons and the first meson, respectively. I and I_3 for the final state are suppressed in denoting the states since they are determined by the initial state.

f	$(f\|T\|1,2)$	$(f\|T\|1,1)$	$(f\|T\|1,0)$	$(f\|T\|0,1)$
$t^-1_1^+1_2^0$	$-\frac{1}{\sqrt{60}}[2\mathcal{X}'-(5/2)\mathcal{Y}_2']$	$-\frac{1}{4}\mathcal{Y}_2'$	$\frac{1}{\sqrt{3}}(\mathcal{X}'-\frac{1}{2}\mathcal{Y}_2')$	0
$t^-1_1^01_2^+$	$\frac{3}{\sqrt{60}}\mathcal{X}'$	$\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1')$	0	0
$t^01_1^-1_2^+$	$\frac{1}{\sqrt{60}}[3\mathcal{X}'-(5/2)\mathcal{Y}_2']$	$-\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1'-\frac{1}{2}\mathcal{Y}_2')$	$\frac{1}{2\sqrt{3}}\mathcal{Y}_2'$	0
$t^01_1^+1_2^-$	$\frac{1}{\sqrt{60}}[3\mathcal{X}'-(5/2)\mathcal{Y}_2']$	$-\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1'-\frac{1}{2}\mathcal{Y}_2')$	$\frac{1}{2\sqrt{3}}\mathcal{Y}_2'$	0
$t^01_1^01_2^0$	$-\frac{4}{\sqrt{60}}\mathcal{X}'$	0	$-\frac{1}{\sqrt{3}}\mathcal{X}'$	0
$s1_1^01_2^0$	0	0	0	0
$s1_1^+1_2^-$	0	0	0	$\frac{1}{\sqrt{2}}(\mathcal{X}'-\mathcal{Y}_1'-\mathcal{Y}_2')$
$s1_1^-1_2^+$	0	0	0	$-\frac{1}{\sqrt{2}}(\mathcal{X}'-\mathcal{Y}_1'-\mathcal{Y}_2')$
$t^+1_1^01_2^-$	$\frac{3}{\sqrt{60}}\mathcal{X}'$	$\frac{1}{2}(\mathcal{X}'-\mathcal{Y}_1')$	0	0
$t^+1_1^-1_2^0$	$-\frac{1}{\sqrt{60}}[2\mathcal{X}'-(5/2)\mathcal{Y}_2']$	$-\frac{1}{4}\mathcal{Y}_2'$	$\frac{1}{\sqrt{3}}(\mathcal{X}'-\frac{1}{2}\mathcal{Y}_2')$	0

IV. THE RELATIVE PROBABILITIES FOR THE POSSIBLE EVENTS

To obtain the relative probabilities for the various processes we need to evaluate

$$\tfrac{1}{4}\,\text{spur}(T^\dagger E_f T E_i),$$

where E_i and E_f are the nucleon spin projection operators onto the initial and final states. In order to evaluate these spurs by the usual techniques we will need to make some assumption about the dependence of the arbitrary unit vector $\mathbf{A}$ upon the spins. However, let us neglect the dependence of the cross sections upon the relative spin orientations of the nucleons in the initial and final states and leave $\mathbf{A}$ arbitrary. Then we have

$$dS_n=\tfrac{1}{4}\langle\text{spur}|T|^2\rangle_{\text{Av}}\rho_F, \tag{44}$$

where the spur is evaluated independently for each nucleon and the indicated average is over the values of $\mathbf{A}$ in the manner

$$\langle\mathbf{A}\cdot\mathbf{B}\mathbf{A}\cdot\mathbf{C}\rangle_{\text{Av}}=\tfrac{1}{3}\mathbf{B}\cdot\mathbf{C}.$$

Using the allowed matrix elements listed in Table III for single production and in Tables IV, V, and VI for double production, the relative probabilities for the processes of interest are easily obtained with the application of Eq. (44) and with the appropriate average and sum over the initial and final states which contribute. For example, the relative probability for the process $n+p\rightarrow n+p+\pi^0$ is obtained by using for $|T|^2$ the sum of the squares of the matrix elements for which the final states are t^01^0 and $s1^0$ for a given initial state, and then averaging over the contributions from the initial states t^0 and s. For processes resulting in the production of two mesons an additional factor of $\frac{1}{2}$ is included in the relative weights to account for the symmetrization of the two meson final states. The relations obtained for the relative weights for the possible one- and two-meson processes are listed for convenience in Table VII with the notation defined in the Appendix.

As a function of the laboratory bombarding energy the relative probabilities[27] for the different events are

[27] For the parameter Ω, the volume in which the primary interaction takes place, we use Fermi's choice of a Lorentz contracted sphere with the meson Compton wavelength as its radius, $\Omega=(4\pi/3)2M/(E+2M)$.

plotted in Figs. 7, 8, and 9 normalized to

$$\sum_{f,n} S_n[i\rightarrow f]=100. \tag{45}$$

Multiplicities higher than two are neglected for the energy range under consideration, while for the case where no mesons are produced we use for the relative weight

$$S_0=\frac{3\pi^2}{2\Omega}\left(1+\frac{\gamma V_0}{E}\right)E^{\frac{1}{2}}\left(4+\frac{E}{M}\right)^{\frac{1}{2}}\left(2+\frac{E}{M}\right), \tag{46}$$

where adjustment has been made for the common factors omitted in S_1 and S_2. The ratio of the total probability for all double meson production to that for all single meson production is also of interest and is shown in Fig. 10 as a function of the bombarding energy for n-p and p-p initiated events. Finally, as an indication of the effect of the final state interactions upon the meson energies, the energy distribution for one of the mesons is shown in Fig. 11 for the process $n+p\rightarrow n+p+\pi^++\pi^-$ at a total available center-of-mass energy of $E=6$. Comparing this with the distribution obtained from just phase space considerations (which is shown normalized to the same total probability), it is seen that the final state interactions spread out the distribution because of the tendency for the two mesons to come off near the resonance energy. Moreover, at high total energies the distribution is thereby shifted toward lower energies with respect to the phase space result.

TABLE VI. Elements of transition matrix T for the production of two mesons when initial state is s. For the various final states the coefficients of the matrix elements of the unspecified production operator, $(T,I'|\mathcal{O}'|s)$, are listed. T and I' are the isotopic spins of the final state nucleons and of the combination of the two nucleons and the first meson, respectively. I and I_3 for the final state are suppressed in denoting the states.

f	$(f\|\mathcal{T}\|1,1)$ $\times(\mathcal{X}'-\mathcal{Y}_1'-\mathcal{Y}_2')$:	$(f\|\mathcal{T}\|1,0)$
$t^-1_1{}^+1_2{}^0$	$1/\sqrt{6}$	0
$t^-1_1{}^01_2{}^+$	$-1/\sqrt{6}$	0
$t^01_1{}^-1_2{}^+$	$1/\sqrt{6}$	0
$t^01_1{}^+1_2{}^-$	$-1/\sqrt{6}$	0
$t^01_1{}^01_2{}^0$	0	0
$s1_1{}^01_2{}^0$	0	$-1/\sqrt{3}$
$s1_1{}^+1_2{}^-$	0	$1/\sqrt{3}$
$s1_1{}^-1_2{}^+$	0	$1/\sqrt{3}$
$t^+1_1{}^01_2{}^-$	$1/\sqrt{6}$	0
$t^+1_1{}^-1_2{}^0$	$-1/\sqrt{6}$	0

V. CONCLUSIONS

A comparison with experiment of the calculated probabilities for the various events shows better agreement, in general, than given by the pure statistical model. In particular, as indicated in Fig. 9, the distribution of the three-pronged events initiated by n-p collisions is calculated to be $(np+-):(pp-0):(pp-)$ $=3.0:1.0:0.9$ at a bombarding energy of 1.7 Bev. The analysis of events of this type by Fowler *et al.*[7] at Brookhaven gives for these ratios 3.3 : 1 : 0.8 for processes initiated by bombarding neutrons in the energy range of 1–2 Bev with a median energy ~1.7 Bev. The improved agreement over the statistical result[3] (~3.3: 1: 20.5) can be attributed to the resonance enhancement of the two-meson states discussed above, and to the suppression of one of the states $(pp-)$ by considerations of conservation of angular momentum and parity (see Table IV). Although experimentally indications are

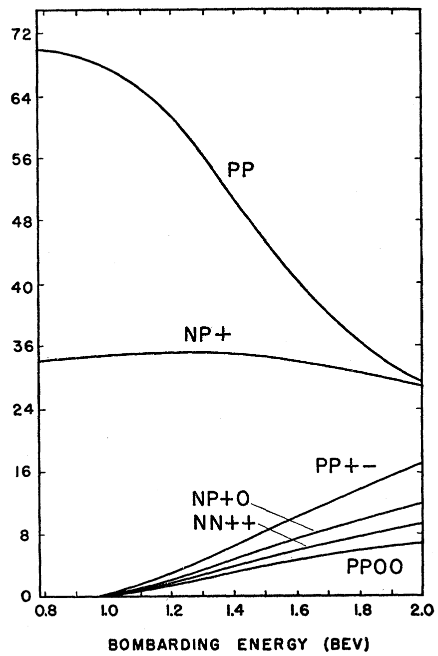

FIG. 7. The relative probability for the occurrence of p-p initiated two- and four-pronged events as a function of the bombarding energy. Curves are labeled by the charges of the final particles.

TABLE VII. Formulas giving the relative probabilities as a function of available energy for single and double meson production in nucleon-nucleon collisions. The notation is defined in Eqs. (49) and (52).

i	f	$S_n[i\rightarrow f]$
pp	$pp0$	0
pp	$np+$	$\Lambda_1\{\mathcal{G}_0+\Delta_1\}$
pn	$pp-$	$\Lambda_1\{\frac{1}{6}\mathcal{G}_0\}$
pn	$np0$	$\Lambda_1\{\frac{2}{3}\mathcal{G}_0+\frac{1}{2}\Delta_1\}$
pn	$nn+$	$\Lambda_1\{\frac{1}{6}\mathcal{G}_0\}$
pp	$pp+-$	$\Lambda_2\{2.4\mathcal{M}_0+6\Delta_2\}$
pp	$pp00$	$\Lambda_2\{0.8\mathcal{M}_0+6\Delta_2\}$
pp	$pn+0$	$\Lambda_2\{1.6\mathcal{M}_0+6\Delta_2\}$
pp	$nn++$	$\Lambda_2\{1.2\mathcal{M}_0+6\Delta_2\}$
pn	$pp-0$	$\Lambda_2\{17/15\mathcal{M}_0+7\Delta_2\}$
pn	$nn+0$	$\Lambda_2\{17/15\mathcal{M}_0+7\Delta_2\}$
pn	$pn-+$	$\Lambda_2\{42/15\mathcal{M}_0+30\Delta_2\}$
pn	$pn00$	$\Lambda_2\{14/15\mathcal{M}_0+4\Delta_2\}$

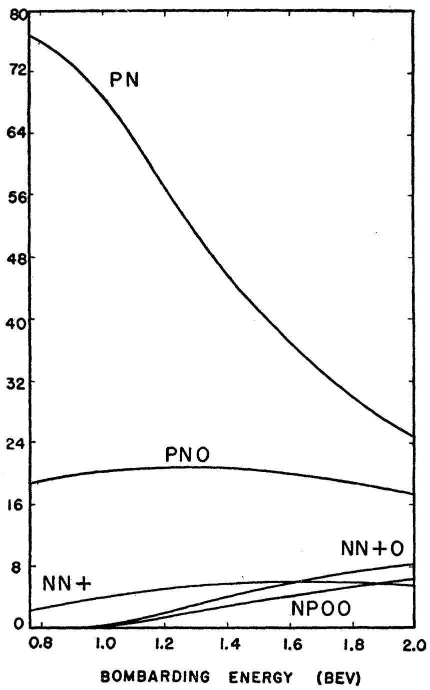

FIG. 8. The relative probability for the occurrence of *n-p* initiated one-pronged events as a function of the bombarding energy. The curves are labeled by the charges of the final particles resulting from the collision.

that these ratios remain fairly constant in the 1–2-Bev range, the calculated ratios run from 2.4:1.0:3.9 to 2.9:1.0:0.7—still good agreement in the upper part of the range.

The ratio of the total cross section for double meson production to that for single meson production is somewhat smaller than the experimentally determined value, although it is ten times greater than that calculated by means of the statistical model. For 1.7-Bev bombarding neutrons, for example, $S_2(\text{total})/S_1(\text{total})$ is calculated to be 1.2, while the experimentally deduced value from 149 analyzed events gives for this ratio the value 2.2. However, this ratio is obtained from observations of three-pronged events only, and from the relation $S_2(\text{total})/S_1(\text{total}) = [S_2(np+-)+S_2(pp-0)]/2.3S_1(pp-)$ which follows from Fermi's[2] weights for the different neutron-proton reactions within each state of meson multiplicity. With the inclusion of the final state interactions such a simple relation involving the three-prong probabilities cannot be obtained, as the relations for these probabilities indicate (Table VII). At ~1.7 Bev an approximate relation of this form is $S_2(\text{total})/S_1(\text{total}) = [S_2(np+-)+S_2(pp-0)]/3.6S_1(pp-)$. With the data from the observations of Fowler *et al.*, this gives $S_2(\text{total})/S_1(\text{total}) = 1.4$.

Preliminary results from *pp*-induced events analyzed at Brookhaven indicate that about one percent of the total interactions result in four-pronged events, $(pp+-)$, at 1.5-Bev bombarding energy. The calculated value for this is 8.5% at 1.5 Bev.

Although detailed agreement is not to be expected in view of the approximations that were made, indications are that results of calculations based on the statistical model can be made to agree with experimental results

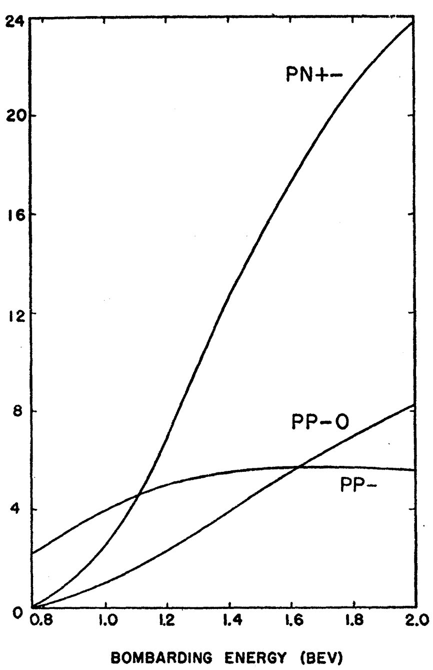

FIG. 9. The relative probability for the occurrence of *n-p* initiated three-pronged events as a function of the bombarding energy. The curves are labeled by the charges of the final particles.

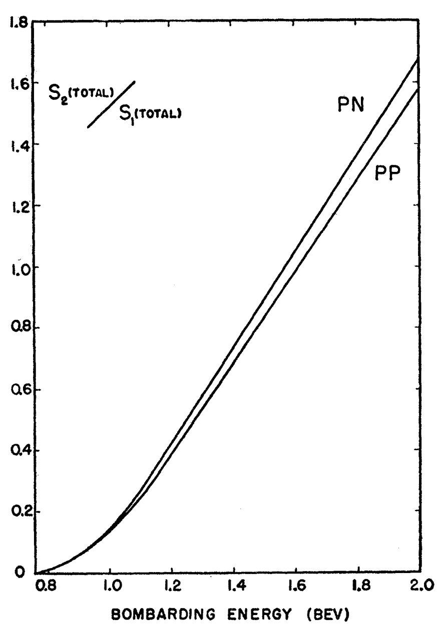

FIG. 10. The ratio of the total probability for the production of two mesons to that for the production of one meson as a function of the bombarding energy for *p-p* and *n-p* collisions.

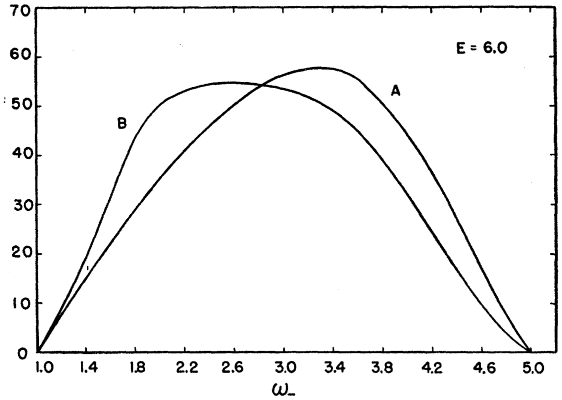

FIG. 11. Momentum distribution in the center-of-mass system of the π^- from the reaction $(pn\rightarrow pn+-)$ at a total available energy $E=6$. Curve A represents the prediction obtained from the statistical theory, while curve B includes the effect of the final state interactions. Both curves are normalized to the same total probability.

when resonance effects are accounted for and conservation laws are applied.

The author wishes to express his grateful appreciation to Professor Keith A. Brueckner for his guidance and suggestions throughout the course of this work.

VI. APPENDIX

Following Eq. (44) and Table III, the probability that the π^0 in the process $n+p\rightarrow n+p+\pi^0$ emerges with energy ω in the interval $d\omega$ is

$$dS_1[np\rightarrow np0]=\{\tfrac{2}{3}-\tfrac{1}{2}[\alpha\omega^{\frac{1}{2}}\,\mathrm{Re}Q(E)-\beta\omega|Q(E)|^2]\}\times|\psi(0)|^2(E-\omega)^{\frac{1}{2}}\frac{q^3}{\omega}d\omega \quad (47)$$

where $\alpha=(5/9\pi)(f^2/4\pi)$ and $\beta=(7/27\pi^2)(f^2/4\pi)^2$. The total weight for the process is

$$S_1[np\rightarrow np0]=\Lambda_1\{\tfrac{2}{3}\mathcal{G}_0(E)+\tfrac{1}{2}\Delta_1(E)\} \quad (48)$$

where

$$\Lambda_1=\langle q/\omega\rangle_{\mathrm{Av}}{}^3, \quad (49a)$$

$$\Delta_1=-(\alpha\mathcal{G}_1(E)\,\mathrm{Re}Q(E)-\beta\mathcal{G}_2(E)|Q(E)|^2), \quad (49b)$$

and

$$\mathcal{G}_n(E)=\int_1^E|\psi(0)|^2(E-\omega)^{\frac{1}{2}}\omega^{(n+4)/2}d\omega. \quad (49c)$$

$\frac{1}{2}\Delta_1$ represents the effect of the final state meson-nucleon interaction. Factors common to all S_1 and S_2 are omitted in Eq. (48). Similarly, for a two-meson process, $n+p\rightarrow n+p+\pi^++\pi^-$, the probability that the negatively charged meson emerges with energy ω_- in the interval $d\omega_-$ is

$$\begin{aligned}dS_2[np\rightarrow np+-]&=\frac{\Omega}{(2\pi)^2}\bigg\{1.4\left\langle\frac{q}{\omega}\right\rangle_{\mathrm{Av}}q_-\omega_-\left[\frac{q_-^2}{\omega_-^2}+\left\langle\frac{q}{\omega}\right\rangle_{\mathrm{Av}}^2\right]\mathcal{G}_0(E-\omega_-)\\&+(5/4)\beta\left\langle\frac{q}{\omega}\right\rangle_{\mathrm{Av}}q_-^3|Q(\omega_-)|^2\mathcal{G}_0(E-\omega_-)\\&+(5/4)\beta\left\langle\frac{q}{\omega}\right\rangle_{\mathrm{Av}}^3q_-\omega_-|Q(E-\omega_-)|^2\mathcal{G}_2(E-\omega_-)\\&-(5/4)\alpha\left\langle\frac{q}{\omega}\right\rangle_{\mathrm{Av}}\frac{q_-^3}{\omega_-^{\frac{1}{2}}}\mathrm{Re}Q(\omega_-)\mathcal{G}_0(E-\omega_-)\\&-(5/4)\alpha\left\langle\frac{q}{\omega}\right\rangle_{\mathrm{Av}}^3q_-\omega_-\,\mathrm{Re}Q(E-\omega_-)\mathcal{G}_1(E-\omega_-)\bigg\}d\omega_-,\end{aligned} \quad (50)$$

where approximations similar to those made in Sec. II were made in carrying out the integrals when the meson energies can be high. The total weight then is

$$S_2[np\rightarrow np+-]=\Lambda_2[(42/15)\mathfrak{M}_0(E)+30\Delta_2(E)], \quad (51)$$

where

$$\Lambda_2=\frac{1}{3\pi}\frac{2M}{E+2M}\left\langle\frac{q}{\omega}\right\rangle_{\mathrm{Av}}^4, \quad (52a)$$

$$\Delta_2=-\frac{1}{12}(\alpha\mathfrak{M}_1(E)-\beta\mathfrak{M}_2(E)), \quad (52b)$$

and

$$\mathfrak{M}_0(E)=\int_1^{E-1}\mathcal{G}_0(E-\omega)\omega^2d\omega, \quad (52c)$$

$$\mathfrak{M}_1(E)=\int_1^{E-1}\mathcal{G}_1(E-\omega)\,\mathrm{Re}Q(E-\omega)\omega^2d\omega, \quad (52d)$$

$$\mathfrak{M}_2(E)=\int_1^{E-1}\mathcal{G}_2(E-\omega)|Q(E-\omega)|^2\omega^2d\omega. \quad (52e)$$

Again the term with the Δ_2 represents the effect of the meson-nucleon interactions in the final state. In the energy range of interest (1–2 Bev), the ratio of this to the first term ranges from about 2.0 to 0.5.