The $\bar{\nu}$ values for the various nuclides calculated from the relative $\bar{\nu}$'s in Table I are based on values for Cf^{252} and Cm^{244} of 3.53 ± 0.15 and 2.60 ± 0.12, respectively.[2–3] All these values are on the average about 7% lower than those determined by Hicks *et al.*,[4,5] who used the $\bar{\nu}$ value[6] of Pu^{240} to determine the counting efficiency of their neutron detector. This discrepancy arises from the difference in neutron counting standardization, and since Pu^{240} was determined relative to the $\bar{\nu}$ for the thermal fission of U^{235}, the discrepancy indicates a difference in the Bureau of Standards neutron source measurement and the U^{235} $\bar{\nu}$ measurement.

Figure 1 is a graphic presentation of these data, and it is interesting to note the regular variation of $\bar{\nu}$ with mass number.

[4] Hicks, Ise, and Pyle, Phys. Rev. **98**, 1521 (1955).
[5] Hicks, Ise, and Pyle (to be published).
[6] Diven, Taschek, Terrel, and Martin (to be published).

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Neutron and Proton Densities in Nuclei*

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A semiempirical investigation of neutron and proton densities in nuclei is made. Experimental values are assumed for the nuclear radius, binding energy, surface energy, surface thickness, and symmetry energy. It is found that the neutron and proton densities extend to approximately the same radius; these results do not depend sensitively on the input data. The nuclear potential extends $\sim 0.7\times 10^{-13}$ cm further than the material radius. An estimate of $\lesssim 100A$ Mev is given for nuclear compressibility.

I. INTRODUCTION

THE excellent experiments on electron scattering[1] and on mu-meson spectroscopy[2] indicate for the charge distribution a surface thickness of $2.2-2.5\times 10^{-13}$ cm and an "equivalent" radius of $1.2A^{1/3}\times 10^{-13}$ cm. (An "equivalent" radius is the radius of a uniform distribution which leads to the same energy for the $2p\rightarrow 1s$ transition in mu-mesonic atoms. The point where the charge density falls to half its central density is more like $1.1A^{1/3}\times 10^{-13}$ cm.) Other experiments[3] which depend upon the nuclear charge distribution do not appear to be in disagreement with these results. Experiments which measure the nuclear potential,[3] however, quite generally lead to larger values for the radius. Attempts to measure the neutron distribution[3] (as opposed to the charge distribution) are hopeful, but as yet inconclusive.

In a previous paper,[4] density distributions in nuclei were calculated neglecting the Coulomb potential. Assuming experimental values for the nuclear radius, binding energy, surface energy, and surface thickness, the calculations yielded the following conclusions: (1) The nuclear potential (at half-maximum) extends $\sim 0.7\times 10^{-13}$ cm beyond the nuclear density. (2) The nuclear compressibility is estimated to be $\lesssim 100A$ Mev.

The present investigation is designed to examine effects arising from the Coulomb potential. The primary effect of the Coulomb potential is to increase the number of neutrons relative to protons, through beta decay. The variation in the potential through the nucleus is also of consequence in *tending* to increase the relative number of protons at the surface.

Johnson and Teller[5] have proposed that the distribution of protons in the nucleus may lie within the neutron distribution by as much as 1/3 to 1/2 of the nuclear surface thickness. Their arguments are based on two consequences of the Coulomb potential: (1) Owing to the larger number of neutrons than protons, the neutrons have, on the average, greater kinetic energy and will extend further than the protons. (2) The Coulomb potential forms a barrier which inhibits penetration of the proton wave functions into the forbidden region. The effectiveness of these arguments depends upon the approximate equality of the nuclear potential for both neutrons and protons.

There are also factors which tend to counter the Johnson-Teller effect. The nuclear symmetry energy tends to resist separation of neutrons and protons, and the decrease in the Coulomb potential toward the edge of the nucleus tends to move protons closer to the surface. The present investigation indicates that the

* Work supported by the U. S. Atomic Energy Commission.
[1] R. Hofstadter *et al.*, Phys. Rev. **95**, 512 (1954). For analyses of the data, see Yennie, Ravenhall, and Wilson, Phys. Rev. **95**, 500 (1954); D. G. Ravenhall and D. R. Yennie, Phys. Rev. **76**, 239 (1954). See also reference 3.
[2] V. F. Fitch and J. Rainwater, Phys. Rev. **92**, 801 (1953). For analyses of the data, see L. N. Cooper and E. M. Henley, Phys. Rev. **92**, 789; D. L. Hill and K. W. Ford, Phys. Rev. **94**, 1617 (1954). See also reference 3.
[3] An excellent summary and analysis of the experiments on nuclear density and potential distribution is given by K. W. Ford and D. L. Hill, Ann. Rev. Nuc. Sci. **5** (1955). Further references to original literature will be found there.
[4] R. A. Berg and L. Wilets, Phys. Rev. **101**, 201 (1956), henceforth referred to as I.
[5] M. H. Johnson and E. Teller, Phys. Rev. **93**, 357 (1954).

net result is that there exists no appreciable separation of neutrons and protons.

The technique used here for determining nuclear densities is semiemperical and statistical in nature. No basic assumptions about the nature of nuclear forces are made, but extensive use is made of the experimental properties of nuclei. The method is a generalization of that used in I, where further details are to be found. Other papers on the statistical method for nuclei have also been published.[6]

II. FORMULATION

It is assumed that the expression for the energy density of the system can be written in the following form[7]:

$$\mathcal{E}(\rho_n,\rho_p)+F(\rho_n)(\nabla\rho_n)^2+F(\rho_p)(\nabla\rho_p)^2. \qquad (1)$$

The first term represents the energy density of a uniform medium. It includes both kinetic and interaction energies, which are not immediately separated, and could include many-body forces, particle correlations, etc. The latter terms represent corrections, under conditions of varying density, to (1) the kinetic energy and (2) the interaction energy, due to the finite range of nuclear forces.

The correction to the kinetic energy can be written in the form

$$\xi(\hbar^2/8M)\rho^{-1}(\nabla\rho)^2. \qquad (2)$$

This "inhomogeneity term" was first proposed by Weizsäcker[8] (with $\xi=1$). Although Weizsäcker's original arguments are not valid,[9] Eq. (2) does have the proper form, and it has been shown for certain nuclear-type potentials[9] that the exact quantum-mechanical densities and energies can be rather well reproduced by $1/8<\xi<1/2$, the precise value of ξ depending on the shape of potential. For convenience we express the finite-range correction in the same form as Eq. (2); this is not critical since the range correction appears to be rather smaller than the kinetic energy correction.

The energy of the nucleus can then be written

$$E=\int\left\{\mathcal{E}(\rho_n,\rho_p)+\zeta\frac{\hbar^2}{8M}\left[\frac{(\nabla\rho_n)^2}{\rho_n}+\frac{(\nabla\rho_p)^2}{\rho_p}\right]\right\}dv, \qquad (3)$$

where ζ (which we expect to be less than unity) includes both the kinetic-energy and finite-range corrections. We seek the functions ρ_n and ρ_p which make the energy a minimum, subject to condition that

$$A=\int(\rho_n+\rho_p)dv, \qquad (4)$$

be a constant. This variational problem leads to the coupled nonlinear differential equations:

$$-\zeta\frac{\hbar^2}{2M}\nabla^2u_\mu+\left(\frac{\partial\mathcal{E}}{\partial\rho_\mu}\right)u_\mu=E_0u_\mu, \qquad (5)$$

where $u_\mu=\sqrt{\rho_\mu}$. The index μ indicates either neutrons (n) or protons (p). E_0 is a Lagrangian multiplier which has the physical meaning of being the binding energy of the last particle.

To fit the experimental data, we have available the constant ζ and the function $\mathcal{E}(\rho_n,\rho_p)$. We write $\mathcal{E}$ in the form:

$$\mathcal{E}(\rho_n,\rho_p)=f(\rho)+k(\rho_n-\rho_p)^2+V_c\rho_p, \qquad (6)$$

with $\rho=\rho_n+\rho_p$. The function $f(\rho)/\rho$ (energy per nucleon) must have a minimum at the observed nuclear density and at a value corresponding to the coefficient of the term linear in A in the semiempirical mass formula[10]; the curvature at the minimum is related to compressibility (see I). The second term on the right-hand side of Eq. (6) represents the symmetry energy; the constant k is determined[11] from the observed ratios of N to Z (or from the semiempirical mass formula). V_c is the Coulomb potential. For small values of the density, the differential equation is governed more by the Laplacian term than by the $\partial\mathcal{E}/\partial\rho_\mu$ term. It is indeed found that the solutions are insensitive to details of $f(\rho)$ beyond those already mentioned.

III. CONSTANT COULOMB POTENTIAL

In order to gain insight into the relative importance of the difference in numbers of neutrons and protons on the one hand, and the variation in Coulomb potential through the nucleus on the other, the Coulomb potential was first set equal to a constant in Eq. (6). This problem is further simplified by considering a semi-infinite nucleus with a plane surface. Such an approximation is justified to the extent that the surface thickness[3] (2.5×10^{-13} cm) is small compared with the radius ($\sim7.5\times10^{-13}$ cm). Equation (6) then becomes an ordinary nonlinear differential equation in the distance variable x,

$$-\zeta\frac{\hbar^2}{2M}u_\mu''+\left(\frac{\partial\mathcal{E}}{\partial\rho_\mu}\right)u_\mu=E_0u_\mu, \qquad (7)$$

subject to the boundary conditions

$$u_\mu,\ u_\mu',\ u_\mu''\to0 \quad \text{as} \quad x\to\infty, \qquad (8a)$$

$$u_\mu\to u_{\mu,0} \quad \text{and} \quad u_\mu',\ u_\mu''\to0 \quad \text{as} \quad x\to-\infty, \qquad (8b)$$

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[6] W. J. Swiatecki, Proc. Phys. Soc. (London) **A63**, 1208 (1950). P. Gombas, Acta Phys. Hung. **1**, 239; **2**, 224 (1952); **3**, 105, 127 (1953). E. Feenberg, Phys. Rev. **59**, 593 (1941).

[7] The quantity $\mathcal{E}(\rho)$ used here is the same as $\mathcal{E}(\rho)+\kappa_F\rho^{5/3}$ used in I.

[8] C. F. von Weizsäcker, Z. Physik **96**, 431 (1935).

[9] R. A. Berg and L. Wilets, Proc. Phys. Soc., (London) **A68**, 229 (1955).

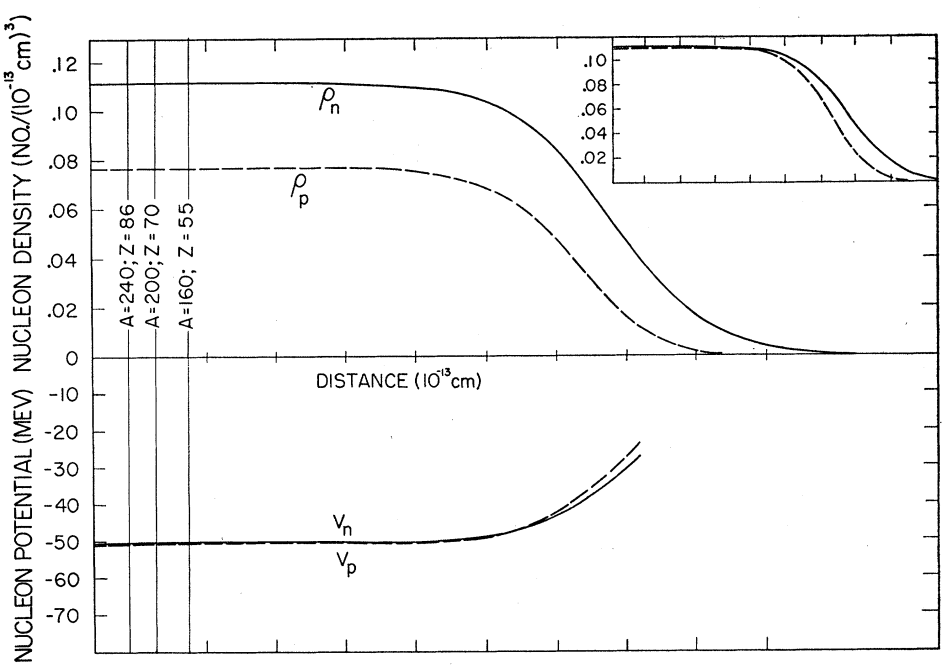
[10] A. E. S. Green, Phys. Rev. **95**, 1006 (1954).

[11] $k(\rho_n-\rho_p)^2$ is not the most general form for the symmetry energy, although this is suggested by the semiempirical mass formula for $\rho_n\approx\rho_{n,0}$ and $\rho_p\approx\rho_{p,0}$. In the surface region the neutron-proton separation is not so great and so the sensitivity of the results on the form of the symmetry energy term is reduced. The primary effect of variation in the form of this term is on the surface energy which is reflected in the estimates of nuclear compressibility (Sec. III. B).

FIG. 1. Semi-infinite nucleus with constant Coulomb potential $V_c=17.34$ Mev. The energy density function used in this example is given by

$$\mathcal{E}=-91.8\rho^{5/3}+134.3\rho^{7/3}+183(\rho_n-\rho_p)^2+V_c\rho_p,$$

in units of 10^{39} Mev cm^{-3}, and ζ was chosen to be 0.7. The nuclear compressibility $K_V=125A$ Mev. Since the unit of distance varies as $\zeta^{1/2}$, the length scale may be altered by changing ζ. The density scale may also be changed by readjusting the constants so as to leave $\mathcal{E}/\rho$ invariant. The inset shows the neutron density (solid line) and proton densities (dashed line) separately scaled to the same asymptotic values.



where the prime denotes derivative. This gives the integrability conditions:

$$\left[\frac{\partial}{\partial\rho_\mu}\left(\frac{\mathcal{E}}{\rho}\right)\right]_{\rho_{n,0},\rho_{p,0}}=0, \tag{9a}$$

$$\left[\frac{\partial\mathcal{E}}{\partial\rho_\mu}\right]_{\rho_{n,0},\rho_{p,0}}=E'. \tag{9b}$$

The solution requires an eigenvalue search with, say, one boundary condition (for some finite x) acting as an eigenvalue. The constant ζ plays the role of a scale factor in Eq. (7). Both the unit of length and the surface energy [I, Eq. (14)] vary as $\zeta^{1/2}$. In the solutions described below, ζ has been adjusted so that the proton surface thickness is 2.5×10^{-13} cm.

A. Density Distributions

The shape and relative separation of the neutron and proton density distributions are found to be quite insensitive to the nuclear compressibility. In a wide variety of cases calculated, the mean proton radius was found to lie inside the neutron radius by a distance only 20% of the proton surface thickness. Figure 1 gives a typical result; other cases calculated are practically indistinguishable.

B. Surface Energy and Compressibility

The surface energy depends sensitively on the nuclear compressibility and thus provides a means of determining this important but elusive quantity. The surface energy also depends upon the neutron-proton separation at the surface; for this reason, discussion of these quantities will be deferred until Sec. IV.

C. Nuclear Potential

In order to discuss the nuclear potential, it is necessary to make further assumptions about the nature of nuclear interactions. The assumptions which will be made now are only for the purpose of obtaining the potential, and are not used in obtaining the density distributions or the surface energy.

Brueckner[12] has shown that a nucleon within a nucleus can be described by the equations of motion of an independent particle if the nucleonic mass M is replaced by an effective mass $M^*\approx0.6M$. The introduction of an effective mass accounts for the velocity dependence of the interaction which arises from interparticle correlations. (Brueckner assumes velocity-independent two-body forces.)

To the extent that the Thomas-Fermi approximation is valid, the wave function of the "last" particle is that of one at the top of a Fermi sea of particles of mass M^*. The "effective kinetic energy" of such a particle is given by $2(3\pi^2)^{2/3}(\hbar^2/M^*)\rho_\mu^{2/3}$, while the total energy of the particle is $\partial\mathcal{E}/\partial\rho_\mu$. Neglecting the Coulomb potential, this permits the identification of the nuclear potential experienced by the last neutron or proton as

$$V_\mu^*=\frac{\partial\mathcal{E}}{\partial\rho_\mu}-2(3\pi^2)^{2/3}(\hbar^2/M^*)\varphi_\mu^{2/3}. \tag{10}$$

[12] K. A. Brueckner, Phys. Rve. 97, 1353 (1955). The introduction of an effective mass in the inhomogeneity correction term, which one might also consider, would result merely in a redefinition of ζ.

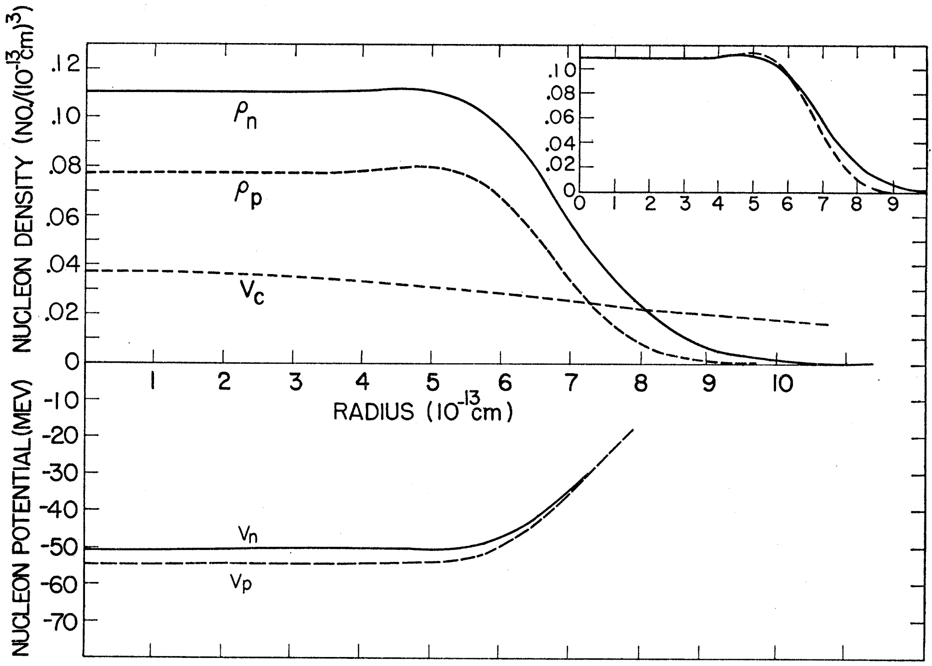


FIG. 2. Spherical nucleus with "realistic" Coulomb potential. $A=225$, $Z=93$. The energy density function used in this example (differing somewhat from the example in Fig. 1) is given by $\mathcal{E}=-67.4\rho^{5/3}+72.5\rho^{7/3}+170(\rho_n-\rho_p)^2+V_c\rho_p$, in units of 10^{39} Mev cm^{-3}, with V_c given by (12) for $R=7.11\times10^{-13}$ cm and $Z=82$. ζ was chosen to be 0.7. The nuclear compressibility is $K_V=125A$ Mev. The scale of length is no longer free because of the spherical geometry and the Coulomb energy. The surface thickness is reduced in this example (2.15×10^{-13} cm) relative to Fig. 1 (2.5×10^{-13} cm) by the Coulomb potential. It still possible to scale the density by readjusting the constants so as to leave $\mathcal{E}/\rho$ invariant. Thus this is also a solution for $A=198$, $Z=82$. The inset shows the neutron density (solid line) and proton density (dashed line) separately scaled to the same central values.

In the analysis of scattering experiments, the true nucleonic mass M is generally used. Schrödinger's equation is invariant with respect to changes in mass so long as the quantity $M(V-E)$ is also invariant. For comparison with current calculations, the quantity

$$V_\mu=(M^*/M)V_\mu^*, \tag{11}$$

is plotted in Figs. 1 and 2 (i.e., zero-energy scattering potentials). The potential is not drawn for low densities, since the approximations involved in Eqs. (6) and (10) may not be valid there. The depth of the potentials are sensitive to the assumed nuclear density.

The potentials for both neutrons and protons are seen to be quite similar in depth and extent. This would not be the case for a different value of M^*. In fact, for $M^*=M$, the proton potential would lie about 10 Mev lower than the neutron potential. In the example discussed in Sec. IV (Fig. 2) the proton potential lies about 4 Mev deeper than the neutron potential even with $M^*=0.6M$.

In the analysis of proton scattering experiments, Melkanoff, Moszkowski, Nodvik, and Saxon[13] report that the energy dependence of the real part of the potential is consistent with an effective mass about half the nucleonic mass. Their extrapolated zero-energy potential seems to lie significantly deeper than the zero-energy potential obtained by Feshbach, Porter, and Weisskopf[14] for neutrons. One possible cause for the discrepancy may lie in the difference in the shapes of the potential wells used in the analysis of the neutron and proton data. It should be noted, however, that the discrepancy is decreased appreciably if the zero-energy neutron potential is compared with the proton potential at an energy corresponding to the Coulomb potential inside the nucleus. At 17 Mev, for example, they give for the proton potential 47 Mev, or only 5 Mev deeper than the zero-energy neutron potential.

The potentials are also seen to extend about 0.7×10^{-13} cm further than the densities (Fig. 1). This can be understood as follows (see I): Near maximum density, the potential falls off less rapidly than the density owing to saturation of nuclear forces; at intermediate densities, the Thomas-Fermi relation $V\sim\rho^{2/3}$ again leads to a less rapid falloff of the potential relative to the density.

While in the right direction, the distance 0.7×10^{-13} cm may not be alone sufficient to account for the greater radii observed for nuclear potential, although it appears to be a substantial contribution. The potentials given here neglect the range of nuclear forces and polarization of the nucleus by the scattered particle.[15]

This treatment differs from that given in I, where an effective mass was not used.

IV. "REALISTIC" COULOMB POTENTIAL, SPHERICAL NUCLEI

For V_c we choose the Coulomb potential of a uniform charge distribution (it is not important to make the potential self-consistent with the resultant charge

[13] Melkanoff, Moszkowski, Nodvik, and Saxon, Phys. Rev. **101**, 507 (1956).

[14] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).

[15] See S. D. Drell, Phys. Rev. **100**, 97 (1955).

distribution):

$$V_c = Ze^2 \times \begin{cases} \frac{1}{2}[3-(r^2/R^2)], & 0<r<R \\ 1/r, & R<r. \end{cases} \tag{12}$$

This introduces an absolute scale of length into the problem, and so ζ was adjusted according to the value obtained in the case V_c = constant.

The differential equations for the case of spherical symmetry are given by

$$-\zeta\frac{\hbar^2}{2M}\left(u_\mu''+\frac{2}{r}u_\mu'\right)+\frac{\partial\mathcal{E}}{\partial\rho_\mu}u_\mu=E_0u_\mu. \tag{13}$$

The boundary conditions as $r\to\infty$ are the same as those of Eq. (8a) (for $x\to\infty$), but for $r=0$ the requirement is $u'(0)=0$. The Lagrangian multiplier E_0 cannot be fixed beforehand, but must be varied as an eigenvalue. The solution thus requires a double eigenvalue search. An example is shown in Fig. 2 with the parameters given in the caption. The results on the density and potential distribution appear to be insensitive to the details of $\mathcal{E}$—that is, various values of the compressibility lead to substantially indistinguishable results.

A. Density Distributions

The variation in Coulomb potential through the nucleus tends to push protons toward the surface of the nucleus. The actual rise in proton density is slight, in general agreement with the analysis of Ford and Hill,[3] who find that the electron scattering data is best fitted, within experimental error, with no rise. The mean proton radius is seen to extend about as far as the mean neutron radius, thus effectively canceling the rather small Johnson-Teller[5] effect discussed in Sec. III. The neutrons do exhibit a longer tail, however.

B. Surface Energy Compressibility

In paper I, where the densities of neutrons and protons were taken as equal, the surface energy was found to arise from equal contributions of (i) the inhomogeneity correction term $(\zeta\hbar^2/8M)\int\rho^{-1}(\rho')^2dv$, and (ii) the loss of binding (interaction) energy of the nucleons in the surface region. When the neutron and proton distributions are unequal, there arises a contribution to the surface energy from the integral of $k(\rho_n-\rho_p)^2$ in the surface region. Comparison of the surface energy obtained in the present case with that obtained for equal neutron and proton distributions, indicate that the symmetry energy increases the surface energy by about 40%.

The free parameter which is available for fitting the surface energy is the nuclear compressibility. The surface energy is an increasing function of the compressibility parameter (I.17):

$$K_V=\left[\rho^{-2/3}\frac{\partial^2\mathcal{E}}{\partial(\rho^{-1/3})^2}\right]_{\rho 0}A. \tag{14}$$

The example which is illustrated in Fig. 2 leads to a value of 1.4×10^{26} Mev cm^{-2} for the surface energy compared with 1×10^{26} Mev cm^{-2} from the semiempirical mass formula,[10] if one uses $1.1A^{1/3}\times10^{-13}$ cm for the nuclear radius rather than Green's[10] value $1.2A^{1/3}\times10^{-13}$ cm, which was used in I. The value of K_V in the example is 125 Mev, and this is clearly too large. From the dependence of the surface energy on K_V deduced from the examples in I, an estimate for the compressibility of $\gtrsim 100A$ Mev seems reasonable. This is to be compared with Brueckner's theoretical value of $\sim 67A$ Mev or the empirical value obtained from isotope shifts of $\sim 50A$ Mev. These are not serious discrepancies considering the difficulty in isolating the effect experimentally, or the general sensitivity of the effect to the theoretical assumptions. Furthermore, the surface energy and compressibility probably vary rather widely with mass number, reaching maximum values near closed shells. The surface energy given by the semiempirical mass formula is only a particular average.

C. Nuclear Potential

The nuclear potential shows the same characteristics discussed in Sec. III.

V. CONCLUSIONS

The neutron and proton densities are found to extend to approximately the same mean radii. There is a slight rise in the proton density at the surface, while the neutron distribution has a longer tail and surface thickness. The nuclear potential for protons and neutrons extend to the same radius, which is about 0.7×10^{-13} cm further than the density distributions. These results appear to be insensitive to be insitive to details of the function $\mathcal{E}(\rho_n,\rho_p)$, apart from the position and value of the minimum $\mathcal{E}/\rho$ which are taken empirically from the observed values of nuclear density and binding energies. The interpretation of the nuclear potential does require the separation of $\mathcal{E}$ into kinetic and potential terms; this procedure is somewhat arbitrary, so that the potentials deduced are somewhat less plausible than the densities.

An estimate of $\gtrsim 100A$ Mev is given for the nuclear compressibility.

VI. ACKNOWLEDGMENTS

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