Single-Body Wave Functions and Two-Body Forces*

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(Received July 25, 1955)

It is found that some of the eigenfunctions of a nuclear Hamiltonian containing Hooke's-law interactions of nucleons can be expressed as independent-particle states. In these states it is more satisfactory to define the nucleons' positions with respect to the centers of mass of the protons and of the neutrons, rather than with respect to a fixed origin. There is some reason to believe that all nuclear bound states can be approximated by means of perturbations from such independent-particle states.

I. PREVIOUS RESULTS

In a previous paper,[1] hereinafter referred to as I, the following nuclear Hamiltonian was studied with the aid of normal coordinates (the notation is that of I):

$$H_0=\sum_{k=1}^{A} p_k^2/2m+\sum_{k>j=1}^{A}[\tfrac{1}{2}b_{kj}(\mathbf{r}_k-\mathbf{r}_j)^2-D_{kj}]. \quad (1)$$

Normal mode vectors $\boldsymbol{\varrho}$ were defined such that

$$H_0=\sum_{\alpha=1}^{A}\pi_\alpha^2/2m+\tfrac{1}{2}\sum_{\alpha=1}^{A}\beta_\alpha\rho_\alpha^2-D; \quad (2)$$

these vectors are related to the nucleons' position vectors by

$$\mathbf{r}_k=\sum_{\alpha=1}^{A}T_{k\alpha}\boldsymbol{\varrho}_\alpha, \quad (3)$$

where T is a real unitary matrix. In the case considered in I, each of the force constants b_{kj} is equal too ne of three different values, b_p, b_n, b_{np}. For that case, $\beta_1=0$ and $\boldsymbol{\varrho}_1$ is proportional to the position vector of the center of mass of the nucleus; $\beta_{Z+1}=Ab_{np}$ and $\boldsymbol{\varrho}_{Z+1}$ is proportional to the position vector of the center of mass of the protons with respect to that of the neutrons. Also,

$$\begin{aligned}\beta_2=\beta_3=\cdots=\beta_Z=Zb_p+Nb_{np},\\ \beta_{Z+2}=\beta_{Z+3}=\cdots=\beta_A=Zb_{np}+Nb_n;\end{aligned} \quad (4)$$

the corresponding normal-mode vectors are highly arbitrary. The sum of coefficients in each of them vanishes, $\boldsymbol{\varrho}_2\cdots\boldsymbol{\varrho}_Z$ depend only on proton coordinates and $\boldsymbol{\varrho}_{Z+2}\cdots\boldsymbol{\varrho}_A$ depend only on neutron coordinates.

The energy levels are those of a set of noninteracting harmonic oscillators, as indicated by Eq. (2). Antisymmetric eigenfunctions can be generated by means of antisymmetrized generating functions such as the G_p given in I:

$$G_p=\exp\left[-\sum_{\alpha=2}^{Z}(U_\alpha^2+\tfrac{1}{2}a_\alpha^2\rho_\alpha^2)\right]\Delta, \quad (5)$$

where Δ is a $Z\times Z$ determinant with elements

$$\Delta_{mn}=s_n(m)\exp\left(2\sum_{\alpha,\beta=2}^{Z}T_{m\beta}T_{n\alpha}a_\beta\mathbf{U}_\alpha\cdot\boldsymbol{\varrho}_\beta\right), \quad (6)$$

$s_n(m)$ being the nth spin function for the mth proton. There is a similar antisymmetrized generating function G_n involving $\boldsymbol{\varrho}_{Z+2}\cdots\boldsymbol{\varrho}_A$. The vectors $\boldsymbol{\varrho}_1$ and $\boldsymbol{\varrho}_{Z+1}$ are not involved in the antisymmetrizing procedure. If G_p (or G_n) is expanded in a power series of the components $(u_\alpha,v_\alpha,w_\alpha)$ of the vectors $\mathbf{U}_\alpha$, the coefficient of each term is an antisymmetric spin-dependent energy eigenfunction for normal modes 2, $3\cdots Z$ (or $Z+2$, $Z+3\cdots A$). As stated in I, the derivatives of G_p with respect to the u_α, v_α, w_α, with those variables all equal to zero, all vanish below a certain total order K_p, which is thus the sum of proton-oscillator excitations in the ground state. For a given number of protons, K_p is equal to what one would get for the total proton quantum number of the ground state by the harmonic-oscillator single-body model. A similar result holds for the neutron oscillators.

II. COMPARISON WITH INDIVIDUAL-PARTICLE WAVE FUNCTIONS

The generating function G_p can be written

$$G_p=\exp\left[-\sum_{k=1}^{Z}(V_k'^2+\tfrac{1}{2}a_p^2r_k'^2)\right]\Delta, \quad (7)$$

and an element of the determinant Δ can be written

$$\Delta_{mk}=s_k(m)\exp(2a_p\mathbf{r}_m'\cdot\mathbf{V}_k'). \quad (8)$$

Here, a_p is the common value $(m\beta_p/\hbar^2)^{\frac{1}{2}}$ of the quantities $a_2\cdots a_Z$;

$$\mathbf{V}_k'=\sum_{\alpha=2}^{Z}T_{k\alpha}\mathbf{U}_\alpha, \quad (9)$$

and

$$\mathbf{r}_m'=\sum_{\beta=2}^{Z}T_{m\beta}\boldsymbol{\varrho}_\beta=\mathbf{r}_m-\frac{1}{Z}\sum_{k=1}^{Z}\mathbf{r}_k=\mathbf{r}_m-\mathbf{R}_p. \quad (10)$$

Thus $\mathbf{r}_m'$ is the position vector of the mth proton with respect to the center of mass of all the protons (given by $\mathbf{R}_p\equiv\sum_{k=1}^{Z}\mathbf{r}_k/Z$). There are Z vectors $\mathbf{V}_k'$ and Z vectors $\mathbf{r}_m'$; each set of Z vectors has been defined in terms of a set of $Z-1$ vectors $(\mathbf{U}_2\cdots\mathbf{U}_Z,\ \boldsymbol{\varrho}_2\cdots\boldsymbol{\varrho}_Z)$. Therefore the $\mathbf{V}_k'$ and the $\mathbf{r}_m'$ cannot be all independent. In fact,

$$\sum_{k=1}^{Z}\mathbf{V}_k'=\sum_{m=1}^{Z}\mathbf{r}_m'=0. \quad (11)$$

* Supported by the National Science Foundation.
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[1] I. Bloch and Y.-C. Hsieh, Phys. Rev. **96**, 382 (1954).

The neutron-oscillator generating function G_n has the same properties, mutatis mutandis.

Evidently if one expands G_p in a power series of the components (f_k', g_k', h_k') of the vectors $\mathbf{V}_k'$, the coefficient of any given combination of powers is an antisymmetric spin-dependent function made up of products of "individual-particle" Hermite functions (i.e., Hermite functions of the variables $a_p x_m'$, $a_p y_m'$, $a_p z_m'$) and the spin functions $s_k(m)$. Such functions suggest the nuclear states commonly postulated in connection with shell phenomena. Furthermore, if one equates this series to the power series for G_p in terms of the $u_\alpha, v_\alpha, w_\alpha$ (whose coefficients are antisymmetric eigenfunctions of the proton part of H_0 which involves normal modes 2 through Z), Eq. (9) enables one to substitute for each product of powers of the f_k', g_k', h_k' a unique sum of products of powers of the u_α, v_α, w_α, each term of which sum has the same total degree as the term in the $\mathbf{V}'$-series from which it arose. This procedure reveals that each coefficient in the $\mathbf{U}$-series is a linear combination of coefficients in the $\mathbf{V}'$-series belonging to the same total power, i.e., that each antisymmetric proton normal-mode eigenfunction is a superposition of antisymmetric combinations of "individual-particle" Hermite functions, with the same total quantum number.

However, the converse, that each individual-particle eigenfunction is a mixture of normal-mode eigenfunctions with the same total excitation (and hence an eigenfunction of the proton part of H_0), does not follow. Because the $\mathbf{V}_k'$ are not independent, it is not justified to equate coefficients of like powers of the f_k', g_k', h_k' in two equal series unless one has first eliminated one $\mathbf{V}'$ by means of Eq. (11), and no expression for one individual-particle eigenfunction in terms of normal-mode eigenfunctions can be obtained.

If one does eliminate one $\mathbf{V}'$ in the expression for G_p, e.g., sets $\mathbf{V}_1' = -\sum_{k=2}^{Z} \mathbf{V}_k'$, then each coefficient in the power series for G_p in terms of the remaining f_k', g_k', h_k' is an eigenfunction of the proton part of H_0. But now these coefficients are no longer readily expressible in terms of individual-particle functions, and are not obviously relevant to the single-body theory of shell structure.

If the linearly dependent vectors $\mathbf{V}_k'$ in G_p are replaced by a set of linearly independent vectors $\mathbf{V}_k$ with components f_k, g_k, h_k (not related to the $\mathbf{U}_\alpha$), G_p becomes a new function G_p' which generates individual-particle wave functions, not necessarily eigenfunctions of any part of H_0. In order to learn more about the properties of these wave functions, one can study the behavior of G_p' when operated on by the proton part of H_0 (with the constant D omitted for convenience). It turns out that

$$\sum_{\alpha=2}^{Z} (\pi_\alpha^2/2m + \beta_p \rho_\alpha^2/2) G_p' = \hbar(\beta_p/m)^{\frac{1}{2}} \Big[\sum_{k=1}^{Z} \mathbf{V}_k \cdot \boldsymbol{\partial}_k + 3(Z-1)/2 + 2\Big(\sum_{k=1}^{Z} \mathbf{V}_k \Big)^2 \Big/ Z \Big] G_p', \quad (12)$$

where β_p is the common value of $\beta_2 \cdots \beta_Z$, and $\boldsymbol{\partial}_k$ is the gradient operator with respect to the components of $\mathbf{V}_k$.

If a power series in the f_k, g_k, h_k is substituted for G_p' in Eq. (12), the operator on the left (the proton part of H_0) merely affects the coefficients of the various powers without acting on the f_k, g_k, h_k. The first sum on the right multiplies $\prod_{k=1}^{Z} f_k^{a_k} g_k^{b_k} h_k^{c_k}$ by the factor $\hbar(\beta_p/m)^{\frac{1}{2}}[\sum_{k=1}^{Z}(a_k+b_k+c_k)]$, without changing the power to which any of the f_k, g_k, h_k is raised. The last term on the right, however, converts each product of powers to a sum of such products, each with its total degree increased by two. If one equates coefficients of like powers in this equation, then, each wave function which G_p' generates is seen to be an eigenfunction of the proton part of H_0 with eigenvalue

$$E_p = \hbar(\beta_p/m)^{\frac{1}{2}}[\textstyle\sum_{k=1}^{Z}(a_k+b_k+c_k)+3(Z-1)/2],$$

except for the influence of the last term, which mixes in other wave functions with total quantum number reduced by two. Thus the wave functions generated by G_p' are *not* in general eigenfunctions of the proton part of H_0, and if combined to give such eigenfunctions are no longer independent-particle functions.

However, if one studies the derivatives of G_p' with respect to the f_k, g_k, h_k with all these quantities equal to zero, one finds that the Pauli exclusion principle makes all the generated wave functions vanish below a certain total quantum number K_p, which, for any given total z-component of proton spin, is identical with the K_p obtained by the normal-mode treatment or by the model in which the protons all move independently in a common oscillator well. For any given S_Z for the protons, any wave function generated by G_p' with total quantum number equal to K_p or K_p+1 *is* an eigenfunction of the proton part of H_0 with appropriate eigenvalue, because the functions introduced by the last term in Eq. (12), with quantum number reduced by two, all vanish. This conclusion applies only to the "ground" and the "first-excited" state functions with a given S_Z, not to the higher ones. A similar result obviously holds for the neutrons.

Therefore a product generating function $G_p'G_n'$ generates eigenfunctions of that part of H_0 which involves normal-mode oscillators 2 to Z and $Z+2$ to A, provided neither the proton function nor the neutron function has a total quantum number more than one above the lowest quantum number appropriate to the relevant value of S_Z. A complete eigenfunction of H_0 is then a free-space function of $\boldsymbol{\rho}_1$, times a Hermite function (of any order) of $\boldsymbol{\rho}_{Z+1}$, times a product of the neutron and proton eigenfunctions mentioned above. H_0 has other (highly excited) eigenfunctions which cannot be so expressed, but even some of those which are thus expressible with the aid of the single-particle functions generated by G_p' and G_n' correspond to quite high excitations of the nucleus, either because neutrons

or protons have large S_Z,[2] or because the "giant dipole" oscillator $(Z+1)$ is highly excited.

It was pointed out in I that the first excited state of the neutron-*vs*-proton or "giant dipole" oscillator $(Z+1)$ seems to be in the continuum, according to evidence connected with nuclear photodisintegration. It was remarked there, too, that the level spacings of the other normal-mode oscillators could not be much less than that of oscillator $(Z+1)$, on account of the approximate equality of *nn*, *np*, and *pp* forces, so that the "first" excited states of the nucleus according to H_0 are probably all in the continuum regardless of which type of oscillator is excited. This argument suggested that the bound excited states which are observed may result from perturbations of H_0—perhaps the cutting off of the forces at finite range, perhaps spin-orbit interactions—which partly resolve the degeneracy of the lowest levels of H_0. If this indication is correct, it would seem that[3] all the bound states of nuclei may arise by perturbations from single-particle states of the sort generated by $G_p'G_n'$ [except for the possible participation of oscillator $(Z+1)$]. It remains to be seen whether perturbation of H_0 can lead to surface waves of the type envisaged by Bohr and Mottelson.[4] Perturbation theory based on the functions generated by $G_p'G_n'$ is likely not to prove straightforward, inasmuch as no Hamiltonian has been found which has all these functions as eigenfunctions.[5]

Perhaps the simplest independent-particle harmonic-oscillator Hamiltonian is

$$H_s=\sum_{k=1}^{A} p_k^2/2m+k_p\sum_{k=1}^{Z} r_k^2/2+k_n\sum_{k=Z+1}^{A} r_k^2/2, \quad (13)$$

where the vectors $\mathbf{r}$ denote the positions of the protons (1 to Z) and the neutrons ($Z+1$ to A) with respect to an arbitrary fixed origin, the $\mathbf{p}_k$ are the conjugate momentum vectors, m is the nucleon mass, and k_p and k_n are constants. The antisymmetric eigenfunctions of H_s are generated by

$$G_s=\exp\Big[-\alpha_p^2\sum_{k=1}^{Z} r_k^2/2-\alpha_n^2\sum_{k=Z+1}^{A} r_k^2/2 - \sum_{j=1}^{A} V_j^2\Big]\Delta_{sp}\Delta_{sn}, \quad (14)$$

where $\alpha_p=(k_pm/\hbar^2)^{\frac{1}{4}}$, $\alpha_n=(k_nm/\hbar^2)^{\frac{1}{4}}$, the $\mathbf{V}_j$ are a set of A independent vectors whose components serve as the variables in the power-series expansion of G_s, and Δ_{sp} and Δ_{sn} are, respectively, a $Z\times Z$ and an $N\times N$ determinant whose kj elements are given by

$$\begin{aligned} &s_j(k)\exp(2\alpha_p\mathbf{r}_k\cdot\mathbf{V}_j),\ j,k=1,2,\cdots Z, \text{ for } \Delta_{sp},\\ &\text{and}\\ &s_j(k)\exp(2\alpha_n\mathbf{r}_k\cdot\mathbf{V}_j),\\ &\qquad j,k=Z+1,Z+2,\cdots A, \text{ for } \Delta_{sn}. \end{aligned} \quad (15)$$

The functions generated are antisymmetrized products of Hermite functions of the $\alpha_p\mathbf{r}_k$ and $\alpha_n\mathbf{r}_k$.

If one defines $\mathbf{R}$ to be the position vector of the nuclear center of mass ($\mathbf{R}=A^{-1}\varrho_1$), and $\mathbf{R}_{np}$ to be the position vector of the center of mass of the protons with respect to that of the neutrons [$\mathbf{R}_{np}=(A/NZ)^{\frac{1}{2}}\varrho_{Z+1}$], and defines the $\mathbf{r}_k'$ by Eq. (10) with a similar equation for the neutrons, one can express G_s as

$$\begin{aligned} G_s=G_p'G_n'\exp\Big[&-(Z\alpha_p^2+N\alpha_n^2)R^2/2\\ &-(NZ/A)(\alpha_p^2-\alpha_n^2)\mathbf{R}\cdot\mathbf{R}_{np}\\ &-(NZ/2A^2)(N\alpha_p^2+Z\alpha_n^2)R_{np}^2\\ &+2\mathbf{R}\cdot\Big(\alpha_p\sum_{j=1}^{Z}\mathbf{V}_j+\alpha_n\sum_{j=Z+1}^{A}\mathbf{V}_j\Big)\\ &+2\mathbf{R}_{np}\cdot\Big(N\alpha_p\sum_{j=1}^{Z}\mathbf{V}_j/A-Z\alpha_n\sum_{j=Z+1}^{A}\mathbf{V}_j/A\Big)\Big], \end{aligned} \quad (16)$$

where $G_p'G_n'$ is the product of individual-proton and individual-neutron generating functions discussed above.

The same substitution applied to H_s yields

$$\begin{aligned} H_s=&\sum_{k=1}^{A} p_k'^2/2m+k_p\sum_{k=1}^{Z} r_k'^2/2+k_n\sum_{k=Z+1}^{A} r_k'^2/2+P^2/2Am\\ &+(Zk_p+Nk_n)R^2/2+P_{np}^2/(2NZm/A)\\ &+(NZ/2A^2)(Nk_p+Zk_n)R_{np}^2\\ &+(NZ/A)(k_p-k_n)\mathbf{R}\cdot\mathbf{R}_{np}, \end{aligned} \quad (17)$$

where $\mathbf{P}$ is the total momentum of the nucleus, $\mathbf{P}_{np}$ is the momentum conjugate to $\mathbf{R}_{np}$, and the $\mathbf{p}_k'$ are the momenta of the individual particles, each with respect to the center of mass of the particles of its type.

This form for H_s shows that each of its eigenfunctions can be expressed in the form $f(\mathbf{r}')g(\mathbf{R}_{np},\mathbf{R})$. Because the $\mathbf{r}_{-k}'$ are not independent, the functions f cannot be

[2] The effect of spins on the quantum number of the lowest energy level and on the results discussed here can be expressed in terms of the total neutron and proton spin values S_n and S_p rather than the S_Z values, inasmuch as the quantum number of the lowest energy level cannot depend on the orientation of a spin vector, but only on its magnitude. Thus, whatever linear combination of (say) proton generating functions G_p or G_p' is needed to generate given states of definite S_p with a given Z-component, all wave functions generated by this linear combination must cancel each other out, up to the excitation K_p appropriate to $S_Z=S_p$. Such a sum of generating functions satisfies the same linear differential equation, such as Eq. (12), as do the separate terms in the sum.

[3] It can be shown that, for any given spin value for either neutrons or protons, the degeneracies of the ground level and the first excited level of particles of that type are the same according to the independent-particle model as they are according to the normal-mode model with ϱ_{Z+1} omitted.

[4] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

[5] See M. H. L. Pryce, Repts. Progr. in Phys. **17**, 1 (1954).

readily calculated. The form (16) for G_s, although it contains $G_p'G_n'$ as a factor, shows that the functions generated by $G_p'G_n'$ are *not* factors in the eigenfunctions of H_s, on account of the presence of the $\mathbf{V}_j$ multiplied by $\mathbf{R}$ and $\mathbf{R}_{np}$. This G_s also fails in general to generate functions $g(\mathbf{R}_{np},\mathbf{R})$ of the right type. However, the ground-state functions generated by G_s *are* of the form $f(\mathbf{r}')g(\mathbf{R}_{np},\mathbf{R})$—or the ground-state eigenfunctions of H_0 (with ϱ_1 omitted) and of H_s (with $\mathbf{R}$ omitted) are identical, subject to Eq. (20) below, and can be taken as (1) independent-particle Hermite functions referred to an arbitrary origin, or (2) independent-particle Hermite functions referred to the neutron and the proton centers of mass, or (3) normal-mode Hermite functions. The first-excited-state eigenfunctions of H_0 and of H_s are not simply related.

It is not difficult to show that

$$H_s-(Zk_p+Nk_n)R^2/2 - (NZ/A)(k_p-k_n)\mathbf{R}\cdot\mathbf{R}_{np}=H_0, \quad (18)$$

if

$$b_p=[(Z+2N)k_p-Nk_n]/A^2, \quad b_n=[(N+2Z)k_n-Zk_p]/A^2, \quad (19)$$

and

$$b_{np}=(b_p+b_n)/2. \quad (20)$$

Thus H_s differs from the particular H_0 to which (20) applies only through those terms which depend on $\mathbf{R}$, or if the independent-particle Hamiltonian H_s is rendered more acceptable through being made invariant under translation, the result is H_0, in which all forces are between nucleon pairs. No such simple relation seems to apply when Eq. (20) does not hold.

Equation (18) resembles a relation pointed out by Post[6] for the special case in which $k_p=k_n=k$, or $\alpha_p=\alpha_n=\alpha$ and $b_p=b_n=b_{np}=k/A$. He goes on to show that each eigenfunction of H_0 (with $\mathbf{R}$ omitted) is equal to an eigenfunction of H_s multiplied by $\exp[\alpha^2(\sum_1^A \mathbf{r}_i)^2/2A]$, but that not all eigenfunctions of H_s can be so used to give eigenfunctions of H_0—only all of those which, when multiplied by the exponential, are invariant under translation.

The eigenfunctions of H_s with this property can be further characterized by means of G_s. In the special case in which the α's are equal, Post's exponential factor simply removes from G_s the term R^2 in the exponent. The term containing $\mathbf{R}$ and the $\mathbf{V}_j$ remains, preventing the modified generating function—let us call it G_s'—from being translation-invariant. The way in which $\mathbf{R}$ enters G_s' indicates that only the ground-state functions have the desired invariance. The fact that G_s' satisfies the equation

$$H_0G_s'=\hbar(k/m)^{\frac{1}{2}}\Big[3(A-1)/2+\sum_{j=1}^{A}\mathbf{V}_j\cdot\boldsymbol{\partial}_j - (2\alpha/A^{\frac{1}{2}})\boldsymbol{\varrho}_1\cdot\sum_{j=1}^{A}\mathbf{V}_j\Big]G_s' \quad (21)$$

shows directly that only its ground-state functions for given S_p and S_n are eigenfunctions of H_0. However, the form of the last term in Eq. (21) leaves open the possibility that mixtures of the independent-particle functions generated by G_s', homogeneous in total quantum number, may be eigenfunctions of H_0. Such mixtures could not properly be called independent-particle functions.

A similar result can be obtained in the more general case in which $\alpha_p\neq\alpha_n$. The obvious generalization of Post's procedure removes from G_s the exponentials involving R^2 and $\mathbf{R}\cdot\mathbf{R}_{np}$, but still leaves in the $\mathbf{R}\cdot\mathbf{V}_j$. Again only the ground states of G_s' are eigenstates of H_0 as they stand; again mixtures of them may be.

If one forces G_s' to be translation-invariant by imposing the restrictions $\sum_1^Z \mathbf{V}_j=\sum_{Z+1}^A \mathbf{V}_j=0$, G_s no longer generates all the eigenfunctions of H_s, but only those mixtures of the original functions which satisfy Post's requirement. That is, G_s' now generates all the eigenfunctions of the proton and neutron parts[7] of H_0 (with given S_Z values for protons and for neutrons), multiplied by the ground-state function of ϱ_{Z+1}.

The relationship between the eigenfunctions of H_0 and those of H_s (with nucleon positions referred to an arbitrary origin) requires the condition (20), whereas no such restriction applied in the comparison of the H_0 eigenfunctions with those generated by $G_p'G_n'$ (with nucleon positions referred to the respective centers of mass). Furthermore, with given S_p and S_n, the latter independent-particle functions match the H_0 eigenfunctions for ground *and* first excited states; those eigenfunctions of H_s which are independent-particle functions match only for ground states. Therefore the $G_p'G_n'$ functions seem somewhat the better set of independent-particle functions to use in conjunction with H_0. It is not known whether similar results obtain in the case of more realistic interparticle forces.

Eden and Francis[8] in a recent paper give a general discussion of nuclear models. They remark that a model by which some correct predictions can be made does not for that reason give wave functions which closely approximate the true wave functions. Instead, they suggest, the wave functions and operators belonging to the model may be related to the true wave functions and operators by a transformation operator M, and the accuracy of the model's results relative to a given dynamical variable ω depends on whether ω commutes with M.

Ideas something like these—though not so generally formulated and completely worked out—motivated the present authors in their earlier work on normal modes. More recently the results here reported have suggested that single-body wave functions are, after all, approximations to the correct ones. However, H_0 itself is far from the correct Hamiltonian, so similarity to its

[6] H. R. Post, Proc. Phys. Soc. (London) **A66**, 649 (1953).

[7] See Eq. (11).

[8] R. J. Eden and N. C. Francis, Phys. Rev. **97**, 1366 (1955).

eigenfunctions is not an adequate criterion of correctness for the energy eigenfunctions of a nuclear model. If H_0 should be improved by perturbation theory, the lowest states would be modified by admixture of excited unperturbed states, of which the higher ones do not seem to be independent-particle states at all. As suggested by Brueckner, Eden, and Francis,[9] such admixture would probably be most obvious in high-energy phenomena of the type they discuss, in which the high-momentum part of the ground state would play the principal role.

The authors are very grateful to the National Science Foundation for supporting this work, and to Mr. W. T. Achor, Mr. D. R. Childs, and Mr. J. E. Turner for checking many of the calculations.

[9] Brueckner, Eden, and Francis, Phys. Rev. **98**, 1445 (1955).

PHYSICAL REVIEW VOLUME 101, NUMBER 1 JANUARY 1, 1956

Cross Section and Angular Distributions of the (d,p) and (d,n) Reactions in C^{12} from 1.8 to 6.1 Mev*

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(Received August 29, 1955)

The reaction $C^{12}(d,p)C^{13}$ has been studied from a deuteron bombarding energy of 1.8 to 6.1 Mev. Resonances were found at 2.47, 2.67, 2.99, 3.39, 4.00, 4.6, 4.8, 5.34, and 5.64 Mev. Angular distributions of protons leaving C^{13} in the ground state show a pronounced Butler peak at 25° over the entire deuteron energy range. The angular distributions can be explained by assuming small amplitudes for compound nucleus formation interfering with large stripping amplitudes. Angular distributions of the lower energy group of protons leaving C^{13} excited to 3.09 Mev show a pronounced Butler peak at 0° and an even smaller contribution of compound nucleus formation. The reaction $C^{12}(d,n)N^{13}$ was also studied, and showed similar resonances and angular distributions. An analysis is made of the phase difference between the resonant and nonresonant parts of the cross section for the (d,p) reaction near the resonance at 4.00 Mev.

INTRODUCTION

In the last few years a great many experiments have been carried out on nuclear reactions of the (d,p) and (d,n) type which have been explained so successfully by the stripping theory of Butler.[1] In the region of deuteron energies of from 6 to 10 Mev the stripping cross section in (d,n) and (d,p) reactions appears to be very nearly the total cross section for these reactions, although in some cases there seems to be a considerable contribution of compound nucleus formation in particles observed at large angles to the direction of the incident deuterons. The purpose of the present experiments was to investigate the relative importance of the stripping reaction and compound nucleus formation in the reactions $C^{12}(d,p)C^{13}$ and $C^{12}(d,n)N^{13}$. Experiments[2] on these two reactions at energies up to 3 Mev indicated pronounced resonances which have been interpreted to be due to a large compound nucleus cross section. The object of this experiment was to investigate the excitation curves and angular distributions of the protons and neutrons produced by deuterons with energies from 1.8 Mev up to 6 Mev, in order to cover the expected transition range of energies where compound nucleus formation would become less important and stripping would become dominant. The reactions in C^{12} and the Q-values for emission of protons and neutrons are: $C^{12}(d,p)C^{13}$, $Q=2.72$ Mev; $C^{12}(d,p)C^{13*}$, $Q=-0.37$ Mev; $C^{12}(d,n)N^{13}$, $Q=-0.28$ Mev; $C^{12}(d,n)N^{13*}$, $Q=-2.65$ Mev.

EXPERIMENTAL METHOD AND RESULTS

$C^{12}(d,p)C^{13}$.—A self-supported carbon foil[3] was bombarded with deuterons from the Rice Institute 6-Mev Van de Graaff accelerator. The foil had a thickness of 155 $\mu g/cm^2$ and was oriented at an angle of 45° to the deuteron beam. The calculated energy loss of the deuterons in the foil varies from 70 kev at 2.5 Mev to 35 kev at a bombarding energy of 6 Mev. The carbon foil was at the center of a scattering chamber 5 inches in diameter that had exit ports every 10°. These ports were covered with thin aluminum foils. The protons from the nuclear reactions passed through the windows and entered scintillation counters which consisted of thin CsI crystals mounted on DuMont 6291 photo-

* Supported in part by the U. S. Atomic Energy Commission; a preliminary report of these results was given by Bonner, Kraus, Eisinger, and Marion in Phys. Rev. **99**, 631(A) (1955).

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[1] S. T. Butler, Proc. Roy. Soc. (London) **A208**, 559 (1951).

[2] Bennett, Bonner, Hudspeth, Richards, and Watt, Phys. Rev. **59**, 781 (1941); Bailey, Freier, and Williams, Phys. Rev. **73**, 274 (1948); Bonner, Evans, Harris, and Phillips, Phys. Rev. **75**, 1401 (1949); G. C. Phillips, Phys. Rev. **80**, 164 (1950); Holmgren, Blair, Simmons, Stratton, and Stuart, Phys. Rev. **95**, 1544 (1954); Takemoto, Dazai, Chiba, Ito, Suganamata, and Watanabe, J. Phys. Soc., Japan **9**, 447 (1954).

[3] Made by the technique described by J. D. Seagrave, Phys. Rev. **85**, 197 (1952); and E. A. Milne, Phys. Rev. **93**, 762 (1954).