It is natural to expect more collective behavior when the deformation is appreciable. In the cases of the double 0^+-2^+ series in Ne^{20} the deformations are apparently not complete (not quite like lines). So, the 4^+ levels may be found at lower energies than are expected from the strong coupling-limit theory. In general there will be more different degrees of deformation, when the nucleus gets heavier, between the least deformed ground state and the highest deformed state. The spectrum of such 0^+ states seems to depend on where the nucleus is in the shell.

ACKNOWLEDGMENTS

It is a pleasure to express my gratitude to Professor E. Bleuler and to Professor D. C. Peaslee for their encouragement and helpful discussions.

PHYSICAL REVIEW VOLUME 101, NUMBER 1 JANUARY 1, 1956

Effect of the Tensor Force on the Level Structure of Li^6 and Li^7†*

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(Received July 13, 1955)

The splittings of the $^3S_1$ and $^1S_0$ states of Li^6 and the P doublet states of Li^7 by the tensor force are calculated by a variational method which includes the effect of configuration interaction. Other forces which would contribute to the splitting, such as spin-dependent central forces or vector spin-orbit forces, are not considered. The method of calculation is based on the use of a variational function of the form $\psi=\psi_0+\lambda t'\psi_0$, where t' is essentially the tensor force, treated as a perturbation on a central force oscillator wave function, ψ_0. The effect of the tensor force is shown to be equivalent to a mixture of ordinary and spin-exchange central forces plus a vector type spin-orbit force of rather complicated structure. Using a Hu-Massey Gaussian shape tensor potential, an S-state splitting of 1.4 Mev is found for Li^6 and an inverted P-doublet splitting of 380 kev is found for Li^7. A Yukawa shape potential would give similar results. In view of the approximations made in the analysis, these results are in reasonable agreement with the experimental splittings of 3.5 Mev and 480 kev for Li^6 and Li^7 respectively. The tensor force is found to contribute about 12 Mev to the binding energy of these nuclei and to introduce a 6 percent admixture of excited states into the ground state. The importance of configuration interaction is shown by a second-order perturbation calculation neglecting configuration interaction which gives entirely different results—a negligible S-state splitting for Li^6 and a normal P doublet structure for Li^7. The effect of the tensor force on the P-doublet separation in Be^7 and the F doublet separation in Li^7 is discussed briefly.

I. INTRODUCTION AND SUMMARY

PRIOR to the discovery of the quadrupole moment of the deuteron[1] and its interpretation in terms of a tensor force,[2] many theoretical studies of the level structure of light nuclei were made on the basis of central forces alone.[3–5] To allow for the observed 2.3-Mev singlet-triplet splitting in the deuteron, central spin-exchange forces were used. In view of the fact that the tensor force can account for the entire amount of the deuteron splitting,[6,7] it is clear that the level structure of other nuclei will also be considerably affected by the presence of the tensor force.

In recent years extensive calculations of the level structure of light nuclei have been made by many authors[8–10] using the vector spin-orbit force of the Mayer-Jensen shell model.[11] Qualitative agreement with the observed level structure of the p-shell nuclei can be obtained in this way. For Li^6 and Li^7 practically pure LS coupling seems indicated.[9,12] In the present work we shall neglect the possible presence of vector-type spin-orbit forces, and assume that the nuclear potential consists solely of a mixture of charge- and spin-independent central forces plus a tensor force.

Previous studies of the effect of the tensor force on light nuclei other than the deuteron have been concerned mainly with its effect on their binding

† Supported in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.
* Based on the author's Ph.D. thesis, Princeton University, 1952. A preliminary account of some of the results of this work was given at the Washington meeting of the American Physical Society [A. M. Feingold and E. P. Wigner, Phys. Rev. **79**, 221(A) (1950)].
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[6] W. G. Guindon, Phys. Rev. **74**, 145 (1948); J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley and Sons, Inc., New York, 1952), p. 103.
[7] R. L. Pease and H. Feshbach, Phys. Rev. **88**, 945 (1952).
[8] See, e.g., I. Talmi, Helv. Phys. Acta **25**, 185 (1952); D. Kurath, Phys. Rev. **88**, 804 (1952); R. Schulten, Z. Naturforsch. **8a**, 759 (1953); G. E. Tauber and Ta-You Wu, Phys. Rev. **93**, 295 (1954).
[9] D. R. Inglis, Revs. Modern Phys. **25**, 390 (1953).
[10] J. P. Elliott, Proc. Roy. Soc. (London) **A218**, 345 (1953).
[11] M. G. Mayer, Phys. Rev. **75**, 1969 (1949); **78**, 16, 22 (1950); Haxel, Jensen and Suess, Z. Physik **128**, 295 (1950).
[12] That Li^6 might be a case of intermediate or jj coupling has been suggested by N. Zeldes [Phys. Rev. **90**, 416 (1953)], T. Regge [Nuovo cimento **11**, 285 (1954)], A. Hitchcock [Phil. Mag. **45**, 385 (1954)], and Otsuki, Sawada, and Suekane [Progr. Theoret. Phys. (Japan) **13**, 79 (1955)].

energy.[13] One would expect the most spectacular evidence of the tensor force to show up in the splitting of levels which in its absence, assuming spin-independent forces alone, would be degenerate. Such levels, aside from the singlet-triplet states of the deuteron, are, assuming the approximate validity of LS coupling, the ^2P states of He^5, the $^{1,3}S$ states and the $^{1,3}D$ states of Li^6, and the ^2P states and the ^2F states of Li^7. We shall primarily be concerned with the splitting of the ^2P states of He^5 and Li^7, and the $^{1,3}S$ state of Li^6,[14] as these are expected to be the lowest levels in these nuclei on the basis of central forces alone.[3]

Dancoff[15] investigated the splitting of the ^2P states in He^5 using the Rarita-Schwinger[2] square well potential and concluded that the tensor force would produce a normal P doublet, in contradiction with the experimental level order, with a splitting that was negligible in comparison with the large experimental value. We shall see below in Sec. V that an improved calculation, with a Gaussian or Yukawa potential, gives results more in accord with experiment.

Calculations on the level structure of Li^6 including the tensor potential have been made by Elliott,[10] Ishidzu and Obi,[16] and Morita and Tamura.[14] These studies neglected the possible effects of configuration interaction. Since the selection rules on the tensor force are such that its first-order perturbation effect vanishes for doublet or S states, which are the states we are concerned with, and very few, if any, states exist in the ground state configuration which can interact with these states in a second-order perturbation calculation, we can expect that configuration interaction will be decisive in estimating the effect of the tensor force. We shall see below that this is indeed the case.

The plan of this paper is as follows. In Sec. II the level structure of Li^6 and Li^7 is calculated by considering the effect of the tensor force as a perturbation of the level structure due to the central force. Single-particle harmonic oscillator wave functions[3] are used and only interactions among the states of the lowest configuration, s^4p^n, are considered. Similar calculations have been performed by Elliott,[10] Ishidzu and Obi,[16] and Morita and Tamura.[13] The results of this method are very unsatisfactory, the calculated splitting of the $^3S_1$ and $^1S_0$ levels of Li^6 being negligible, and the calculated order of the ^2P levels of Li^7 being the reverse of the experimental order.

In Sec. III a perturbation method to take configuration interaction into account is developed. The method is based on a variational procedure first considered by Lennard-Jones,[17] who suggested the use of a variational function of the form $\psi_0+\lambda V\psi_0$, where ψ_0 is the unperturbed wave function, that due to the central force in our case, V is the perturbation potential, the tensor force, and λ is the variational parameter.[18] We shall use a modification of this method, replacing V by a function with a more suitable radial dependence.[19] The method is then applied to the deuteron as a test, and the necessary formalism for applying the method to other nuclei is developed along the lines used by Dancoff.[15]

To gain further insight into the variational method, it is applied in Sec. IV to the α particle to find the contribution of the tensor force to its binding energy. The splitting of the ^2P states of He^5 is then calculated in Sec. V, and it is now found that the $^2P_{3/2}$ level lies below the $^2P_{1/2}$ state, with a level separation of 1.0 Mev, to be compared with the observed splitting of at least 2.5 Mev.[20] In Sec. VI and Sec. VII the variational method is applied to the $^{1,3}S$ states of Li^6 and the ^2P states of Li^7 respectively. In contrast to the results of Sec. II where configuration interaction was neglected, qualitative agreement with the experimental data is now obtained. An inverted P doublet is now found for Li^7 with a splitting of the same magnitude as the experimental value of 480 kev, while for Li^6 a splitting of 1.4 Mev for the S states is obtained, which, while less than one-half the experimental value of 3.5 Mev, is some ten times the value obtained in Sec. II.

Most of the calculations are performed using a Gaussian potential for ease in computing matrix elements. The potential we shall use is one given by Hu and Massey,[21]

$$V_{12}=V_{C12}+t_{12}=V_0\exp(-\beta^2r_{12}^2)+T_0S_{12}\exp(-\tau^2r_{12}^2), \quad (1)$$

where V_{C12} and t_{12} are the central and tensor potentials respectively, V_0 and T_0 their respective strengths, β and τ their inverse characteristic ranges, and S_{12} is the tensor operator,

$$S_{12}=[3(\mathbf{r}_{12}\cdot\boldsymbol{\sigma}_1)(\mathbf{r}_{12}\cdot\boldsymbol{\sigma}_2)/r_{12}^2]-(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2). \quad (2)$$

The parameters have the values

$$V_0=-29.5\text{ Mev},\quad T_0=-17.4\text{ Mev},\quad \beta=\tau=1.29, \quad (3)$$

[13] E. Gerjuoy and J. Schwinger, Phys. Rev. **61**, 138 (1942); J. Irving, Proc. Phys. Soc. (London) **A66**, 17 (1953); M. Morita and T. Tamura, Progr. Theoret. Phys. (Japan) **12**, 653 (1954).

[14] The effect of the tensor force on the D states of Li^6 has been investigated using an extension of the methods of this paper by D. H. Lyons, Ph.D. thesis, University of Pennsylvania, 1954 (unpublished). For a summary of the results see D. H. Lyons and A. M. Feingold, Phys. Rev. **95**, 606 (1954).

[15] S. M. Dancoff, Phys. Rev. **58**, 326 (1940).

[16] T. Ishidzu and S. Obi, Progr. Theoret. Phys. (Japan) **10**, 690 (1953).

[17] J. E. Lennard-Jones, Proc. Roy. Soc. (London) **A129**, 598 (1930).

[18] This method has been applied to nuclear problems by, e.g., E. Wigner, Phys. Rev. **43**, 252 (1933); E. Feenberg, and J. K. Knipp, Phys. Rev. **48**, 906 (1935); E. Feenberg and S. S. Share, Phys. Rev. **50**, 253 (1936).

[19] The advantage of such a procedure is implicit in the work of H. R. Hassé, Proc. Cambridge Phil. Soc. **26**, 542 (1930). See also L. Pauling and E. B. Wilson, *Introduction to Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1935), p. 383 ff.

[20] Unless otherwise stated, all experimental data will be taken from the compilation of F. Ajzenberg and T. Lauritsen, Revs. Modern Phys. **27**, 77 (1955).

[21] T. Hu and H. S. W. Massey, Proc. Roy. Soc. (London) **A196**, 135 (1949).

where β and τ are in units of mc^2/e^2.[22] The central force is further assumed to be of Majorana space-exchange character, so that the lowest nuclear states will be those having maximum spatial symmetry.[3,23] Knowledge of the shape and strength of the central potential will not be necessary for most of the calculations except insofar as it determines the shape of the wave functions (especially the nuclear radius), quantities which we shall determine or assume from other considerations, or use as adjustable parameters. The use of an exchange tensor operator, $S_{12}P_{12}$, where P_{12} is the Majorana space-exchange operator,[24] instead of S_{12} in the tensor potential, will be shown below to give practically the same results as a nonexchange tensor force. The results also will be found to be largely independent of the sign of the tensor potential.

II. p-SHELL CALCULATIONS

We first calculate the level structure of Li^6 and Li^7, confining ourselves to the states of the s^4p^n configuration.[25] We assume independent particle wave functions with a Gaussian radial dependence of the form $\exp(-\alpha^2r^2)$, with α being the nuclear radius parameter.[3] Since we are assuming here that the s shell is closed and since we are interested only in the relative spacing of the levels, we can neglect completely the particles in the closed s shell.

An estimate of α can be made from the Coulomb energy differences of the pairs He^6-Li^6 and Be^7-Li^7. Using the formulas of Feenberg and Wigner,[3] this leads to the value $\alpha=1.0$ for Li^6 and $\alpha=1.2$ for Li^7. If we identify the root-mean-square value of the radius with the value $r=1.2A^{1/3}\times10^{-13}$ cm,[26] we find $\alpha=1.2$ for both Li^6 and Li^7. We therefore adopt the values $\alpha=1.1$ for Li^6 and $\alpha=1.2$ for Li^7 as the best estimates for α. Most of the results will be presented as a function of α.

The Majorana central force matrix elements for Li^6 and Li^7 have been given by Feenberg and Wigner[3] and are reproduced in Table I. The first column of Table I lists the various states, the first superscript being the spin multiplicity for the protons, the second superscript the multiplicity for the neutrons; the second column gives the position of the levels with respect to the position of the lowest state; the last column gives the partition of the symmetric group to which the spatial wave function of the p particles belongs. The integrals L and K are defined[3] as $L=V_0L^0$, $K=V_0K^0$, where

$$L^0=c^2\int\cdots\int x_1^2x_2^2R^2(r_1)R^2(r_2)J(r_{12})dv_1dv_2,$$
$$K^0=c^2\int\cdots\int x_1y_1x_2y_2R^2(r_1)R^2(r_2)J(r_{12})dv_1dv_2, \tag{4}$$

where $R(r)$ and $J(r_{12})$ are the radial dependences of the single-particle wave function and the central potential respectively, and c is the normalization constant for a p-particle wave function. For the Gaussian choice of $R(r)$ and $J(r_{12})$, as we are assuming, the integrals L^0 and K^0 are given by[3]

$$K^0=\gamma^{3/2}/4(\gamma+1)^{7/2},\quad L^0=(4\gamma^2+4\gamma+3)\gamma^{3/2}/4(\gamma+1)^{7/2}, \tag{5}$$

where $\gamma=\alpha^2/\beta^2$, the square of the ratio of the central force range to the nuclear radius. Graphs of K^0 and L^0 are given in Fig. 1.

The tensor force matrix elements for all p-shell nuclei can also be expressed in terms of two integrals, M and N, and the pertinent tensor matrix elements for

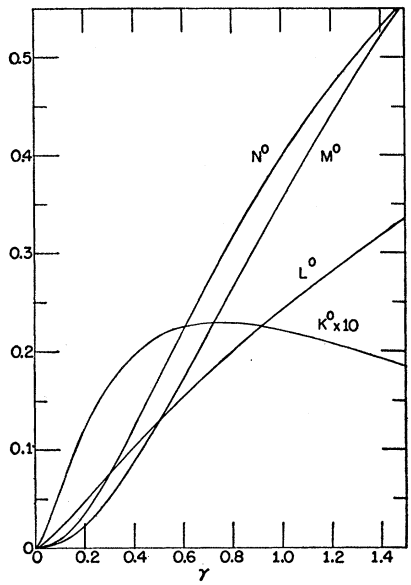

FIG. 1. p-shell integrals as a function of γ, the square of the ratio of the range of the nuclear potential to the nuclear radius. Based on a Gaussian shape for both the potential and the wave function. The central force matrix elements are functions only of K^0 and L^0, while the tensor force matrix elements are functions only of M^0 and N^0.

TABLE I. Majorana central force matrix elements for Li^6 and Li^7 (Feenberg and Wigner).[3] The energy of the lowest state has been subtracted from all matrix elements.

Li^6			Li^7		
^{22}P	$-2L+K$	(1+1)	^{23}S	$-6L+7K$	(1+1+1)
^{22}D	$-3K$	(2)	^{23}D	$-3L+K$	(2+1)
^{22}S	0	(2)	^{21}D	$-3L+K$	(2+1)
			^{21}P	$-3L+3K$	(2+1)
			^{23}P	$-3L+3K$	(2+1)
			^{21}F	$-5K$	(3)
			^{21}P	0	(3)

[22] Unless specified otherwise, all lengths are given in units of $e^2/mc^2=2.82\times10^{-13}$ cm.

[23] E. Wigner, Phys. Rev. **51**, 106 (1937); E. Wigner and E. Feenberg, Repts. Progr. Phys. **8**, 274 (1942).

[24] L. Eisenbud and E. Wigner, Proc. Nat. Acad. Sci. U. S. **27**, 281 (1941) have shown that S_{12} and $S_{12}P_{12}$ are the only types of two-body, charge-independent tensor operators that can be formed which do not involve the nucleon momenta.

[25] Similar calculations have been reported by Elliott,[10] Ishidzu and Obi,[16] and Morita and Tamura.[13]

[26] B. C. Carlson and I. Talmi, Phys. Rev. **96**, 436 (1954); J. S. Levinger, Phys. Rev. **97**, 122 (1955).

Li^6 and Li^7 are listed in Table II for the case of an exchange tensor force as well as for an ordinary tensor force. As was mentioned earlier, the first-order tensor matrix elements vanish for doublet or S states. Table II contains for Li^7 the matrix elements only for those states that interact directly with the ^2P or ^2F states, as the latter pairs of states are the ones we are interested in. The integrals M and N are defined by $M=T_0M^0$, $N=T_0N^0$, with

$$M^0=\frac{c^2}{3}\int\cdots\int(\mathbf{r}_1\times\mathbf{r}_2)^2R^2(r_1)R^2(r_2)J(r_{12})dv_1dv_2, \tag{6}$$

$$N^0=\frac{c^2}{3}\int\cdots\int[r_1^2r_2^2-3(\mathbf{r}_1\cdot\mathbf{r}_2)(\mathbf{r}_{12}\cdot\mathbf{r}_1)(\mathbf{r}_{12}\cdot\mathbf{r}_2)/r_{12}^2]\times R^2(r_1)R^2(r_2)J(r_{12})dv_1dv_2,$$

where $J(r_{12})$ is now the radial dependence of the tensor potential. For the case of Gaussian wave functions and potential, the integrals are given by

$$M^0=2\gamma^{5/2}/(\gamma+1)^{5/2},\quad N^0=(2\gamma+7)\gamma^{5/2}/2(\gamma+1)^{7/2}, \tag{7}$$

where now $\gamma=\alpha^2/\tau^2$, the square of the ratio of the tensor force range to the nuclear radius. If $J(r_{12})$ is the same for both the central and tensor potentials, as they are for the Hu-Massey potential, then it is easily seen that

$$M^0=2(L^0-3K^0). \tag{8}$$

Graphs of the functions M^0 and N^0 are given in Fig. 1. For the value $\alpha=1.1$, we have for the Hu-Massey potential

$$K=-0.68\text{ Mev},\quad M=-4.0\text{ Mev},\qquad L=-5.4\text{ Mev},\quad N=-4.9\text{ Mev}. \tag{9}$$

TABLE II. Tensor force matrix elements for Li^6 and Li^7.

	Ordinary Tensor	Exchange Tensor
	Li^6	
$\langle^3P_0\|t\|^3P_0\rangle$	M	$-M$
$\langle^3P_1\|t\|^3P_1\rangle$	$-(1/2)M$	$(1/2)M$
$\langle^3P_2\|t\|^3P_2\rangle$	$(1/10)M$	$-(1/10)M$
$\langle^3D_1\|t\|^3D_1\rangle$	$-(1/10)(7M-4N)$	same
$\langle^3D_2\|t\|^3D_2\rangle$	$+(1/10)(7M-4N)$	same
$\langle^3D_3\|t\|^3D_3\rangle$	$-(1/35)(7M-4N)$	same
$\langle^3S_1\|t\|^3D_1\rangle$	$-(2/\sqrt{5})(M-N)$	same
	Li^7	
$\langle^2P_{1/2}\|t\|^4P_{1/2}\rangle$	$(1/2\sqrt{10})(3M-2N)$	same
$\langle^2P_{3/2}\|t\|^4P_{3/2}\rangle$	$-(1/20)(3M-2N)$	same
$\langle^2P_{1/2}\|t\|^4D_{1/2}\rangle$	$(3/10\sqrt{2})(M-2N)$	same
$\langle^2P_{3/2}\|t\|^4D_{3/2}\rangle$	$-(3/20)(M-2N)$	same
$\langle^2F_{5/2}\|t\|^4P_{5/2}\rangle$	$[(21)^{\frac{1}{2}}/70](7M-8N)$	same
$\langle^2F_{5/2}\|t\|^4D_{5/2}\rangle$	$(3/35)(7M-4N)$	same
$\langle^2F_{7/2}\|t\|^4D_{7/2}\rangle$	$-(3/70\sqrt{2})(7M-4N)$	same
$\langle^4P_{1/2}\|t\|^4P_{1/2}\rangle$	$-(1/2)N$	$(1/4)(3M-2N)$
$\langle^4P_{3/2}\|t\|^4P_{3/2}\rangle$	$(2/5)N$	$-(1/5)(3M-2N)$
$\langle^4P_{5/2}\|t\|^4P_{5/2}\rangle$	$-(1/10)N$	$(1/20)(3M-2N)$
$\langle^4D_{1/2}\|t\|^4D_{1/2}\rangle$	$(3/10)N$	$-(3/20)(7M-2N)$
$\langle^4D_{3/2}\|t\|^4D_{3/2}\rangle$	0	0
$\langle^4D_{5/2}\|t\|^4D_{5/2}\rangle$	$-(3/14)N$	$(3/28)(7M-2N)$
$\langle^4D_{7/2}\|t\|^4D_{7/2}\rangle$	$(3/35)N$	$-(3/70)(7M-2N)$

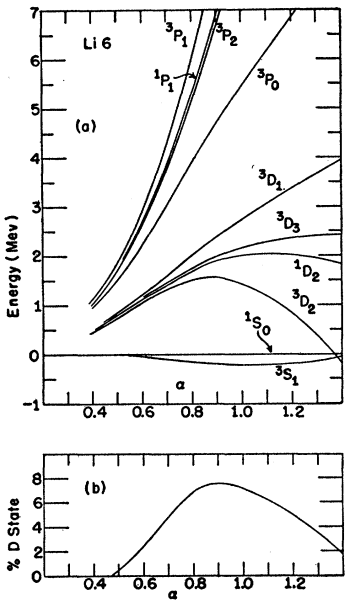

FIG. 2. (a) Position of the states of the s^4p^2 configuration of Li^6 with respect to the position of the $^1S_0$ state, as a function of α, the nuclear radius parameter. Based on the Hu-Massey Gaussian potential and neglecting configuration interaction. The experimental value of α is $\sim$1.1. (b) The percent admixture of $^3D_1$ state introduced into the $^3S_1$ ground state by the tensor force.

By using the matrix elements of Tables I and II, the level structure of Li^6 can then be readily computed. The tensor force mixes the $^3S_1$ and $^3D_1$ states, the other states being left pure. The resultant level separations for Li^6, assuming a nonexchange tensor force, are given in Fig. 2(a) as a function of α. For an exchange tensor potential, the P-state levels should be displaced an equal amount below the position of the $^1P_1$ state.

A comparison of Fig. 2(a) with the known level structure[20] of Li^6 shows that, on this model, the tensor force cannot account for the main features of Li^6. The calculated $^3S_1-{}^1S_0$ splitting is only 210 kev, 6% of the experimental value, and the order of the D states is incorrect. (Experimentally, the $^3D_3$ state lies lowest.) Only in the case of the magnetic moment is there improvement over the pure LS coupling value. The experimental value of the magnetic moment, 0.822 nuclear magnetons, indicates a D-state admixture of $\sim$10%, which is in rough agreement with the calculated value of 6.4% as shown in Fig. 2(b).[27] However, the calculated quadrupole moment is -10×10^{-27} cm^2, some 20 times the experimental value of $\sim|0.5|\times10^{-27}$ cm^2.[28]

The calculated splittings of the ^2P and ^2F states of Li^7 are also in strong disagreement with the experimental values. Using the matrix elements of Table II we find, for $\alpha=1.2$, a splitting of 82 kev for the ^2P states, with the $^2P_{1/2}$ state *below* the $^2P_{3/2}$ state, in

[27] That the tensor force might account for the magnetic moment of Li^6 has been pointed out by K. Komoda and M. Sasaki, Progr. Theoret. Phys. (Japan) **8**, 669 (1952).

[28] N. A. Schuster and G. E. Pake, Phys. Rev. **81**, 157 (1951).

contrast to the experimental value of 480 kev with the reverse order for the levels. The calculated result that the $^2P_{1/2}$ state should lie below the $^2P_{3/2}$ state results from the fact that the only states in the s^4p^3 configuration with which the tensor force permits the ^2P states to interact are the ^4P and ^4D states, and in both cases the $^2P_{1/2}$ state has larger matrix elements with these states than does the $^2P_{3/2}$ state (see Table II). Similar results hold for the ^2F states. The calculated splitting of the ^2F states is 98 kev with the $^2F_{5/2}$ state below the $^2F_{7/2}$ state. If we identify the 4.6 Mev and 7.4-Mev levels of Li^7 with the $^2F_{7/2}$ and $^2F_{5/2}$ states, respectively, then we have as striking a disagreement with the calculated values as exists with the P states. The use of an exchange tensor force leads to almost identical results since the ^2P and ^2F states are symmetric in the spatial coordinates of the three p-particles.

III. CONFIGURATION INTERACTION—VARIATIONAL METHOD

A. Variational Method

To include configuration interaction in computing the effect of the tensor force we shall use a variational method. We use as the variational wave function

$$\psi=\psi_0+\lambda t'\psi_0, \tag{10}$$

where ψ_0 is our assumed independent particle oscillator wave function, the same wave function used in Sec. II to represent the effect of the central forces, but now with the closed s-shell wave function included. The function t' is given by

$$t'=\tfrac{1}{2}\sum_{i\neq j} t_{ij}', \quad t_{ij}'=(\alpha r_{ij})^n t_{ij}, \tag{11}$$

and λ is the variational parameter to be varied so as to minimize the energy of ψ with respect to the total Hamiltonian. If the factor $(\alpha r_{ij})^n$ were omitted, then t' would be just t, the total tensor potential, and (10) would be of the form $\psi=\psi_0+\lambda V\psi_0$, where V is the perturbation potential,[29] a variational method that has been much used in problems where configuration interaction is of importance.[5,17,18] The nondimensional factor $(\alpha r_{ij})^n$ is introduced so that by adjusting the integral parameter n, $t'\psi_0$ in (10) will have approximately the proper radial dependence to represent the effect of the perturbation potential, t.[30] From the variational viewpoint, ψ_0 need not be the solution of the unperturbed Hamiltonian, and indeed it is best if the nuclear radius parameter, α, be so chosen to represent the experimental radius.

If the total Hamiltonian, H, be written in the form $H=H_0+t$, where H_0 is the central part of H, i.e., the kinetic energy plus the central potential, we have

$$\begin{aligned}\langle\psi_0|t|\psi_0\rangle&=\langle\psi_0|t'|\psi_0\rangle=0,\\ \langle\psi_0|H_0t|\psi_0\rangle&=\langle\psi_0|H_0t'|\psi_0\rangle=0,\end{aligned} \tag{12}$$

if ψ_0 is a pure S state or a doublet spin state, which are the only states we are concerned with here. The triplet D states of Li^6 require a modified treatment and will not be considered further in this paper.[14] For $\lambda=0$ we have the unperturbed energy

$$E_0=\langle\psi_0|H_0|\psi_0\rangle, \tag{13}$$

ψ_0 being assumed normalized. For another values of λ, we have

$$E(\lambda)=\langle\psi|H_0+t|\psi\rangle/\langle\psi|\psi\rangle. \tag{14}$$

The addition to the energy due to the introduction of a nonzero λ is then

$$\begin{aligned}\Delta E(\lambda)&=E(\lambda)-E_0\\ &=(2\lambda\langle\{t't\}\rangle+\lambda^2\langle e^3\rangle)/(1+\lambda^2\langle t't'\rangle),\end{aligned} \tag{15}$$

where we have used the notation $\langle A\rangle\equiv\langle\psi_0|A|\psi_0\rangle$ since ψ_0 is the only wave function appearing explicitly in the matrix elements, and where[31]

$$\{t't\}=\tfrac{1}{2}(t't+tt'), \tag{16}$$

$$e^3=t'H_0t'+t'tt'-E_0t't'. \tag{17}$$

The value of λ that minimizes $\Delta E(\lambda)$ is then obtained by differentiating (15), and is given by

$$\lambda_0=\frac{\langle e^3\rangle}{2\langle\{t't\}\rangle\langle t't'\rangle}\{1-(1+4k)^{\frac{1}{2}}\}, \tag{18}$$

where

$$k\equiv\langle\{t't\}\rangle^2\langle t't'\rangle/\langle e^3\rangle^2. \tag{19}$$

If we assume $k<\frac{1}{4}$ then the square root in (18) may be expanded in a power series in k and we obtain

$$\lambda_0=-\frac{\langle\{t't\}\rangle}{\langle e^3\rangle}\{1-k+2k^2-5k^3+\cdots\} \tag{20}$$

$$\approx-\langle\{t't\}\rangle/\langle e^3\rangle \quad (k \text{ small}). \tag{20a}$$

Substituting (19) into (15) we obtain

$$\begin{aligned}\Delta E\equiv\Delta E(\lambda_0)&=\lambda_0\langle\{t't\}\rangle\\ &=-\frac{\langle\{t't\}\rangle^2}{\langle e^3\rangle}\{1-k+2k^2-5k^3+\cdots\}\end{aligned} \tag{21}$$

$$\approx-\langle\{t't\}\rangle^2/\langle e^3\rangle \quad (k \text{ small}). \tag{21a}$$

The quantity ΔE[32] represents the additional variational

[29] Strictly speaking, the perturbation potential should also include the difference between the central potential and the "effective" oscillator potential for which the Gaussian functions are the eigenfunctions. However, this central perturbation potential will be neglected as it affects the splitting of the levels only in higher order.

[30] The introduction of t' in place of t is similar to the frequently used variational function $\psi=\psi_0[1+\lambda Vf(r)]$ where $f(r)$ is a function containing additional variational parameters [see, e.g., references in footnote 19]. Our choice of the form (10) has the advantage that it is no more difficult to compute with than the poorer function $\psi_0+\lambda t\psi_0$.

[31] Note that in general t' does not commute with t. This can be seen most easily from the fact that, e.g., t_{12} does not commute with t_{13}. While t, t', $\{t't\}$ are Hermitian, $t't$ is not.

[32] For the special case $t'=t$, Eq. (21) is identical with the perturbation expression given by Wigner, reference 18.

energy contributed by the tensor force.[33] The splitting produced by the tensor force of a pair of central force degenerate levels is then given by the difference between their respective values of ΔE. We designate this splitting by $\Delta\Delta E$.[34]

The amount of excited state, $\lambda_0 t'\psi_0$, is then given by

$$\lambda_0^2\langle t't'\rangle/(1+\lambda_0^2\langle t't'\rangle)=k(1-3k+10k^2-\cdots), \quad (22)$$

$$\approx k \quad (k \text{ small}). \quad (22a)$$

Thus the expansions in powers of k are justified provided the tensor force does not introduce more than 25% excited state into ψ [see Eq. (18)].

Equation (21a) for ΔE is closely related to the usual second-order perturbation formula

$$\Delta E=\sum_k{}'|\langle\psi_k|t|\psi_0\rangle|^2/(E_0-E_k). \quad (23)$$

By defining an $\bar{E}$, an effective average energy for the states ψ_k, Eq. (23) can be rewritten as

$$\Delta E=\langle\psi_0|t^2|\psi_0\rangle/(E_0-\bar{E}). \quad (24)$$

To see more clearly the relation between Eqs. (21a) and (24) we make the restriction $t'=t$. Equation (21a) is then exactly in the same form as (24) with $\bar{E}$ being just the energy of the state $t\psi_0$ with respect to the total Hamiltonian. In the general case where $t'\neq t$, Eq. (21a) may be put in the form

$$\Delta E=\frac{\langle\{t't\}\rangle^2/\langle t't'\rangle}{E_0-\bar{E}}, \quad (25)$$

where $\bar{E}$ is now the energy of the state $t'\psi_0$.

B. Application to the Deuteron

As a test of the variational method we shall first calculate the deuteron binding energy. We use a Yukawa potential suggested by Pease and Feshbach[35] which gives the correct binding energy, 2.23 Mev, when the wave equation is solved exactly. The potential is[35]

$$V_{12}=V_C+t=V_0e^{-\beta r_{12}}/\beta r_{12}+T_0S_{12}e^{-\tau r_{12}}/\tau r_{12}, \quad (26)$$

$$V_0=-46.1 \text{ Mev}, \quad \beta=2.38,$$

$$T_0=-24.9 \text{ Mev}, \quad \tau=1.69.$$

For ψ_0 we choose the simple function

$$\psi_0=(\alpha^3/\pi)^{\frac{1}{2}}e^{-\alpha r}\alpha_1\alpha_2, \quad (27)$$

where $\alpha_1\alpha_2$ is a triplet spin function. The assumed variational function is then of the form

$$\psi=[1+\lambda(\alpha r)^n t]\psi_0. \quad (28)$$

The true D state part of ψ has a node at the origin, so that we can expect that the optimum choice for the exponent n to be at least 2 for the Yukawa potential. The unperturbed Hamiltonian, H_0, is the sum of the kinetic energy operator, K, plus V_C. For a given choice of n and α the various matrix elements are readily computed since they involve only elementary integrals. The maximum binding energy for various choices of n, and the associated values of α, ΔE, and the percent D-state admixture, are tabulated in Table III. We see from Table III that for the choice $n=2$ the associated binding energy is 1.9 Mev, close to the experimental value of 2.2 Mev. This is in spite of our poor choice for ψ_0 which has a sharp peak at the origin.[36] The value of ΔE for $\alpha=1.6$ is about twice the additional binding energy contributed by the tensor force.[33] The D-state admixture is found to be 3.7% which agrees with the magnetic moment data. Since the value $\alpha=1.6$ is much larger than the correct asymptotic value, $\alpha=0.65$, ψ is too compact and hence the computed quadrupole moment is found to be about one-half the experimental value.

TABLE III. Deuteron binding energy.

n	α	$\langle H_0\rangle$ Mev	Binding energy Mev	Percent D state
0	0	0	0	0
1	1.25	1.8	0.1	0.3
2	1.60	3.1	1.9	3.7
3	1.25	1.8	0.5	5.2

The various matrix elements for $n=2$ are given as a function of α in Table IV. We first note that the energy of the D-state part of ψ, $t'\psi_0$, is very high, 142 Mev for $\alpha=1.6$, thus indicating considerable configuration interaction. This is due both to the complicated angular dependence of the tensor force and to the poor choice for the shape of ψ_0. We also see that the value of $\langle e^3\rangle$ [see Eq. (17)] is determined almost completely by the kinetic energy term. For this reason, in evaluating $\langle e^3\rangle$ for other nuclei we shall make the approximation that

$$\langle e^3\rangle\approx\langle K\rangle. \quad (29)$$

The kinetic energy operator, K, will be taken to be of the form

$$K=-(\hbar^2/2M)\sum_i\Delta_i, \quad (30)$$

where the sum extends over all N nucleons, and $M=M_0/(1-1/N)$, where M_0 is the mass of a nucleon,

[33] The additional binding energy due to the tensor force is somewhat less than the magnitude of ΔE since if the tensor force were absent the appropriate value of α for ψ_0 would be somewhat smaller than for ψ. This effect is particularly pronounced for the deuteron [see Sec. III(b)] where all the binding energy comes from the tensor force, but should be negligible for the lithium nuclei. On the other hand, the contribution of the tensor force to the energy of the state ψ, $\langle\psi|t|\psi\rangle/\langle\psi|\psi\rangle$, can be shown to be approximately $2\Delta E$.

[34] It is perhaps worthwhile to point out that while the variational value of ΔE must be smaller in magnitude than the accurate value (assuming that α is chosen properly) this need not be true for $\Delta\Delta E$, since the latter is an energy difference.

[35] R. L. Pease and H. Feshbach, Phys. Rev. **81**, 142 (1951). We neglect a very small amount of spin-exchange force in this potential.

[36] A. Wilson [Proc. Cambridge Phil. Soc. **34**, 365 (1938)] has shown that a trial function of the form $\psi=e^{-\alpha r}$ gives a good estimate of the binding energy of the deuteron when pure central forces are used. Our value of $\alpha=1.6$, while considerably different from the correct asymptotic value, $\alpha=0.65$, is approximately the same value as found by Wilson. The large departure from the asymptotic value is simply a reflection of the incorrect shape assumed for ψ_0.

TABLE IV. Deuteron matrix elements for $n=2$.

α	$\langle H_0\rangle$ Mev	$\frac{\langle t'Kt'\rangle}{\langle t't'\rangle}$ Mev	$\frac{\langle t'V_c t'\rangle}{\langle t't'\rangle}$ Mev	$\frac{\langle t'tt'\rangle}{\langle t't'\rangle}$ Mev	$\frac{\langle e^3\rangle}{\langle t't'\rangle}$ Mev	$-\Delta E$ Mev	Binding energy Mev	Per-cent D state
1.25	1.79	105	−7.3	15.8	111	3.35	1.56	3.1
1.50	2.68	125	−8.7	18.4	132	4.49	1.81	3.5
1.60	3.14	132	−9.2	19.4	139	5.00	1.86	3.7
1.75	3.96	144	−10.1	21.2	151	5.72	1.76	3.9
2.00	5.63	166	−11.7	23.9	172	6.96	1.33	4.2

and the factor $(1-1/N)$ allows approximately for the spurious kinetic energy of the center-of-mass motion resulting from our use of independent-particle wave functions.[3,4]

C. Operator Formalism

The matrix elements $\langle\{t't\}\rangle$, $\langle t't'\rangle$, and $\langle t'Kt'\rangle$ contain many terms since t' and t are sums of two-particle operators. Thus $\langle\{t't\}\rangle$, for example, contains three distinct types of terms, $\langle t_{ij}'t_{ij}\rangle$, $\langle(t_{ij}'t_{ik}+t_{ij}t_{ik}')/2\rangle$, and $\langle t_{ij}'t_{kl}\rangle$ where i, j, k, l are different particle labels. Following Dancoff[15] we refer to such terms as two-, three-, or four-particle terms, respectively, and similarly for the terms in $\langle t't'\rangle$ and $\langle t'Kt'\rangle$.[37] The same nomenclature will be used also for the respective operators themselves. It is also convenient[16] to resolve each operator into parts which behave as pure scalars, vectors, tensors, etc., under separate spatial or spin space rotations, each part transforming as a scalar under combined space and spin rotations. We shall call such parts scalars, vectors, etc., respectively, and shall also so designate their resultant matrix elements. For doublet states, as occur in the lowest levels of He^5 and Li^7, only the scalar and vector parts of the various operators give nonzero contributions, while for S states, which occur in He^4 and Li^6, only the scalar parts give nonvanishing contributions.

The resolution of the various operators is readily accomplished by standard group-theoretical techniques,[15,38,39] and we find the following. The two-particle terms in $\{t't\}$ have the scalar part

$$S(t_{12}'t_{12})=2f(r_{12})f'(r_{12})(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2), \tag{31}$$

where

$$t_{12}=f(r_{12})S_{12}, \quad t_{12}'=f'(r_{12})S_{12}. \tag{32}$$

The scalar part of the three-particle terms in $\{t't\}$ is given by

$$S(\{t_{12}'t_{13}\}+\{t_{13}'t_{12}\})=[f'(r_{12})f(r_{13})+f'(r_{13})f(r_{12})]\times[3(\mathbf{r}_{12}\cdot\mathbf{r}_{13})^2/r_{12}^2r_{13}^2-1](\boldsymbol{\sigma}_2\cdot\boldsymbol{\sigma}_3). \tag{33}$$

The scalar parts of the corresponding terms of $t't'$ can be obtained from Eqs. (31) and (33) by replacing $f(r_{ij})$ by $f'(r_{ij})$. Because of the factor $[3(\mathbf{r}_{12}\cdot\mathbf{r}_{13})^2/r_{12}^2r_{13}^2-1]$ the matrix element of (33) can be expected to be an order of magnitude smaller than the matrix element of (31). We shall therefore neglect all three-particle terms: in computing the scalar matrix elements of $\{t't\}$ and $t't'$. For a similar reason, the contribution of the four-particle terms in $\{t't\}$ and $t't'$ can be expected to be even smaller and hence will also be neglected. The same reasoning can be applied to the operator $t'Kt'$ and we shall retain only its two-particle terms. For $t'Kt'$ we find

$$S(t_{12}'Kt_{12}')=-(\hbar^2/M)(3+\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2)\{f'(r_{12})^2\textstyle\sum_k\Delta_k +2f'(r_{12})r_{12}^2\Delta+2f'(r_{12})r_{12}^2\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_2) -8f'(r_{12})\mathbf{r}_{12}\cdot\boldsymbol{\nabla}-4[f'(r_{12})^2/r_{12}^2]\mathbf{r}_{12}\cdot(\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_2)\}, \tag{34}$$

where

$$\boldsymbol{\nabla}=\boldsymbol{\nabla}_1[f'(r_{12})/r_{12}^2], \quad \Delta=\Delta_1[f'(r_{12})/r_{12}^2], \tag{35}$$

Δ_k and $\boldsymbol{\nabla}_k$ being the Laplacian and gradient operators respectively.

The vector part of $t_{12}'Kt_{12}'$ is[15]

$$V(t_{12}'Kt_{12}')=(9i\hbar^2/M)[f'(r_{12})^2/r_{12}^2]\times(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot(\mathbf{r}_{12}\times[\boldsymbol{\nabla}_1-\boldsymbol{\nabla}_2])). \tag{36}$$

We neglect the vector parts of the three- and four-particle terms of $t'Kt'$. To compute the vector parts of $\{t't\}$ and $t't'$ it is necessary to consider the three-particle terms in these operators as their two particle terms do not possess a vector part.[15] The vector part of the three-particle terms of $\{t't\}$ is given by[15]

$$V(\{t_{12}'t_{13}\}+\{t_{13}'t_{12}\})=(9/2)[f'(r_{12})f(r_{13})+f'(r_{13})f(r_{12})]\times[(\mathbf{r}_{12}\cdot\mathbf{r}_{13})(\mathbf{r}_{12}\times\mathbf{r}_{13})/r_{12}^2r_{13}^2]\cdot(\boldsymbol{\sigma}_2\times\boldsymbol{\sigma}_3). \tag{37}$$

For $t't'$, replace $f(r_{ij})$ by $f'(r_{ij})$ in Eq. (37). We neglect the vector parts of the four-particle terms in $\{t't\}$ and $t't'$. In applying the above formulas we shall use the abbreviated notation

$$\begin{aligned} &S_1=\langle S(\{t't\})\rangle, \quad S_2=\langle S(t'Kt')\rangle, \quad S_3=\langle S(t't')\rangle,\\ &V_1=\langle V(\{t't\})\rangle, \quad V_2=\langle V(t'Kt')\rangle, \quad V_3=\langle V(t't')\rangle. \end{aligned} \tag{38}$$

The use of an exchange tensor, $t_{ij}P_{ij}$, instead of the ordinary tensor force assumed above does not change any of the above formulas for the two-particle operators, since $P_{ij}P_{ij}=1$. Formula (37) will be changed, but for the spatially symmetric wave functions we are concerned with, it can be shown that the value of V_1 so obtained is the same as for a nonexchange tensor force. Thus within the approximations we are making, all results obtained with the variational method are independent of the exchange nature assumed for the tensor force.

IV. BINDING ENERGY OF He^4

As an additional test of the variational method we shall calculate in this section the contribution of the tensor force to the binding energy of He^4 and shall compare the results with the more accurate calculations of Irving.[13] The matrix elements for He^4 also occur as part of the later calculations on He^5, Li^6, and Li^7.

[37] The one-particle sum in the kinetic energy operator, K, is kept intact in this connection.

[38] E. Wigner, *Gruppentheorie* (Friedrich Vieweg und Sohn, Braunschweig, 1931).

[39] A. M. Feingold, Princeton thesis, 1952 (unpublished).

For ψ_0 we assume the wave function

$$\psi_0(He^4)=cA(12)A(34)\exp\{-\alpha^2\textstyle\sum_j r_j^2\}, \quad (39)$$

where the particle labels 1, 2 refer, say, to the neutrons and 3, 4 to the protons. The functions $A(12)$ and $A(34)$ are the antisymmetric singlet spin functions, and c is the appropriate normalization constant. The initial wave function, ψ_0, is a pure $^1S_0$ state, and $t'\psi_0$ is thus a pure $^5D_0$ state. The various matrix elements $\langle|t'|t|\rangle$, $\langle t't'\rangle$, and $\langle t'Kt'\rangle$ are readily evaluated subject to the approximations given in Sec. III (only the scalar parts of the matrix elements are nonzero). Some details of the calculations together with final explicit formulas for the matrix elements are given in the Appendix. The final results for ΔE for three values of n, the integral parameter in the definition of t', which determines the radial behavior of the D state [see Eq. (11)], are shown in Fig. 3 as a function of α, the radius parameter. We see that the value $n=1$ is the optimum choice, as might have been anticipated from the previous deuteron calculations since we are now using the Hu-Massey Gaussian potential instead of the singular Yukawa potential. For $\alpha=1.2$, which we assume gives the proper radius for the α-particle, we find $\Delta E=-7.9$ Mev, with a D-state admixture of 6 percent.

Figure 3 also includes the value of ΔE obtained by Werner[40] using the same method as above but with the Pease-Feshbach potential of Eq. (26) (with $n=2$) instead of the Hu-Massey potential. For $\alpha=1.2$, Werner[40] finds $\Delta E=-10.7$ Mev, with a D-state admixture of 9 percent. These values can be compared with the more elaborate variational calculations of Irving,[13] who found, using the same Pease-Feshbach potential, a D state admixture of 2.6%, with the tensor force increasing the binding energy by 6.1 Mev.[41] Our larger value of the D-state admixture is undoubtedly due to our poorer shape for ψ_0, and consequently also the poorer form for $t'\psi_0$ as compared to Irving's more accurate wave function.

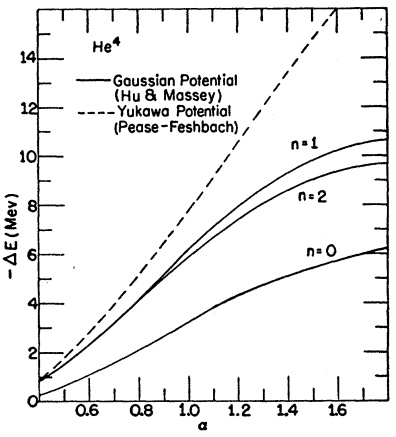

FIG. 3. Contribution of the tensor force to the binding energy of He^4 as a function of α, the nuclear radius parameter. The experimental radius is given by $\alpha\approx1.2$. The solid curves are based on the Hu-Massey Gaussian potential of Eqs. (1) and (3) for various values of the shape parameter, n. The dashed curve is based on the Pease-Feshbach Yukawa potential of Eq. (26) as calculated by Werner.[40]

[40] F. Werner, Masters thesis, University of Pennsylvania, 1954 (unpublished).

V. He^5 P-DOUBLET SPLITTING

Although the ground state of He^5 is a virtual state, we assume, following Dancoff,[15] a bound state form for ψ_0. For the $^2P_{3/2}$ state ($m_j=3/2$) we take for ψ_0 the antisymmetrized form

$$\psi_0=cA(12)[A(34)\alpha_5P(5)+A(45)\alpha_3P(3)+A(53)\alpha_4P(4)]\exp\{-\alpha^2\textstyle\sum r_j^2\}, \quad (40)$$

where the particle labels 1, 2 refer to the protons and 3, 4, 5, to the neutrons, and, e.g., $P(5)=(x_5+iy_5)$. The remainder of the notation is the same as in Eq. (39). The $^2P_{1/2}$ state need not be considered explicitly as the scalar parts of its matrix elements are identical with the corresponding ones for the $^2P_{3/2}$ state, while the vector parts of its matrix elements are twice (with a change in sign) the corresponding $^2P_{3/2}$ matrix elements (the Landé interval rule).

The splitting, $\Delta\Delta E$, of the two P states is given by the difference between the values of ΔE for the two states. Using the approximation (21a) for ΔE, an approximate formula for $\Delta\Delta E$ in terms of the matrix elements for the $^2P_{3/2}$ state alone is then

$$\Delta\Delta E\approx-6V_1(S_1/S_2)+3V_2(S_1/S_2)^2, \quad (41)$$

where we have assumed that V_1/S_1 and V_2/S_2 are small quantities. A negative value for $\Delta\Delta E$ corresponds to the $^2P_{3/2}$ state being below the $^2P_{1/2}$ state. The average (weighted) variational displacement of the levels, due to the tensor force, is then given by

$$\langle\Delta E\rangle_{Av}\approx-S_1^2/S_2. \quad (42)$$

The splitting $\Delta\Delta E$ will in general be much smaller than $\langle\Delta E\rangle_{Av}$.

Some details of the calculations and the final formulas for the matrix elements for the $^2P_{3/2}$ state are given in the Appendix. The resultant value of ΔE as a function of α, for several choices of the parameter n, is shown in Fig. 4(a). The choice $n=1$ is again found to minimize ΔE for the Gaussian tensor potential and thus presumably gives also the most reliable value for the level splitting. The calculated splitting for $n=1$ is shown in Fig. 4(b). For $\alpha=1.2$, which again we take as the most appropriate value, we have a splitting of 1.0 Mev, with the $^2P_{3/2}$ state lying below the $^2P_{1/2}$ state. Figure 4 also includes the results of a similar calculation using the Pease-Feshbach potential of Eq. (26) as calculated by Werner.[40] Werner finds a splitting of 1.3 Mev, again with the $^2P_{3/2}$ state below the $^2P_{1/2}$ state, and a 9% excited state admixture in the $^2P_{3/2}$ state.

[41] This latter value is not given explicitly by Irving but is easily deduced from his matrix elements.

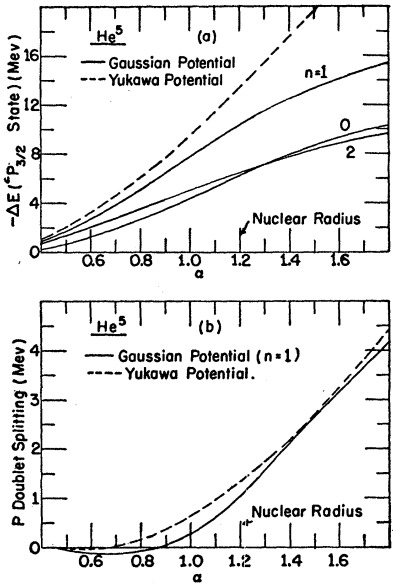

FIG. 4. (a) Contribution of the tensor force to the binding energy of the $^2P_{3/2}$ ground state of He^5 as a function of α, the nuclear radius parameter. The experimental "radius" is given by $\alpha \approx 1.2$. The solid curves are based on the Hu-Massey Gaussian potential for various values of the shape parameter, n. The dashed curve is based on the Pease-Feshbach Yukawa potential as calculated by Werner.[40] (b) The $^2P_{3/2}-^2P_{1/2}$ separation as a function of α. The solid curve is based on the Hu-Massey potential with $n=1$, while the dashed curve is based on the Pease-Feshbach potential as calculated by Werner.[40] A positive value of the separation corresponds to the $^2P_{3/2}$ state being below the $^2P_{1/2}$ state.

While the order of the states agrees with experiment, the calculated splittings are much smaller than the experimental value of at least 2.6 Mev.[42]

The numerical results for the Gaussian potential, with $n=1$, are given in more detail in Table V.[43] The second and third columns give the kinetic energy of the states ψ_0 and $t'\psi_0$, respectively. The great difference in kinetic energy (over 100 Mev for $\alpha=1.2$) shows the importance of states from highly excited configurations, and also justifies the approximation of Eq. (29). Because of this great energy difference the percent excited state admixture into ψ_0, given in column 6 of Table V, is small even though the values of ΔE are quite large. The last two columns of Table V illustrate the fact that the splitting of the P states results from two competing effects[15] [see Eq. (41)]. The first term on the righthand side of Eq. (41) favors the $^2P_{3/2}$ state as the ground state and is due to the $^2P_{3/2}$ state having larger matrix elements with states of higher configurations than does the $^2P_{1/2}$ state. The second term of Eq. (41) favors the $^2P_{1/2}$ state as the ground state and is a reflection of $\bar{E}$ being larger for the $^2P_{3/2}$ state than for the $^2P_{1/2}$ state [see Eq. (25)].

The large discrepancy between our results and those obtained by Dancoff,[15] who found a normal doublet structure with a small level splitting, seems to be due to a number of factors. Firstly, the use of a variational function $\psi_0+\lambda t'\psi_0$ instead of $\psi_0+\lambda t\psi_0$, which is what Dancoff essentially used, increases the calculated effect of the tensor force considerably. (See Fig. 4(a) where the curve for $n=0$ corresponds to setting $t'=t$.) Secondly, we have used a Gaussian shaped potential and Gaussian wave function instead of the square well potential and exponential wave function used by Dancoff. This has the same effect as using a much smaller nuclear radius, which, as we see from Table V and Eq. (41), favors an inverted doublet structure and a large level splitting.

VI. S STATES OF Li^6

To calculate the separation of the $^1S_0$ and $^3S_1$ states of Li^6 due to the tensor force we calculate the value of ΔE for each state separately, the difference between the two value of ΔE then being the splitting, $\Delta\Delta E$, of the states. The calculation of ΔE for each state is similar to the He^4 calculation, as again only the scalar parts of the operators give nonvanishing contributions to the matrix elements, but the calculations are now much more complicated since we have six nucleons to deal with instead of four. For ψ_0 we use the same wave functions as in Sec. II, but now include the α core. Thus we take for ψ_0,

$$\psi_0(^3S_1)=c_1\exp\{-\alpha^2\textstyle\sum r_j^2\}\sum_p(\mathbf{r}_3\cdot\mathbf{r}_4)A(12)A(56)a_3a_4,$$
$$\psi_0(^1S_0)=c_2\exp\{-\alpha^2\textstyle\sum r_j^2\}\times\sum_p(\mathbf{r}_3\cdot\mathbf{r}_4)A(12)A(34)A(56), \quad (43)$$

where $\sum_p$ means the proper sum over the separate permutations of the neutron labels 1, 2, 3 and the

TABLE V. He^5 P doublet states as effected by Gaussian tensor force (calculated for $n=1$).

α	$\langle K\rangle$[a] Mev	$\langle t'Kt'\rangle$[b] / $\langle t't'\rangle$ Mev	$-\Delta E$[c] Mev	$-\Delta\Delta E$[d] Mev	Percent exc. state[e]	$6V_1(S_1/S_2)$ Mev	$3V_2(S_1/S_2)^2$ Mev
0.6	13	98	2.7	−0.10	2.6	0.05	0.16
0.8	23	117	5.1	−0.08	4.1	0.35	0.44
1.0	36	139	7.9	0.27	5.1	1.15	0.85
1.2	51	165	10.6	1.04	5.7	2.46	1.30
1.4	67	194	12.7	2.10	5.7	4.12	1.70
1.6	90	226	14.2	3.16	5.6	5.68	2.05
1.8	110	264	15.7	4.57	5.3	7.00	2.27

[a] Kinetic energy of unperturbed initial state ψ_0.
[b] Kinetic energy of state $t'\psi_0$ admixed into $^2P_{3/2}$ state.
[c] Variational contribution of the tensor force to the binding energy of the $^2P_{3/2}$ state.
[d] Splitting between the $^2P_{1/2}$ and $^2P_{3/2}$ states. A positive value for $-\Delta\Delta E$ means that the $^2P_{3/2}$ state lies below the $^2P_{1/2}$ state.
[e] Percent excited state admixed into the $^2P_{3/2}$ state by the tensor force.

[42] If the virtual nature of the levels were taken into account, the predicted value of the splitting might differ considerably from the values given above, but it does not seem likely that it would be increased sufficiently to agree with the very large experimental value.

[43] The results of Table V are based on the Gaussian potential of Eq. (1) using the parameters of Eq. (3). To find the effects of changing the parameters T_0 and τ we need only note that the dependence of ΔE (and also $\Delta\Delta E$) on T_0 and τ is of the form $(T_0^2/\alpha^2)F(\alpha/\tau)$. The value of ΔE (or $\Delta\Delta E$) for a different choice, T_0', τ', can then be obtained from Table V by using the formula, $\Delta E(\alpha,T_0',\tau')=(\tau/\tau')^2(T_0'/T_0)^2\Delta E(\alpha\tau/\tau',T_0,\tau)$.

proton labels 4, 5, 6 so that the resultant wave functions will be antisymmetrical with respect to the neutrons and protons separately.

The resultant formulas for the matrix elements S_1, S_2, and S_3 are given in the Appendix. On the basis of the He^4 and He^5 results, the calculations have been restricted to the value $n=1$. The numerical results are summarized in Table VI and Fig. 5. For $\alpha=1.1$, we have a splitting of 1.4 Mev and a value of $\Delta E=-11.9$ Mev for the $^3S_1$ state. While this value of the splitting is less than half the experimental separation of 3.5 Mev, it is some 7× the value found in Sec. II where configuration interaction was neglected. Figure 5 also includes the results obtained by Lyons[14] using the Pease-Feshbach potential of Eq. (26). Lyons finds for $\alpha=1.1$ a splitting of 1.9 Mev, with a ΔE for the $^3S_1$ state of -13.3 Mev. With respect to the magnetic moment, the calculations of Sec. II give better agreement with the experimental value of 0.822 nm than the variational method which yields a magnetic moment of 0.922 nm, a value which deviates in the wrong direction from the pure $^3S_1$ state value of 0.879 nm. This behavior is due to the fact that the $t'\psi_0$ state in ψ contains some $^7D_1$ component which gives a large positive contribution to the magnetic moment, while the $^3D_1$ component, which is the only state mixed in with ψ_0 by the method of Sec. II, gives a negative contribution. This indicates that our variational method overemphasizes the role of very highly excited configurations. Thus it appears reasonable to accept the splitting given by the variational method while using the magnetic moment value given by the low states alone, i.e., the value given by the method of Sec. II. The quadrupole moment given by the variational method is, for $\alpha=1.1$, $+10\times10^{-27}$ cm^2, in contrast to the value of -10×10^{-27} cm^2 found in Sec. II. These values are to be compared with the experimental value of $\sim|0.5|\times10^{-27}$ cm. Evidently a reliable calculation of the quadrupole moment would require an elaborate second-order perturbation calculation taking states from many configurations into account.

TABLE VI. The $^1S_0$ and $^3S_1$ states of Li^6 as effected by the tensor force. Based on Hu-Massey Gaussian tensor potential.

α	$\langle K\rangle$[a] Mev	$\frac{\langle t'Kt'\rangle}{\langle t't'\rangle}$[b] Mev	$-\Delta E$[c] Mev	$-\Delta\Delta E$[d] Mev	Percent exc. state[e]	Magn. mom.[f] nm	Quad. mom.[g] 10^{-27} cm^2
0.6	17	96	3.8	0.65	3.6	0.896	7.0
0.8	31	113	7.0	1.03	5.5	0.908	9.4
1.0	48	135	10.3	1.31	6.6	0.918	9.9
1.2	69	161	13.3	1.48	7.1	0.926	9.3
1.4	93	192	15.8	1.57	7.0	0.931	8.1
1.6	122	226	17.7	1.62	6.7	0.933	6.9
1.8	155	267	19.0	1.63	6.2	0.933	5.7

[a] Kinetic energy of unperturbed initial state ψ_0.
[b] Kinetic energy of $t'\psi_0$ state admixed into $^3S_1$ state.
[c] Contribution of the tensor force to the binding energy of the $^3S_1$ state.
[d] Splitting between the $^3S_1$ and $^1S_0$ states. In all cases the $^3S_1$ state lies below the $^1S_0$ state.
[e] Percent excited state admixed into the $^3S_1$ state by the tensor force.
[f] Magnetic moment of the $^3S_1$ state in nuclear magnetons including the effect of the tensor force.
[g] Quadrupole moment of $^3S_1$ state as effected by tensor force.

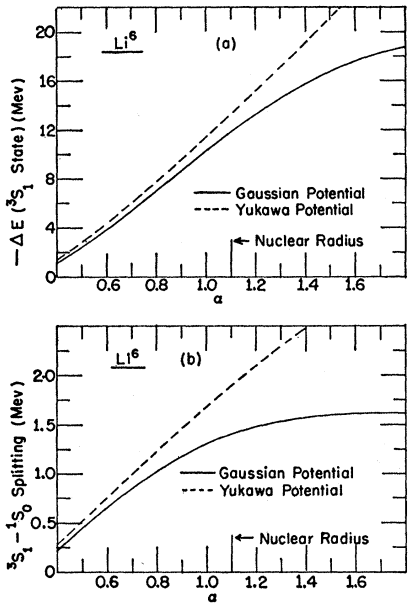

FIG. 5. (a) Contribution of the tensor force to the binding energy of the $^3S_1$ ground state of Li^6 as a function of α, the nuclear radius parameter. The experimental radius is given by $\alpha\approx1.1$. The solid curve is based on the Hu-Massey Gaussian potential, while the dashed curve is based on the Pease-Feshbach Yukawa potential as calculated by Lyons.[15] (b) The $^3S_1-{}^1S_0$ level separation for the Hu-Massey potential (solid curve) and the Pease-Feshbach potential (dashed curve) as calculated by Lyons.

The calculations above use the full 6-particle wave function for ψ_0, and hence the variational function ψ includes quintet states (and also septet states for the $^3S_1$ level) which can only arise from configurations in which the α-particle core is broken up. An alternative procedure to the above is to use the variational method but restrict the excited states in ψ to those that leave the α core intact. The variational calculation for this "α-d" model[44] is extremely simple to carry out as only the two p-particles need be considered in forming ψ_0, and the operators $\{t't\}$ and $t'Kt'$ now consist of only one term each. The results for the depression of the $^3S_1$ state (which is now the same as the splitting between the $^3S_1$ and $^1S_0$ states since the $^1S_0$ state is unaffected by the tensor force in this model), the percent admixture of $^3D_1$ state introduced by the tensor force, and the resultant magnetic moment, are given in Table VII. The values of ΔE are of course now much smaller since they do not include the effect of the tensor force on the α core, but the splitting is very similar to that obtained using the full 6-particle wave functions.[45]

[44] See J. Dabroswki and J. Sawicki, Phys. Rev. **97**, 1002 (1955) for a discussion of the validity of the α-d model as a description of Li^6.

[45] D. H. Lyons (reference 14) finds however that the $\alpha-d$ model cannot explain the experimental structure of the D states of Li^6, while the use of the full 6-particle wave function does give reasonable agreement.

TABLE VII. The "α-d" model for Li^6.

α	$-\Delta E$ Mev	Percent D state	Magn. moment nm
0.6	0.66	0.6	0.876
0.8	1.17	1.0	0.873
1.0	1.69	1.3	0.872
1.2	2.18	1.6	0.870
1.4	2.64	1.9	0.869
1.6	3.10	2.1	0.867
1.8	3.54	2.3	0.866

VII. P STATES OF Li^7

The variational calculation for the Li^7 P states proceeds in exact analogy with the He^5 calculations of Sec. V. As in the case of He^5, the $^2P_{1/2}$ wave function need not be considered explicitly. For the $^2P_{3/2}$ state we take for ψ_0 the properly antisymmetrized and normalized function formed from

$$\exp\{-\alpha^2 \sum r_i^2\} P(123) A(12) \alpha_3 A(45) A(67), \quad (44)$$

where 1, 2, 6, 7 are the neutron labels, and 3, 4, 5, are the proton labels, and where $P(123)$ is the symmetric spatial function $\sum_p (x_1+iy_1)(\mathbf{r}_2\cdot\mathbf{r}_3)$, where $\sum_p$ means the sum over the cyclic permutation of the labels 1, 2, 3. If the α core be neglected, this is just the wave function used in Sec. II. Formulas for the various matrix elements for the $^2P_{3/2}$ state (we take $n=1$ in the calculations) are given in the Appendix. The corresponding matrix elements for the $^2P_{1/2}$ state can then be immediately determined as in the case of He^5.

Numerical results as a function of α, the nuclear radius parameter, are given in Table VIII and in Fig. 6. The last column of Table VIII gives the resultant magnetic moment for the $^2P_{3/2}$ state, to be compared with the experimental value of 3.257 nm, the value for ψ_0, the pure $^2P_{3/2}$ state, being 3.126 nm. The same general remarks apply to this table as to Table V for He^5 (see Sec. V). It is seen that the splitting of the P states is a very sensitive function of the nuclear radius, as it also was for He^5. For $\alpha=1.2$, which we assume gives the proper radius for Li^7 (see Sec. II), we have the $^2P_{3/2}$ state below the $^2P_{1/2}$ state, in agreement with experiment, and a splitting of 380 kev, in good agreement with the experimental value of 480 kev. The corresponding value of ΔE is -11.6 Mev. Judging from the results obtained for He^5, it can be expected that the use of the Pease-Feshbach potential of Eq. (26) would have given somewhat larger values for ΔE and the splitting.

If the above calculations are repeated using an "$\alpha-H_3$" model, i.e., neglecting the α core of Li^7, a normal doublet structure is found, with the $^2P_{1/2}$ state 80 kev below the $^2P_{3/2}$ state, a result similar to that found in Sec. II. Together with Lyons' results for the D states of Li^6 [45] we can conclude that it is essential to include the excitation of the α core in evaluating the effect of the tensor force.

The results given in Table VIII (excluding the last column) can be applied equally well to the mirror nucleus Be^7, where the P state splitting is known to be 434 kev, some 10% smaller than the Li^7 P-state splitting. This reduced splitting could be explained on our model by requiring the nuclear radius of Be^7 to be 2% larger than the Li^7 radius, which is not unreasonable in view of the larger Coulomb energy of Be^7.[46]

The variational method can also be applied to the calculation of the splitting of the ^2F states of Li^7. A rough calculation indicates that the separation of the ^2F states is about $3\times$ the separation of the ^2P states, with the $F_{7/2}$ state below the $F_{5/2}$ state.[47] If we identify the 4.65 Mev and 7.4 Mev levels of Li^7 with the $F_{7/2}$ and $F_{5/2}$ states respectively (the 7.4-Mev level is known to have a spin of 5/2) we have an experimental value of 6 for the ratio of the F-state splitting to the P-state splitting.

VIII. CONCLUSIONS

We have seen that the variational method of computing the effect of the tensor force gives results in qualitative agreement with the experimental data on the separation of the $^1S_0$ and $^3S_1$ states of Li^6 and the separation of the P doublet states of He^5 and Li^7. Quantitative agreement could not be expected due to the crudeness of the variational method used to approximate the all-important effect of configuration interaction. In addition to the uncertainty in shape and strength of the nuclear potential and the uncertainty in the choice of the nuclear radius (to which the He^5 and Li^7 results are particularly sensitive), we have also introduced a large error by neglecting many 3-particle and all 4-particle terms in carrying out the variational calculations. The most important term neglected in

TABLE VIII. Li^7 P doublet states as effected by the tensor force. Based on Hu-Massey Gaussian potential.

α	$\langle K\rangle$[a] Mev	$\langle l'Kl'\rangle$[b] $\langle l'l'\rangle$ Mev	$-\Delta E$[c] Mev	$-\Delta\Delta E$[d] Mev	Percent exc. state[e]	$6V_1(S_1/S_2)$ Mev	$3V_2(S_1/S_2)^2$ Mev	Magn. mom.[f] nm
0.6	22	101	3.1	−0.03	2.9	0.02	0.06	3.205
0.8	39	127	5.8	−0.02	4.2	0.11	0.15	3.260
1.0	60	160	8.8	0.11	4.9	0.36	0.28	3.310
1.2	87	200	11.6	0.38	5.2	0.78	0.41	3.348
1.4	120	248	14.1	0.74	5.1	1.24	0.53	3.369
1.6	156	305	16.0	1.11	4.8	1.77	0.62	3.378
1.8	196	370	17.3	1.45	4.3	2.22	0.68	3.375

[a] Kinetic energy of unperturbed ψ_0 state.
[b] Kinetic energy of $l'\psi_0$ state admixed into $^2P_{3/2}$ state.
[c] Contribution of the tensor force to the binding energy of the $^2P_{3/2}$ state.
[d] Splitting between the $^2P_{1/2}$ and $^2P_{3/2}$ states. A positive value of $-\Delta\Delta E$ means that the $^2P_{3/2}$ state lies below the $^2P_{1/2}$ state.
[e] Percent excited state admixed into the $^2P_{3/2}$ state by the tensor force.
[f] Magnetic moment in nuclear magnetons of the $^2P_{3/2}$ state as effected by the tensor force.

[46] See E. Feenberg, Phys. Rev. **81**, 644 (1951), for a discussion of the compressibility of these nuclei. A study of the difference between the Be^7 and Li^7 P doublet splittings from the viewpoint of a vector spin-orbit force has been made by D. R. Inglis, Phys. Rev. **82**, 181 (1951).

[47] Inglis (reference 9) finds a theoretical value of 7/3 for this ratio, on the assumption that the splitting is due to a vector spin-orbit force.

the He^5 and Li^7 calculations is probably the vector part of the 3-particle terms of $t'Kt'$. A crude calculation indicates that inclusion of this term would reduce the splitting found for He^5 by perhaps 15%, with possibly a larger effect for Li^7. The most important terms neglected in the Li^6 calculations are probably the scalar parts of the 3-particle terms of $\{t't\}$ and $t'Kt'$. Here a rough calculation indicates that inclusion of these terms would increase the splitting by about 40% (for $\alpha=1.1$, with a larger effect for larger values of α). The calculation of these terms is, however, so complicated that it has not been carried to completion. All in all, we estimate, taking the above effects into consideration, that the numerical results given in this paper are probably reliable to within a factor of 2. However, this is sufficiently accurate to conclude that the tensor force is responsible for a considerable part, if perhaps not all, of the experimentally observed separations of the P doublet states of He^5 and Li^7 and the S states of Li^6.

Perhaps the most striking result is the importance of considering configuration interaction in calculating the effect of the tensor force. This has two main consequences. Firstly, the effect of the tensor force is largely independent of the exchange nature and even the sign of the tensor force. Secondly, in considering the effect of the tensor force on the splitting of singlet and triplet states, the tensor force is equivalent to a mixture of ordinary and spin-exchange forces [see Eq. (31) and (33)], while its action on the splitting of doublet states is similar to the effect produced by a vector spin-orbit force in first order. The equivalent vector spin-orbit force is a mixture of both two- and three-particle forces [see Eqs. 36) and (37)] and thus is much more complicated in form than the vector spin-orbit force usually taken as the basis of the shell model. While it is attractive to speculate that the vector spin-orbit force of the shell model may possibly be a manifestation of the tensor force[9,48] the type of coupling introduced by the tensor force seems to be of a far more complicated nature than the simple and successful jj coupling scheme of the shell model.

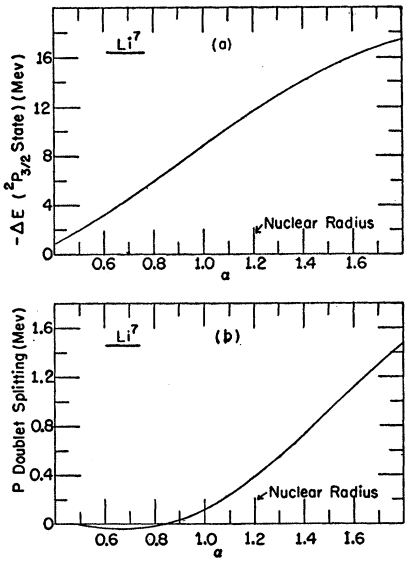

FIG. 6. (a) Contribution of the tensor force to the binding energy of the $^2P_{3/2}$ ground state of Li^7. The experimental radius is given by $\alpha\approx 1.2$. Based on the Hu-Massey Gaussian potential. (b) The splitting between the $^2P_{3/2}$ and $^2P_{1/2}$ states. A positive value for the separation corresponds to the $^2P_{3/2}$ level being below the $^2P_{1/2}$ level.

IX. ACKNOWLEDGMENTS

It is a pleasure to thank Professor E. Wigner for suggesting this problem, and for his continuous advice and encouragement throughout all phases of the work. Many helpful discussions were also had with Dr. T. Teichmann, Dr. E. Jaynes, Dr. S. Rubinow, and Dr. F. R. Innes. The author is indebted to the National Research Council for a predoctoral fellowship at Princeton University where this work was begun.

APPENDIX

He^4 Matrix Elements

To calculate S_1 we first note that for He^4 the operator $\{t't\}$ reduces to $\{t't\}=4t_{13}'t_{13}$ if we neglect the three- and four-particle terms. Using the scalar part of $t_{13}'t_{13}$ given in Eq. (31) we obtain for S_1, after integrating over the spin variables and the coordinates of particles 2 and 4,

$$S_1=\frac{3\times 2^6 T_0^2\alpha^6}{\pi^3}\int\cdots\int(\alpha r_{13})^n \times\exp\{-2\alpha^2(r_1^2+r_3^2)-2\tau^2 r_{13}^2\}dv_1dv_3. \quad \text{(A1)}$$

The six-dimensional integral is readily evaluated by switching to r_1 and r_{13} as the independent variables and introducing polar coordinates. We find

$$\begin{aligned} S_1&=48\pi^{-\frac{1}{2}}T_0^2\zeta^{n+3}[\tfrac{1}{2}(n+1)]! && n \text{ odd}\\ &=3\times 2^{3-n}T_0^2\zeta^{n+3}(n+1)!/(\tfrac{1}{2}n)! && n \text{ even}, \end{aligned} \quad \text{(A2)}$$

where $\zeta^2=\alpha^2/(\alpha^2+2\tau^2)$. The evaluation of S_2 and S_3 proceeds in the same manner, and we obtain

$$S_3=3\times 2^{3-2n}T_0^2\zeta^{2n+3}, \quad \text{(A3)}$$

$$\begin{aligned} S_2&=36(\hbar^2/M)T_0^2\alpha^2\zeta(3+\zeta^2) && n=0\\ &=6(\hbar^2/M)T_0^2\alpha^2\zeta^3(31+27\zeta^2) && n=1\\ &=45(\hbar^2/M)T_0^2\alpha^2\zeta^5(7+9\zeta^2) && n=2. \end{aligned} \quad \text{(A4)}$$

[48] E. P. Wigner, in *Symposium on New Research Techniques in Physics*, July 15–29, 1952 (Academia Brasileira de Ciências, Rio de Janeiro, 1954). The results in this paper on the repulsion between a nucleon and a half-filled shell due to the tensor force interaction depend on the assumption of a nonexchange tensor force. Otherwise there is an attraction. Dr. H. Horie has recently confirmed and extended these calculations (private communication from Professor Wigner). This is in contrast to the results of the present paper, which are essentially independent of the exchange character of the tensor force.

He^5 Matrix Elements

The calculation of S_1, S_2, and S_3 proceeds in exactly the same fashion as for He^4. We obtain

$$\begin{aligned} S_1&=6T_0^2\zeta^3(5+3\zeta^2) && n=0\\ &=12\pi^{-\frac{1}{2}}T_0^2\zeta^4(5+4\zeta^2) && n=1\\ &=40T_0^2\zeta^5(1+\zeta^2) && n=2, \end{aligned} \tag{A5}$$

$$\begin{aligned} S_3&=6T_0^2\zeta^3(5+3\zeta^2) && n=0\\ &=40T_0^2\zeta^5(1+\zeta^2) && n=1\\ &=(45/2)T_0^2\zeta^7(5+7\zeta^2) && n=2, \end{aligned} \tag{A6}$$

$$\begin{aligned} S_2&=(3\hbar^2/M)T_0^2\alpha^2\zeta(135+109\zeta^2+21\zeta^4) && n=0\\ &=(3\hbar^2/2M)T_0^2\alpha^2\zeta^3(155+349\zeta^2+210\zeta^4) && n=1\\ &=(225\hbar^2/4M)T_0^2\alpha^2\zeta^5(7+29\zeta^2+14\zeta^4) && n=2. \end{aligned} \tag{A7}$$

The calculation of V_2 is similar since we are retaining only the two-particle terms in the operator. The formulas are

$$\begin{aligned} V_2&=36(\hbar^2/M)T_0^2\alpha^2\zeta^3 && n=0\\ &=54(\hbar^2/M)T_0^2\alpha^2\zeta^5 && n=1\\ &=135(\hbar^2/M)T_0^2\alpha^2\zeta^7 && n=2. \end{aligned} \tag{A8}$$

The calculations of V_1 and V_3 are somewhat different since now we must use the 3-particle terms [see Eq. (37)]. As an example, $V_1(n=1)$ becomes, after performing the spin sum and integrating over the two particles not involved in the potential,

$$V_1=\frac{5\times 2^{11/2}T_0^2\alpha^{12}}{\pi^{9/2}}\int\int\int(r_{12}+r_{13})(\mathbf{r}_{12}\cdot\mathbf{r}_{13})$$
$$\times[(\mathbf{r}_{12}\times\mathbf{r}_{13})^2/r_{12}^2r_{13}^2]\exp\{-2\alpha^2(r_1^2+r_2^2+r_3^2)$$
$$-\tau^2(r_{12}^2+r_{13}^2)\}dv_1dv_2dv_3. \tag{A9}$$

Integrating over r_1 by switching to r_1, r_{12}, and r_{13} as the independent variables, we obtain

$$V_1=\frac{5\times 2^4T_0^2\alpha^9}{3^{\frac{3}{2}}\pi^3}\int\int(r_2+r_3)(\mathbf{r}_2\cdot\mathbf{r}_3)[(\mathbf{r}_2\times\mathbf{r}_3)^2/r_2^2r_3^2]$$
$$\times\exp\{-(4/3)\alpha^2[r_2^2+r_3^2-(\mathbf{r}_2\cdot\mathbf{r}_3)]$$
$$-\tau^2(r_2^2+r_3^2)\}dv_2dv_3. \tag{A10}$$

This can be reduced to a two-dimensional integral by using polar coordinates and integrating over the angles:

$$V_1=\frac{5\times 3^3T_0^2}{\pi}\int\int\frac{\exp[-(r_2^2+r_3^2)/\eta^2]}{r_2}$$
$$\times[(r_2^2r_3^2-3r_2r_3+3)e^{r_2r_3}$$
$$-(r_2^2r_3^2+3r_2r_3+3)e^{-r_2r_3}]dr_2dr_3, \tag{A11}$$

where $\eta^2=4\alpha^2/(4\alpha^2+3\tau^2)$. Although (A11) can be integrated exactly in closed form, it is convenient for computational purposes to expand the exponentials in power series before integrating. The final formulas thus obtained for V_1 and V_3 are

$$\begin{aligned} V_1&=\frac{5\times 3^{5/2}T_0^2\eta^{10}}{2^4}\sum_{k=0}^{\infty}\frac{(k+2)(2k+3)!\eta^{4k}}{(4k^2+18k+20)k!(k+1)!2^{4k}} && n=0\\ &=\frac{135T_0^2\eta^{11}}{8\pi^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(k^2+3k+2)\eta^{4k}}{(2k+5)2^{2k}} && n=1\\ &=\frac{5\times 3^{7/2}T_0^2\eta^{12}}{2^8}\sum_{k=0}^{\infty}\frac{(2k+3)!\eta^{4k}}{k!(k+1)!2^{4k}} && n=2, \end{aligned} \tag{A12}$$

$$\begin{aligned} V_3&=V_1 && n=0\\ &=\frac{5\times 3^{7/2}T_0^2\eta^{12}}{\pi}\sum_{k=0}^{\infty}\frac{[(k+2)!]^3\eta^{4k}}{(2k+5)!k!} && n=1\\ &=\frac{5\times 3^{9/2}T_0^2\eta^{14}}{2^{12}}\sum_{k=0}^{\infty}\frac{(2k+5)!\eta^{4k}}{k!(k+2)!2^{4k}} && n=2. \end{aligned} \tag{A13}$$

Li^6 Matrix Elements

$$\begin{aligned} S_1(^1S_0)&=24\pi^{-\frac{1}{2}}T_0^2\zeta^4(3+4\zeta^2),\\ S_1(^3S_1)&=4\pi^{-\frac{1}{2}}T_0^2\zeta^4(23+16\zeta^2+8\zeta^4), \end{aligned} \tag{A14}$$

$$\begin{aligned} S_3(^1S_0)&=3T_0^2\zeta^5(3+5\zeta^2),\\ S_3(^3S_1)&=T_0^2\zeta^5(69+60\zeta^2+35\zeta^4)/6, \end{aligned} \tag{A15}$$

$$\begin{aligned} S_2(^1S_0)&=(3\hbar^2/M)T_0^2\alpha^2\zeta^3(93+300\zeta^2+255\zeta^4),\\ S_2(^3S_1)&=(\hbar^2/2M)T_0^2\alpha^2\zeta^3(713+1827\zeta^2\\ &\qquad+1335\zeta^4+525\zeta^6). \end{aligned} \tag{A16}$$

Li^7 Matrix Elements

$$S_1=(2/15\pi^{\frac{1}{2}})T_0^2\zeta^4(525+760\zeta^2+264\zeta^4), \tag{A17}$$

$$S_3=(3/2)T_0^2\zeta^5(35+38\zeta^2+11\zeta^4), \tag{A18}$$

$$S_2=(\hbar^2/4M)T_0^2\alpha^2\zeta^3(75+4450\zeta^2+4257\zeta^4+1484\zeta^6), \tag{A19}$$

$$V_2=(6\hbar^2/M)T_0^2\alpha^2\zeta^5(3+2\zeta^2), \tag{A20}$$

$$V_1=(3/5600\pi^{\frac{1}{2}})T_0^2\eta^{11}(4088+1232\eta^2+2670\eta^4$$
$$+900\eta^6+1225\eta^8+\cdots), \tag{A21}$$

$$V_3=(13/45)V_3(He^5)+(4/525)T_0^2\eta^{12}(14+42\eta^2$$
$$+27\eta^4+36\eta^6+20\eta^8+\cdots). \tag{A22}$$