Effect of the Failure of Isotopic Spin Conservation on the Pion-Nucleon S Waves*

H. P. NOYES†
The University of Rochester, Rochester, New York
(Received September 6, 1955)

The charge-dependent effects of the $\pi^- - \pi^0$ mass difference, the Coulomb interaction, and the $\pi^- + p \rightarrow \gamma + n$ radiative transition are calculated phenomenologiaclly using a charge-independent potential model for the pion-nucleon nuclear S-state interaction. It is found that the charge-exchange S-wave-scattering is little affected but that the transition rate for a bound π^- meson (Panofsky effect) is suppressed about 10%, removing the discrepancy between the difference in scattering lengths as calculated from scattering and from the Panofsky effect. However, it is also found that there can be 25% corrections to the π^- elastic scattering amplitude of either sign, making the precise analysis of low-energy π^- elastic scattering experiments difficult.

INTRODUCTION

THE over-all success of the hypothesis of the conservation of isotopic spin by the pion nucleon interaction is marred by the discrepancy between the difference in the S-wave scattering lengths as determined directly, $(\delta_1 - \delta_3)/k = 0.27$,[1] and as evaluated from photomeson experiments and the Panofsky effect using detailed balancing, $(\delta_1 - \delta_3)/k = 0.19$.[2] Part of this discrepancy may be experimental,[3] but there are three ways that the $\pi^- - p$ system fails to conserve isotopic spin and which could also be large enough to affect the analysis of these experiments. In the first place, the Panofsky effect itself shows that the neutral pion is 9 electron masses lighter than its charged counterparts.[4] This means that the eigenscattering states are not pure isotopic spin states, so that both the eigen phase shifts and the asymptotic ratios of the scattered waves will differ from those calculated using the isotopic spin formalism. Further, the phase shifts of the $\pi^0 - n$ system must be evaluated at a different momentum than those in the $\pi^- - p$ system. In the second place, the Coulomb interaction between the negative pion and the proton not only modifies the $\pi^- - p$ phase shifts, but also changes the amplitude of the incident wave at the proton and hence affects the $\pi^0 - n$ scattering as well. Finally, the fact that the negative pion can undergo radiative capture with a probability comparable to that for charge exchange scattering[5] again modifies the eigenstate phase shifts and amplitudes.

Ideally, one would like to calculate all these effects, including the mass difference itself, by including the electromagnetic interaction in the Hamiltonian for the pion-nucleon system and solving the scattering problem for this system. This would in all likelihood lead to a charge dependent modification of the interaction as well as to the kinematic effects listed above. However, the attempts to date to calculate pion-nucleon S-wave scattering from field theory, even in the absence of electromagnetic effects, have not been very convincing or successful. Consequently the approach used in this paper is to use a charge-independent potential model for the pion-nucleon interaction and to include only those isotopic-spin-dependent terms already required by experiment: the $\pi^- - \pi^0$ and $n - p$ mass difference, the Coulomb interaction, and a radiative capture interaction adjusted to fit the experimental transition ratio $\Gamma_{\pi^0}/\Gamma_\gamma = 0.94 \pm 0.20$.[5]

CHARGE INDEPENDENT POTENTIAL MODEL

The fact that the early measurements of δ_3 did not extrapolate linearly to zero at zero energy led Marshak[6] to postulate a potential model for the S-state interaction consisting of a repulsive core surrounded by a longer range attractive potential. Orear[7] has since shown that the scattering experiments are compatible with a linear momentum dependence for the S-phase shifts up to 200 Mev, and that agreement with experiment is improved by the addition of some d-wave scattering. However, since there is some slight theoretical justification for the Marshak model in terms of a strong meson-meson interaction,[8] this model is used here.

In order to determine the parameters of the model, it is convenient to make use of the effective range analysis[9] suitably modified to take account of the repulsive core. This is done by plotting $k \cot(\delta + kR)$ against k^2 for some assumed core radius R. Since the present data are very insensitive to the choice of core radius,[10] it is convenient to assume the same core for both isotopic spin states. For simplicity in the calculations that follow, it is useful to use a square well for the attractive part of the

* Research supported by the U. S. Atomic Energy Commission.
† Now at the University of California Radiation Laboratory, Livermore, California.
[1] W. Spry, Phys. Rev. 95, 1295 (1954); J. H. Tinlot and A. Roberts, Phys. Rev. 95, 137 (1954).
[2] H. L. Anderson and E. Fermi, Phys. Rev. 86, 794 (1952); J. Leiss and C. Robinson, Proceedings of the Fifth Annual Rochester Conference on High-Energy Physics (Interscience Publishers, New York, 1955), p. 4.
[3] At the Pisa Conference, Bernardini reported a new result of 1.7 for the minus-plus photo-pion production ratio in deuterium, which raises the Panofsky effect value to $(\delta_1 - \delta_3)/k = 0.22 \pm 0.02$.
[4] W. Chinowsky and J. Steinberger Phys. Rev. 93, 586 (1954).
[5] Panofsky, Aamodt, and Hadley, Phys. Rev. 81, 565 (1951).
[6] R. E. Marshak, Phys. Rev. 88, 1208 (1952): H. P. Noyes and A. E. Woodruff, Phys. Rev. 94, 1401 (1954).
[7] J. Orear, Nevis Laboratory Report No. 11 (unpublished).
[8] A. N. Mitra and F. J. Dyson, Phys. Rev. 90, 372(A) (1952); M. Ross, Phys. Rev. 95, 1687 (1954).
[9] J. M. Blatt and J. D. Jackson, Phys. Rev. 76, 18 (1949); H. A. Bethe, Phys. Rev. 76, 38 (1949).
[10] H. P. Noyes, Phys. Rev. 99, 618 (A) (1955).

potential with the same range in both states. The scattering length and effective range are then given in terms of the range and depth of the potential by

$$-a=[\tan\lambda(r-R)]/\lambda-r, \quad (1)$$

and

$$r_{\text{eff}}=r-R+1/[(R-a)\lambda^2]-(r-R)^3/3(R-a)^2, \quad (2)$$

where

$$V(x)=\infty,\ x<R;\quad V(x)=-\lambda^2,\ R<x<r;$$
$$V(x)=0,\ r<x.$$

The effective range plots are given in Fig. 1 for the well parameters given in Table I, and the momentum variation of the phase shifts is plotted in Fig. 2. (The unit of length is $\hbar/m_\pi-c$.) All these models suffer from the defect of not giving a δ_1 as large as is apparently indicated by experiment around 100 Mev, but (except for II) are not ruled out by existing data. Since, however, they all are fitted to the same scattering lengths, they agree with experiment at low energy and serve to test how dependent the effects considered here are on the details of the model. It is, perhaps, of interest that the more reasonable models (I, III) require a proton of about the size indicated by the electron scattering experiments.[11]

TABLE I. Well parameters for the various models. The unit of length is $\hbar/m_\pi-c$.

Model	Core (R)	Well (r)	r_{eff}^1	r_{eff}^3	λ_1	λ_3
I	0.3	0.8	0.714	1.002	2.2982	1.7997
II	0.3	1.3	1.856	2.885	0.9486	0.6995
III	0.4	0.7	0.364	0.421	4.3807	3.8852
IV	0.5	0.5	0	0	∞	∞

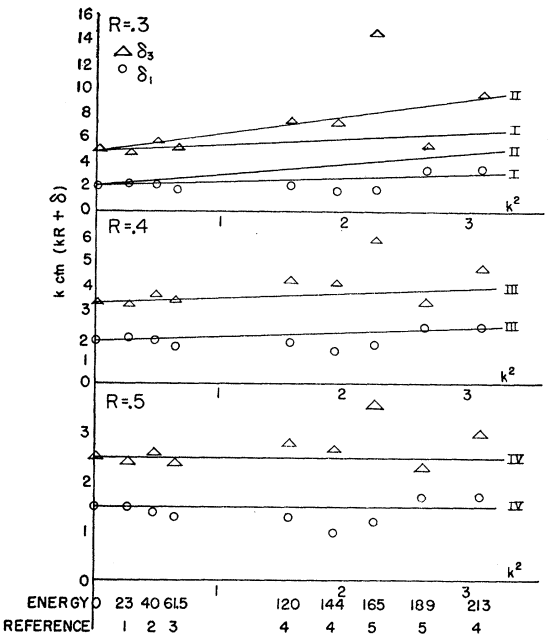

FIG. 1. Effective-range plots for three core radii. Roman numerals refer to models given in Table I. All models have scattering lengths $a_3=0.10$, $a_1=-0.17$ [see Rinehart, Rogers, and Lederman, Nevis Laboratory Report No. 12 (unpublished); J. Orear, Phys. Rev. **96**, 176 (1954)]. Phase shifts come from the following authors: 1. J. Orear *et al.*, Phys. Rev. **96**, 174 (1954), and Proceedings of the Fifth Annual Rochester Conference on High-Energy Physics (Interscience Publishers, Inc., New York, 1955), p. 18. 2. H. A. Bethe and F. de Hoffmann, Phys. Rev. **95**, 1100 (1954). 3. E. Fermi *et al.*, Phys. Rev. **95**, 1581 (1954). 4. F. de Hoffmann *et al.*, Phys. Rev. **95**, 1586 (1954), Track I. 5. H. L. Anderson *et al.*, Phys. Rev. **100**, 267, 278, 338 (1955).

INTRODUCTION OF ISOTOPIC SPIN DEPENDENT EFFECTS

The above potential models can be considered as arising from an interaction term of the form

$$\boldsymbol{\phi}\cdot[2\lambda_3^2+\lambda_1^2+(\lambda_3^2-\lambda_1^2)(\mathbf{t}\cdot\mathbf{T})]\boldsymbol{\phi}/3, \quad (3)$$

where $\mathbf{t}$ is the isotopic spin of the nucleon, $\mathbf{T}$ the isotopic spin of the meson, and $\boldsymbol{\phi}$ a six-component wave function for the pion-nucleon system. If we let the total isotopic spin $\mathbf{I}=\mathbf{t}+\mathbf{T}$, this interaction is diagonal in the representation diagonal in I^2, I_z, and gives the usual Klein-Gordon equation. If the free-particle equation of motion is to agree with the experimentally determined masses, we must add to the usual free-particle Hamiltonian a term of the form

$$\delta m^2(T_3^2+at_3+bt_3T_3^2). \quad (4)$$

Any field-theoretic treatment which is to agree with experiment must lead to this result, although it might also modify the interaction (3); the latter effect is ignored here. If y_- and y_0 represent the π^--p and the π^0-n wave functions, the equations of motion inside the range of forces are then

$$\begin{aligned} y_-''+k_-^2y_-+(\lambda_3^2+2\lambda_1^2)y_-/3+\sqrt{2}(\lambda_3^2-\lambda_1^2)y_0/3&=0,\\ y_0''+k_0^2y_0+(2\lambda_3^2+\lambda_1^2)y_0/3+\sqrt{2}(\lambda_3^2-\lambda_1^2)y_-/3&=0, \end{aligned} \quad (5)$$

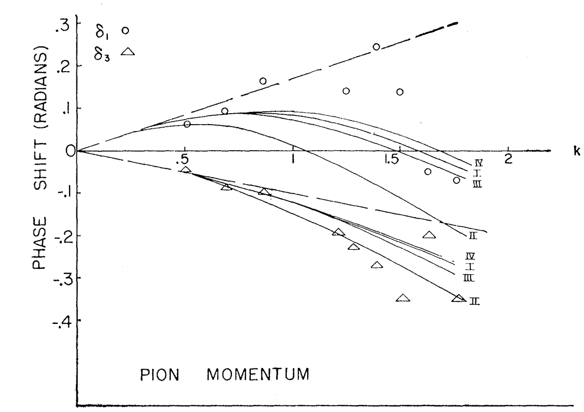

FIG. 2. Comparison of models given in Table I with data. Phase shifts calculated from the models are indistinguishable on this plot from those given by the effective range formula $k\cot(\delta+kR)=(R-a)^{-1}+\frac{1}{2}r_{\text{eff}}k^2$.

[11] R. Hoffstatder and R. W. McAllister, Phys. Rev. **98**, 217 (1955).

where k_- and k_0 are the center-of-mass momentum of the charged and neutral pion respectively. Coulomb effects can be included simply by using the appropriate Coulomb wave functions for the negative pion outside the range of interaction.

The inclusion of the possibility of radiative capture is not so straightforward, since this involves the question of the charge and current distribution inside the range of interaction, which is unknown. However, at a given energy, this transition can clearly be simulated phenomenologically simply by including an interaction term coupling the $\pi^- - p$, $\pi^0 - n$ system to the $\gamma - n$ system. The energy dependence of this term could then be fitted by using data on the S-wave photoproduction of negative and neutral pions from neutrons. Rather than attempt such an elaborate treatment here, we simply adjust the coupling to the $\pi^- - p$ system to give the observed Panofsky ratio, and assume it will not change significantly over the small energy range ($129 < E_\gamma < 161$ Mev) considered. For simplicity we also assume that the direct coupling to the $\pi^0 - n$ system is negligible and that the gamma ray is not scattered once it is produced. If $y\gamma_1$ and $y\gamma_2$ are the wave functions for S-wave gamma rays of two different polarizations, our equations of motion inside the range of interaction then become:

$$\begin{aligned} &y_-'' + k_-^2 y_- + (\lambda_3^2 + 2\lambda_1^2) y_-/3 \\ &\qquad + \sqrt{2}(\lambda_3^2 - \lambda_1^2) y_0/3 - J^2 y_{\gamma 1} - J^2 y_{\gamma 2} = 0, \\ &y_0'' + k_0^2 y_0 + (2\lambda_3^2 + \lambda_1^2) y_0/3 + \sqrt{2}(\lambda_3^2 - \lambda_1^2) y_0/3 = 0, \qquad (6) \\ &y_{\gamma 1}'' + k_\gamma^2 y_{\gamma 1}^2 - J^2 y_- = 0 \quad y_{\gamma 2}'' + k_\gamma^2 y_{\gamma 2} - J^2 y_- = 0, \end{aligned}$$

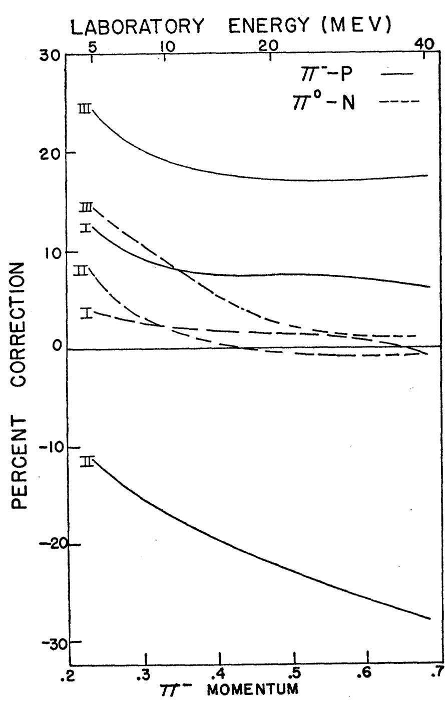

FIG. 3. Comparison of the absolute magnitude of the scattering amplitudes calculated including the charge-dependent effects with those calculated from the same model ignoring these effects.

where k_γ is the center-of-mass momentum of the gamma ray, and J^2 is to be adjusted to give the experimental ratio of 0.94 for π^0 to γ yield in the Panofsky effect.

Using the obvious fact that we can let $y_{\gamma 1} = y_{\gamma 2} = y_\gamma$, there are three normal mode solutions to the above equations, which we label P, N, and G. These solutions are:

$$\begin{aligned} &y_I{}^j = 0, && x < R, && j = -, 0, \gamma, \\ &y_I{}^j = \alpha_I{}^j \sin K_I(x - R), && R < x < r, && I = P, N, G, \qquad (7) \\ &y_I{}^j = \xi_I{}^j \sin(k_j x + \delta_I{}^j), && r < x, && \end{aligned}$$

where the normal mode frequencies K_I and ratios $\alpha_I{}^j$ are determined by solving Eqs. (6). The nine phase shifts are given by

$$k_j \cot(k_j r + \delta_j) = K_I \cot[K_I(r - R)]. \qquad (8)$$

Since we are interested in transitions starting with the $\pi^- - p$ system, we pick our normalization such that $\xi_I{}^- = 1$. Then continuity of the wave functions gives

$$\xi_I{}^j = \alpha_I{}^j \sin(k_- r + \delta_I{}^-) / \alpha_I{}^- \sin(k_j r + \delta_I{}^j). \qquad (9)$$

If we now impose the usual boundary condition of an incident plane wave and outgoing spherical waves (with appropriate momenta), the differential cross sections starting with negative mesons incident on hydrogen are

$$\frac{d\sigma^-}{d\Omega} = |s_-|^2, \quad \frac{d\sigma^0}{d\Omega} = \frac{v_0}{v_-}|s_0|^2, \quad \frac{d\sigma^\gamma}{d\Omega} = \frac{2c}{v_-}|s_\gamma|^2, \qquad (10)$$

where

$$s_j = \frac{1}{2ik_-}[(\textstyle\sum_I \xi_I{}^j \exp(i\delta_I{}^j) F_I / \exp(-i\delta_I{}^-) F_I) - \delta_{-,j}], \qquad (11)$$

and we have defined

$$\begin{aligned} F_P = \xi_N{}^0 \xi_G{}^\gamma \exp[-i(\delta_N{}^0 + \delta_G{}^\gamma)] - \xi_G{}^0 \xi_N{}^\gamma \\ \times \exp[-i(\delta_G{}^0 + \delta_N{}^\gamma)], \quad (P, N, G) \text{ cyclic}. \qquad (12) \end{aligned}$$

If we ignore the gamma-ray transition and the mass difference, $K_p{}^2 = k_-{}^2 + \lambda_3{}^2$ and $K_N{}^2 = k_-{}^2 + \lambda_1{}^2$ so that $\delta_p{}^- = \delta_p{}^0 = \delta_3$ and $\delta_N{}^- = \delta_N{}^0 = \delta_1$. Further, since under these conditions $\xi_p{}^0 = \sqrt{2}$ and $\xi_N{}^0 = -1/\sqrt{2}$, $F_p = -(1/\sqrt{2}) \exp(-i\delta_1)$ and $F_N = -\sqrt{2} \exp(-i\delta_3)$, this expression reduces to the usual result.

If we introduce the Coulomb interaction in the $\pi^- - p$ system, the phase shifts for this system are now defined by

$$\frac{k_- F_0'(k_- r) \cos\delta_I{}^- + k_- G_0'(k_- r) \sin\delta_I{}^-}{F_0(k_- r) \cos\delta_I{}^- + G_0(k_- r) \sin\delta_I{}^-} = K_I \cot[K_I(r - R)], \qquad (13)$$

where F_0 and G_0 are the regular and irregular Coulomb functions as defined by Yost, Wheeler, and Breit.[12]

[12] Yost, Wheeler, and Breit, Phys. Rev. **49**, 174 (1936).

The asymptotic coefficients of the scattered waves are now

$$\xi_I^j=\alpha_I^j[F_0(k_-r)\cos\delta_I^j+G_0(k_-r)\sin\delta_I^j]/\alpha_I^-\sin(k_jr+\delta_I^j),\quad j=0,\gamma;\quad \xi_I^-=1, \quad (14)$$

and in the $\pi^- - p$ system we must add the Coulomb amplitude

$$-\alpha\exp[i\alpha\ln\sin^2(\theta/2)]/2k\sin^2(\theta/2),\quad \alpha=e^2/\hbar v_{\mathrm{LAB}},$$

to the nuclear term s_- defined above.

In the Panofsky effect we must replace the Coulomb scattering functions with a bound-state wave function outside the range of forces. It is a very good approximation to ignore the level shift and simply use the π^- mesonic hydrogen atom wave function $xu_c(x)$ for this. We therefore take $y_-=\xi_I^-xu_c(x)$ outside the range of interaction. The π^0 and γ phase shifts are defined as before, but the asymptotic ratios now become

$$\xi_I^j=\alpha_I^jru_c(r)/\alpha_I^-\sin(k_jr+\delta_I^j),\quad j=0,\gamma,\quad \xi_I^-=1. \quad (15)$$

If we normalize to one bound π^- meson, the amplitudes of the outgoing waves are

$$p_j=\textstyle\sum_I\xi_I^j\exp(i\delta_I^j)F_I/2i\sum_IF_I, \quad (16)$$

TABLE II. Results of the various models for the Panofsky effect.

Model	J	$\Gamma_{\pi^0}/\Gamma_\gamma$	% Correction
I	0.5479	0.945	−9.81
II	0.09426	0.950	−8.14
III	1.6314	0.954	−11.07

with the F_I defined as before; the ratio of the yield of neutral mesons to gamma rays is $v_0|p_0|^2/2c|p_\gamma|^2$. We can compare the transition rate to that calculated by Fermi[2] simply by taking the ratio of p_0 to $p_{\mathrm{Fermi}}=u_c(0)\sqrt{2}(a_3-a_1)/3$, where a_3 and a_1 are the scattering lengths for S-states of isotopic spin $\frac{3}{2}$ and $\frac{1}{2}$.

RESULTS AND CONCLUSIONS

If one solves Eqs. (6) and fits the gamma-ray coupling parameter J^2 to give the observed ratio of π^0 to γ yield, all models show that the π^0 transition rate is depressed by about 10% compared to that given by Fermi. The exact values are given in Table II. Consequently these models, with a difference in scattering lengths of 0.27, predict that the difference in scattering lengths as calculated from the Panofsky effect in the usual way should be 0.243, in agreement with the latest experimental result.[3] It remains to show that they are also in agreement with the charge-exchange scattering between 20 and 40 Mev. The corrections to the charge-exchange scattering amplitude are given in Fig. 3, and are at most a couple of percent for all models over this energy region, so that it can be safely concluded that the charge dependent corrections considered here do remove the discrepancy between charge exchange scattering and the Panofsky effect.

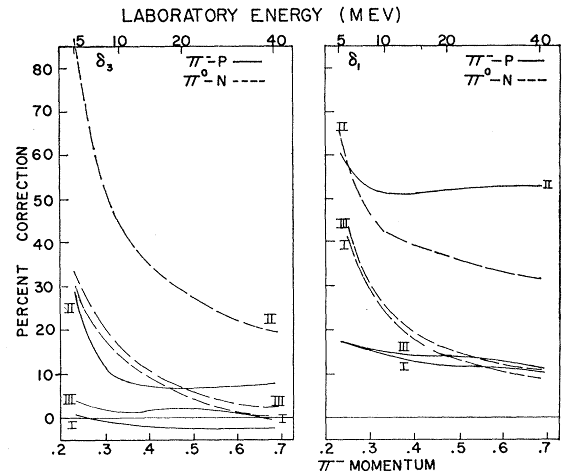

FIG. 4. Comparison of the phase shifts calculated including the charge-dependent effects with those calculated from the same model ignoring these effects. The large rise of the π^0-n phase shift corrections at low energy is due primarily to the fact that the π^0 momentum remains finite as k_- approaches zero.

The situation with regard to elastic scattering is not so straightforward. As can be seen from Fig. 3, the corrections to the elastic scattering amplitude can be as much as 25% and go in opposite directions for a small core and large range (II) and for a larger core and smaller range (III). That this change is not just due to changes in the phase shifts can be seen from Fig. 4, where it is seen that the corrections to the phase shifts that replace δ_3 and δ_1 (P and N normal modes in the notation of equation 7) are very similar for models I and III. It is clear from the numerical work leading to these results that both the changes in asymptotic ratios and the role played by the normal mode in which the radiative transition is dominant are sensitive to the choice of model. Consequently it will be necessary not just to reinterpret the phase shifts in the usual isotopic-spin-independent scattering amplitude but to use expression (11) for the analysis of π^- elastic scattering at low energy when results of precision greater than 25% are desired. Since this expression contains 15 experimental parameters (9 phase shifts and 6 ratios), it will be hopeless to attempt this analysis directly at a single energy even if precise photoproduction and radiative capture data are available in addition to scattering experiments. However, if one is guided by model calculations to assume that certain of the parameters are energy-independent, analysis should become possible, if tedious.