Dynamics of a Disordered Linear Chain

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By a disordered linear chain, we mean a chain of one-dimensional harmonic linear oscillators, each coupled to its nearest neighbors by harmonic forces, with the mass of each oscillator and the coupling parameters taken to be random variables with known distributions. The problem of calculating the distributions. The problem of calculating the distribution function of the frequencies of the normal modes of vibration of the chain in the limit as the chain becomes infinitely long was resolved by Dyson. In this paper, we present a simple algebraic proof of the essential limit relation in Dyson's paper.

I. INTRODUCTION

DYSON[1] considered the problem of determining the distribution function of the characteristic frequencies of a chain of N masses, each coupled to its nearest neighbors by elastic springs, in the case where the masses and coupling forces are random variables.

After some elementary transformations, the problem reduces to determining the distribution of the characteristic roots of the Hermitian matrix

$$H_N=\begin{bmatrix} 0 & i\lambda_1 & 0 & 0 & \cdots & \\ -i\lambda_1 & 0 & i\lambda_2 & 0 & \cdots & \\ 0 & -i\lambda_2 & 0 & i\lambda_3 & \cdots & \\ & \cdots & & -i\lambda_{N-2} & 0 & i\lambda_{N-1} \\ & & & & -i\lambda_{N-1} & 0 \end{bmatrix},$$

where the λ_k are real random variables.

The essential step in the derivation of the limiting distribution is the proof of the relation

$$\lim_{N\to\infty}\frac{1}{N}\log|H_N+\lambda I|=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\log[\lambda+z_n(\lambda)], \quad (1)$$

where $z_n(\lambda)$ represents the infinite continued fraction

$$z_n(\lambda)=\lambda_n^2/(\lambda+\lambda_{n+1}^2/(\lambda+\cdots. \quad (2)$$

This result was obtained by Dyson using recurrence relations for trace H_N^{2k}, for $k=1, 2, \cdots$, obtained from combinatorial arguments. Here we shall present a simple algebraic proof.

II. PROOF OF LIMIT RELATION

Consider the determinant

$$H_N(\lambda)=\begin{vmatrix} \lambda & i\lambda_1 & 0 & 0 & \cdots & \\ -i\lambda_1 & \lambda & i\lambda_2 & 0 & \cdots & \\ & & \ddots & & & \\ \cdots & & & -i\lambda_{N-2} & \lambda & i\lambda_{N-1} \\ \cdots & & & 0 & -i\lambda_{N-1} & \lambda \end{vmatrix} =H_N(\lambda;\lambda_1,\lambda_2,\cdots,\lambda_{N-1}). \quad (1)$$

Expanding in terms of the elements of the first row, we obtain the recurrence relation

$$H_N(\lambda;\lambda_1,\lambda_2,\cdots,\lambda_{N-1})=\lambda H_{N-1}(\lambda;\lambda_2,\lambda_3,\cdots,\lambda_{N-1}) +\lambda_1^2 H_{N-2}(\lambda;\lambda_3,\lambda_4,\cdots,\lambda_{N-1}) \quad (2)$$

for $N\geq 3$, with $H_1(\lambda)=\lambda$, $H_2(\lambda;\lambda_2)=\lambda^2+\lambda_1^2$. From (2), we obtain

$$\frac{H_N(\lambda;\lambda_1,\cdots,\lambda_{N-1})}{H_{N-1}(\lambda;\lambda_2,\cdots,\lambda_{N-1})}=\lambda+\lambda_1^2/(H_{N-1}/H_{N-2}) =\lambda+\cfrac{\lambda_1^2}{\lambda+\cfrac{\lambda_2^2}{\ddots\,\cfrac{\lambda_{N-1}^2}{\lambda}}}, \quad (3)$$

and thus

$$\lim_{N\to\infty}\frac{H_N(\lambda;\lambda_1,\cdots,\lambda_{N-1})}{H_{N-1}(\lambda;\lambda_2,\cdots,\lambda_{N-1})} =\lambda+\lambda_1^2/(\lambda+\lambda_2^2/\lambda+\cdots). \quad (4)$$

To obtain the relation given in (1) of I, let us write

$$H_N(\lambda;\lambda_1,\lambda_2,\cdots,\lambda_{N-1})=\frac{H_N(\lambda;\lambda_1,\lambda_2,\cdots,\lambda_{N-1}}{H_{N-1}(\lambda;\lambda_2,\lambda_3,\cdots,\lambda_{N-1})} \times\frac{H_{N-1}(\lambda;\lambda_2,\lambda_3,\cdots,\lambda_{N-1})}{H_{N-2}(\lambda;\lambda_3,\lambda_4,\cdots,\lambda_{N-1})}\times\cdots\times\frac{H_2(\lambda;\lambda_1)}{H_1(\lambda)}\lambda.$$

Taking logarithms and replacing each ratio H_k/H_{k-1} by its equivalent in (3), we obtain the desired relation upon letting $N\to\infty$.

[1] F. J. Dyson, Phys. Rev. **92**, 1331 (1953).