Nucleon-Core Polarization in the Calculation of the Strength Function*

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The model of Lane, Thomas, and Wigner, for the excited states of nuclei, is generalized by the introduction of a variable amount of nucleon-core polarization into the states of the independent-particle model. The mixing of each of the polarized states, by configuration interaction, into a small neighborhood near the initial energy of the state forms the basis of our model. The energy moments of the strength function are calculated. The second moment of the strength function, which may be roughly equal to the square of the full width at half-maximum of the strength function, is calculated approximately using a two-body Yukawa potential without exchange. By the choice of a suitable amount of core polarization the large value obtained by Lane, Thomas, and Wigner for the second moment [(22 Mev)²] is reduced by a factor of 23 to the reasonably small value of (4.8 Mev)².

1. INTRODUCTION

IN the recent work of Lane, Thomas, and Wigner[1] the behavior of the excited states of nuclei was considered to be intermediate between the behavior of the states of the independent-particle model and the states of the uniform model.[2] In their model the single-particle states of the independent-particle model were mixed, by configuration interaction, into the many other states of the independent-particle model. The configuration interaction is due to the difference between the actual nucleon-core potential, V, and the average potential, $\bar{V}$, used in the definition of the shell model. The basic assumption of the aforementioned model was that the mixing of the single-particle states occurred, principally, into a relatively small neighborhood near the initial energy of these states. A direct consequence of this assumption is a resonance-like behavior of the average reduced widths, $\langle\gamma_{\lambda c}^2\rangle$: that is, $\langle\gamma_{\lambda c}^2\rangle$ has maxima near the positions of the single-particle levels of the independent-particle model. The giant resonances of the average total neutron cross sections[3] can be described in terms of the average behavior of the reduced widths.

The average total neutron cross sections had been described with considerable success by the cloudy crystal ball model.[4] This model assumed that the average neutron cross sections could be described in terms of the scattering of neutrons by a complex potential well, $V_0(1+i\zeta)$. The success of the cloudy crystal ball model in describing the average neutron cross sections can be understood in terms of the average behavior of the excited states of nuclei. Thomas[5] showed that the average total cross sections obtained with the model of Lane, Thomas, and Wigner are identical to those of the cloudy crystal ball model if the strength function, $s(E_\lambda)$, (the average reduced width divided by the average level spacing) is assumed to be

$$s(E_\lambda)=\sum_p s^{(p)}(E_\lambda)=\sum_p \frac{\hbar^2 R_p(a)}{\pi m\zeta V_0}\,\frac{1}{1+[(E-\mathcal{E}_p)/\tfrac{1}{2}\zeta V_0]^2}, \quad (1)$$

where $R_p(a)$ is the pth radial wave function in the real part of the complex potential well, evaluated at the nuclear radius a, and $\mathcal{E}_p$ is the position of the pth single-particle level in V_0. The partial strength function $s^{(p)}$ is the contribution of the pth single-particle level to the strength function. The strength function refers to a given orbital angular momentum of the channel under consideration (a nucleon and the corresponding target nucleus in its ground state).

Lane, Thomas, and Wigner assumed that the square root of the second moment of the partial strength function of their model was approximately equal to the full width at half-maximum of this function. They calculated the second moment to be the expectation value, taken in the ground-state channel, of the square of the configuration mixing potential, $V-\bar{V}$. This expectation value was evaluated by them using a sum of two-body Yukawa potentials, without exchange, for V. The result they obtained. $M_2\sim(22\text{ Mev})^2$, was much too large. If the square root of the second moment is approximately equal to the full width at half-maximum of the partial strength function, their result implied that the various maxima of the strength function overlapped hopelessly. (The spacing of the single-particle levels in the shell model is of the order of 10 Mev.)

If the model of Lane, Thomas, and Wigner is generalized by including nucleon-core polarization in the initial shell-model wave functions, the value of the second moment may be reduced. Such a calculation is carried out in the succeeding sections of this paper. The nucleon-core polarization is brought about by introducing an adjustable amount of the actual nucleon-

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[1] Lane, Thomas, and Wigner, Phys. Rev. **98**, 693 (1955).

[2] A similar picture was proposed by J. M. C. Scott, Phil. Mag. **45**, 1322 (1954), and E. P. Wigner, Science **120**, 790 (1954).

[3] H. H. Barschall, Phys. Rev. **86**, 431 (1952), and subsequent papers.

[4] Feshbach, Porter, and Weisskopf, Phys. Rev. **96**, 448 (1954).

[5] R. G. Thomas, Phys. Rev. **97**, 224 (1955).

core potential into the definition of the core states.[6] The resulting first-order states are a closer approximation to the actual states, so that one would expect the configuration interaction to be smaller. If the square root of the second moment is approximately equal to the full width at half-maximum of the partial strength function, then the second moment calculated using the polarized single-particle wave functions should, in general, be smaller than the second moment calculated by Lane, Thomas, and Wigner.

The detailed relation of the second moment to the width of the partial strength function, $s^{(p)}$, depends on the form of $s^{(p)}$. Let us assume that $s^{(p)}$ may be written as

$$s^{(p)}=\frac{c}{1+\alpha_1(E-\mathcal{E}_p)^2+\alpha_2^2(E-\mathcal{E}_p)^4+\cdots}, \quad (2)$$

where c, α_1, α_2, etc. are constants. Then the square root of the second moment of $s^{(p)}$ will be approximately equal to its full width at half-maximum unless the constants α_n satisfy the following inequality:

$$\alpha_1 \gg \alpha_2+\alpha_3+\alpha_4+\cdots. \quad (3)$$

If this inequality is satisfied the asymptotic behavior of the partial strength function, at large energies, may be proportional to $1/E_\lambda^2$. Then the second moment becomes, principally, a measure of how far the $1/E_\lambda^2$ behavior of the partial strength function extends as we increase E_λ. Thus the partial strength functions (1), corresponding to the cloudy crystal ball model, have infinite second moments. As far as our model is concerned, there is, at present, no reason why the strength functions should have the particular form corresponding to the cloudy crystal ball model.[7]

2. CORE POLARIZATION IN THE MODEL OF LANE, THOMAS, AND WIGNER

In constructing the wave functions to be used in generalization of the model of references 1 and 2, two requirements are to be kept in mind. First of all, the position of the maxima in the observed total neutron cross sections correspond, roughly, to the position of the single-particle levels in the shell model. Therefore the traces of the shell model must be retained—the wave functions to be constructed must contain a factor corresponding to an extra nucleon wave function whose eigenvalues are roughly those given by the shell model. The other requirement that we impose on the generalization is that the expressions for the reduced widths remain simple. In constructing the wave functions of the generalized model we shall take polarization into account by assuming that the core functions depend parametrically on the position of the extra nucleon. The total wave function will be the product of a polarized core function, $\Psi_c(\mathbf{x}_1, \cdots, \mathbf{x}_{A-1}; \mathbf{x}_A)$, and a single-nucleon wave function $u_{cp}(\mathbf{x}_A)$ which satisfy, respectively, the following equations:

$$\Big[\sum_{i=1}^{A-1}-(\hbar^2/2m)\Delta_i+\sum_{i=1}^{A-1}\sum_{j=i+1}^{A-1}v(|\mathbf{x}_i-\mathbf{x}_j|) +\beta\sum_{i=1}^{A-1}v(|\mathbf{x}_i-\mathbf{x}_A|)\Big]\Psi_c(\mathbf{x}_1, \cdots, \mathbf{x}_{A-1}; \mathbf{x}_A) =\epsilon_c(\mathbf{x}_A)\Psi_c(\mathbf{x}_1, \cdots, \mathbf{x}_{A-1}; \mathbf{x}_A), \quad (4)$$

and

$$[-(\hbar^2/2m)\Delta_A+(1-\beta)\bar{V}(\mathbf{x}_A)+\epsilon_c(\mathbf{x}_A)] \times u_{cp}(\mathbf{x}_A)=E_{cp}u_{cp}(\mathbf{x}_A), \quad (5)$$

where β is an adjustable parameter whose value lies between 0 and 1; Δ_i is the Laplacian with respect to the coordinates of the ith nucleon, and $\bar{V}(\mathbf{x}_A)$ is an average of $\sum_{i=1}^{A-1}v(|\mathbf{x}_i-\mathbf{x}_A|)$ with regard to the unpolarized ($\beta=0$) ground state ($c=0$) core function $\Psi_0^{(0)}$: that is,

$$\bar{V}(\mathbf{x}_A)=\int|\Psi_0^{(0)}|^2\sum_{i=1}^{A-1}v(|\mathbf{x}_i-\mathbf{x}_A|)d\tau, \quad (6)$$

where $d\tau=d\mathbf{x}_1\cdots d\mathbf{x}_{A-1}$.

The wave functions Ψ_c contain the coordinates of the Ath nucleon parametrically, inasmuch as no derivatives with regard to the coordinates of this nucleon occur in the definition of Ψ_c. Therefore, for every value of $\mathbf{x}_A$, the set of functions Ψ_c can be made orthonormal with regard to all the coordinates of the other $A-1$ nucleons. That is, we can require,

$$\int\Psi_c\Psi_{c'}^*\,d\tau=\delta_{cc'}. \quad (7)$$

Similarly we can choose

$$\int u_{cp}(\mathbf{x}_A)u_{cp'}^*(\mathbf{x}_A)d\mathbf{x}_A=\delta_{pp'}, \quad (8)$$

where the integral over the coordinates of the Ath nucleon is chosen to extend over the volume occupied by the core nucleons. Together, (7) and (8) yield

$$\int\Psi_c\Psi_{c'}^*u_{cp}u_{c'p'}^*d\tau d\mathbf{x}_A=\delta_{cc'}\delta_{pp'}, \quad (9)$$

where the integral now extends over the coordinates of all the nucleons. The wave functions $\Psi_c u_{cp}$ form a

[6] A core polarization of this kind has been discussed by K. M. Watson, Phys. Rev. **89**, 575 (1953); N. C. Francis and K. M. Watson, Phys. Rev. **92**, 291 (1953); Brueckner, Levinson, and Mahmoud, Phys. Rev. **95**, 655 (1954); G. Takeda and K. M. Watson, Phys. Rev. **97**, 1336 (1955).

[7] Various forms of the strength function and the corresponding total cross sections, for a typical case, are discussed in a Ph.D. thesis by Erich Vogt, Princeton University, 1955 (unpublished). It was found that the form of the strength function (for a given half-width) could be varied considerably without changing the total cross sections by a great amount.

complete orthonormal set of functions in the space given by the coordinates of the A nucleons.

The potential $\beta\sum_{i=1}^{A-1} v(|\mathbf{x}_i-\mathbf{x}_A|)$ in (4) will influence the wave function $\Psi_c(\mathbf{x}_1, \cdots, \mathbf{x}_{A-1}; \mathbf{x}_A)$ only if one of the $\mathbf{x}_1, \cdots, \mathbf{x}_{A-1}$ is in the neighborhood of $\mathbf{x}_A$. Similarly, $\epsilon_c(\mathbf{x}_A)$ will depend on $\mathbf{x}_A$ only if $\mathbf{x}_A$ is close to the boundary of the nucleus. One can think of the nucleon A as immersed in a sea of nuclear matter; the energy of this sea will be the same no matter where the perturbation is located as long as it is further from the nuclear surface than the range of nuclear forces. Hence the potential $\epsilon_c(\mathbf{x}_A)$ will be constant over most of the nucleus and change (increase) only near its surface; it will have the general shape of a nuclear "well" potential. The E_{cp}, as far as their dependence on p is concerned, are characteristic values of a single-particle Eq. (5) in which the potential $\epsilon_c(\mathbf{x}_A)+(1-\beta)\bar{V}(\mathbf{x}_A)$ is of the usual well type. Hence E_{cp} will be, apart from a constant which is the binding energy of the core, equal to the energy of a single particle in a suitable potential well.

The shell-model wave functions, whose mixing formed the basis of the model of Lane, Thomas, and Wigner, correspond to $\beta=0$. The maximum amount of core polarization is obtained by including all of V in the definition of the core states. For this case, β has its maximum value, $\beta=1$.

In order to obtain the reduced widths in terms of our Ψ_c and u_{cp}, we have to define the functions X_λ of the compound nucleus which satisfy the equation

$$HX_\lambda=E_\lambda X_\lambda, \tag{10}$$

where H is the entire Hamiltonian for the A nucleons. Certain fixed boundary conditions are imposed on the states X_λ at the nuclear surface.[8] Since both the set X_λ and the set $\Psi_c u_{cp}$ are complete orthonormal sets, we have the following real orthogonal transformation:

$$X_\lambda=\sum_{c,p} C_{\lambda;cp}\Psi_c u_{cp}, \tag{11}$$

with

$$\sum_\lambda C_{\lambda;cp}C_{\lambda;c'p'}=\delta_{cc'}\delta_{pp'}, \tag{12}$$

and

$$\sum_{c,p} C_{\lambda;cp}C_{\lambda';cp}=\delta_{\lambda\lambda'}. \tag{13}$$

The square root of the reduced width is given by

$$\gamma_{\lambda c}=(\hbar^2/2m)^{\frac{1}{2}}\int X_\lambda\Psi_c^*(\mathbf{x}_1, \cdots, \mathbf{x}_{A-1}; a)\times u_{cp}^*(a)d\tau d\Omega, \tag{14}$$

where the integral extends over the coordinates of the core nucleons and the angular coordinates of the extra nucleon at the surface of the core, $|\mathbf{x}_A|=a$. When the extra nucleon, A, is at the surface of the core, Ψ_c is independent of $\mathbf{x}_A$ and is assumed to be equal to the unpolarized core wave function.

The core is assumed to be spherically symmetric so that the potential $(1-\beta)\bar{V}+\epsilon_c(\mathbf{x}_A)$ is spherically symmetric. We may therefore write u_{cp} as a product of a radial function and a spherical harmonic,

$$u_{cp}(\mathbf{x}_A)=R_{cp}(|\mathbf{x}_A|)Y_{cp}(\mathbf{\Omega}_A). \tag{15}$$

Using (15) and (11) in (14) we obtain

$$\gamma_{\lambda c}=\sum_p\gamma_{\lambda;cp}, \tag{16}$$

where

$$\gamma_{\lambda;cp}=(\hbar^2/ma)^{\frac{1}{2}}C_{\lambda;cp}R_{cp}(a). \tag{17}$$

The basis of the model which underlies the present article is the assumption that for each λ one term in the expansion (16) is much larger than the others. The reduced width, $\gamma_{\lambda c}^2$, of the compound state X_λ thus arises from the "nearest" single-particle state, u_{cp}, of the shell model. This is possible if $\gamma_{\lambda;cp}$, considered as a function of λ, has a maximum near E_{cp} and falls rapidly to a very small value in a distance which is small compared to the spacing of the single-particle levels ($\sim$10 Mev) but large compared to the spacing of nuclear resonance lines. It is this assumption which allows us to decompose the strength function into partial strength functions as was done in (1).

The energy moments of $C_{\lambda;cp}^2$ can be obtained from the formal decomposition, (11), of the compound states. Let us write

$$H=H_0+(1-\beta)(V-\bar{V}), \tag{18}$$

which defines H_0. We obtain, from (11),

$$E_\lambda X_\lambda=\sum_{c,p} C_{\lambda;cp}[H_0+(1-\beta)(V-\bar{V})]\Psi_c u_{cp}. \tag{19}$$

However, $\Delta_A\Psi_c(\mathbf{x}_1, \cdots, \mathbf{x}_{A-1}; \mathbf{x}_A)$ does not vanish identically so that the functions $\Psi_c u_{cp}$ are not eigenfunctions of the operator H_0. In fact,

$$H_0\Psi_c u_{cp}=E_{cp}\Psi_c u_{cp}-2(\hbar^2/2m)\nabla_A\Psi_c\cdot\nabla_A u_{cp}-(\hbar^2/2m)(\Delta_A\Psi_c)u_{cp}. \tag{20}$$

If we multiply each side of (19) by $\Psi_c u_{cp}$ and integrate over the coordinates of all the nucleons, using the decomposition (11) on the left-hand side of (19) and the expression (20) on the right-hand side, we obtain

$$E_\lambda C_{\lambda;cp}=E_{cp}C_{\lambda;cp}+\sum_{c',p'}C_{\lambda;c'p'}\times\{(\Psi_c u_{cp}, (1-\beta)(V-\bar{V})\Psi_c u_{c'p'})-2(\hbar^2/2m)(\Psi_c u_{cp}, \nabla_A\Psi_{c'}\cdot\nabla_A u_{c'p'})-(\hbar^2/2m)(\Psi_c u_{cp}, u_{c'p'}\Delta_A\Psi_{c'})\}. \tag{21}$$

Multiplication by $C_{\lambda;cp}$ and summation over λ yields for the first moment, by means of the orthogonality properties (12) and (13),

$$M_1\equiv\sum_\lambda E_\lambda C_{\lambda;cp}^2=E_{cp}+(\Psi_c u_{cp}, (1-\beta)(V-\bar{V})\Psi_c u_{cp})-2(\hbar^2/2m)(\Psi_c u_{cp}, \nabla_A\Psi_c\cdot\nabla_A u_{cp})-(\hbar^2/2m)(\Psi_c u_{cp}, u_{cp}\Delta_A\Psi_c). \tag{22}$$

One can see that the third term on the right-hand side

[8] L. Eisenbud and E. P. Wigner, Phys. Rev. **72**, 29 (1947).

vanishes by taking the gradient of the normalization condition (7). The sum of the second and the fourth terms on the right-hand side is the correlation energy, E_{corr}, that is, the contribution of the core polarization to the first energy moment of the strength function,

$$E_{\text{corr}}\equiv M_1-E_{cp}=(\Psi_c u_{cp},\ (1-\beta)(V-\bar{V})\Psi_c u_{cp}) - (\hbar^2/2m)(\Psi_c u_{cp},\ u_{cp}\Delta_A\Psi_c). \quad (23)$$

The correlation energy, E_{corr}, will be calculated in the next section and compared to a similar calculation by Wigner.[9]

The higher moments, M_ν, will be defined in terms of the average position of the giant resonance as given by M_1. Thus we define

$$M_\nu=\sum_\lambda C_{\lambda;cp}^2(E_\lambda-M_1)^\nu. \quad (24)$$

If the inverse of the real orthogonal transformation (11) is used in calculating the expectation value of the operator $(H-M_1)^\nu$ taken in the state $\Psi_c u_{cp}$, it follows at once that M_ν is given by

$$M_\nu=(\Psi_c u_{cp},\ (H-M_1)^\nu\Psi_c u_{cp}). \quad (25)$$

Since the operator $H-M_1$ is Hermitian, a simple expression may be obtained for the second moment, M_2;

$$\begin{aligned}M_2&=(\{H_0+(1-\beta)(V-\bar{V})-M_1\}\\&\quad\times\Psi_c u_{cp},\ \{H_0+(1-\beta)(V-\bar{V})-M_1\}\Psi_c u_{cp})\\&=\int|(1-\beta)(V-\bar{V})\Psi_c u_{cp}-2(\hbar^2/2m)\\&\quad\times\nabla_A\Psi_c\cdot\nabla_A u_{cp}-(\hbar^2/2m)u_{cp}\Delta_A\Psi_c\\&\quad-E_{\text{corr}}\Psi_c u_{cp}|^2 d\tau d\mathbf{x}_A\\&=\int|(1-\beta)(V-\bar{V})\Psi_c u_{cp}-2(\hbar^2/2m)\\&\quad\times\nabla_A\Psi_c\cdot\nabla_A u_{cp}-(\hbar^2/2m)u_{cp}\Delta_A\Psi_c|^2 d\tau d\mathbf{x}_A\\&\quad-E_{\text{corr}}^2. \quad (26)\end{aligned}$$

The moments higher than M_2 will involve higher order derivatives of Ψ_c and u_{cp} with regard to $\mathbf{x}_A$. The general expression for the second moment, (26), is considerably more complicated than the special case ($\beta=0$) of Lane, Thomas, and Wigner. In the limit of $\beta=0$, (26) reduces to their expression:

$$M_2=\int|(V-\bar{V})\Psi_c u_{cp}|^2 d\tau d\mathbf{x}_A,\quad(\beta=0). \quad (27)$$

The second moment with core polarization contains, in addition to the potential energy, $(V-\bar{V})$, a number of "kinetic energy" terms. By varying the amount of core polarization we can vary the relative magnitudes of the potential energy and the kinetic energy terms in the square root of the integrand of (26). Thus we can vary, to some extent, the value of M_2.

The moments of $C_{\lambda;cp}^2$ bear a simple relation to the moments of the strength function. Using (15) and (17) in the definition, (24), of M_ν, we obtain,

$$M_\nu=\sum_\lambda(1/\zeta_p^2)\gamma_{\lambda;0p}^2(E_\lambda-M_1)^\nu, \quad (28)$$

where

$$\zeta_p^2=(\hbar^2/2m)R_{0p}^2(a). \quad (29)$$

In (28) and (29) the particular channel corresponding to the ground state of the core ($c=0$) has been chosen. The sum in (28) may be changed to an integral,

$$M_\nu=(1/\zeta_p^2)\int s^{(p)}(E_\lambda)(E_\lambda-M_1)^\nu dE_\lambda, \quad (30)$$

in which the integral extends from the ground-state energy to infinity.

3. THE VALUE OF THE SECOND MOMENT

An expression for the second moment with core polarization [given by (26)] was calculated in the preceding section. In this section we shall evaluate this expression approximately. We assume at the outset that the extra nucleon A is at the center of the core, or rather, calculate the integral $d\tau$, of (26), for $\mathbf{x}_A=0$ and consider this an average of the integrand with respect to $\mathbf{x}_A$. The validity of this approximation will be discussed at the end of this section. We choose u_{cp} to be an s-state in order that it not vanish at the center of the core. In this case, however, the terms of (26) involving $\nabla_A u_{cp}$ vanish.

Using the approximations just made, M_2 may be written

$$M_2=\int|(1-\beta)(V-\bar{V})\Psi_0 - (\hbar^2/2m)\Delta_A\Psi_0|^2 d\tau-E_{\text{corr}}^2, \quad (31)$$

where we have chosen that core function ($c=0$) corresponding to the ground state of the core. We shall calculate (31) by obtaining approximate expressions for Ψ_0. To obtain the approximate ground-state core function we shall treat the nucleon-core potential, βV, as a perturbation. We assume that the unperturbed core function, $\Psi_0^{(0)}$, can be written as a determinant of single-particle wave functions,

$$\Psi_0^{(0)}=\frac{1}{[(A-1)!]^{\frac{1}{2}}}\begin{vmatrix} w_1(\mathbf{x}_1) & \cdots & w_1(\mathbf{x}_{A-1})\\ \vdots & & \vdots\\ w_{A-1}(\mathbf{x}_1)\cdots & & w_{A-1}(\mathbf{x}_{A-1})\end{vmatrix}. \quad (32)$$

If these single-particle wave functions belong to a square well of radius a, with the boundary condition that the wave functions vanish at the radius a, then the normalized s-wave single-particle wave functions are

$$w_s(\mathbf{x}_i)=(2\pi a)^{-\frac{1}{2}}(1/r_i)\sin(s\pi r_i/a). \quad (33)$$

[9] E. P. Wigner, "On the Shell Model for Nuclei," L. Farkas Memorial Volume, Research Council of Israel special publication, No. 1, Jerusalem (1952).

The subscript s on $w_s(\mathbf{x}_i)$ will be used to indicate that (33) is an s-wave function.

The perturbing potential,

$$\beta V=\beta\sum_{i=1}^{A-1} v(|\mathbf{x}_i-\mathbf{x}_A|),$$

is chosen to be a sum of two-body Yukawa potentials without exchange. That is,[10]

$$v(|\mathbf{x}_i-\mathbf{x}_A|)=-\tfrac{1}{2}Ce^2\frac{e^{-\kappa|\mathbf{x}_i-\mathbf{x}_A|}}{|\mathbf{x}_i-\mathbf{x}_A|}, \tag{34}$$

with $C\approx 40$ and $\kappa=2.5mc^2/e^2$ cm^{-1}. We may write the perturbed single-particle wave function, $\omega_n(\mathbf{x}_i,\mathbf{x}_A)$, up to first order, as

$$\omega_n(\mathbf{x}_i,\mathbf{x}_A)=w_n(\mathbf{x}_i)+\omega_n{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A), \tag{35}$$

where the first-order term $\omega_n{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A)$ is

$$\omega_n{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A)=\beta\sum_{n'}{}'\frac{v_{nn'}}{\mathcal{E}_n-\mathcal{E}_{n'}}w_{n'}(\mathbf{x}_i), \tag{36}$$

with

$$v_{nn'}=-\tfrac{1}{2}Ce^2\int w_n w_{n'}\frac{e^{-\kappa|\mathbf{x}_i-\mathbf{x}_A|}}{|\mathbf{x}_i-\mathbf{x}_A|}d\mathbf{x}_i. \tag{37}$$

The first-order term, $\Psi_0{}^{(1)}$, in the perturbation calculation for Ψ_0 may therefore be written as

$$\Psi_0{}^{(1)}=\frac{1}{[(A-1)!]^{\frac{1}{2}}}\times\sum_{i=1}^{A-1}\begin{vmatrix} w_1(\mathbf{x}_1) & \cdots & \omega_1{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A) & \cdots & w_1(\mathbf{x}_{A-1})\\ \vdots & & \vdots & & \vdots\\ w_{A-1}(\mathbf{x}_1)\cdots & & \omega_{A-1}{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A) & & \cdots w_{A-1}(\mathbf{x}_{A-1})\end{vmatrix}. \tag{38}$$

Each determinant of this sum contains one column of first-order single-particle wave functions.

Since $\Psi_0{}^{(0)}$ does not depend on the coordinates $\mathbf{x}_A$, $\Delta_A\Psi_0{}^{(0)}=0$, while

$$\Delta_A\Psi_0\doteq\Delta_A\Psi_0{}^{(1)}=\frac{1}{[(A-1)!]^{\frac{1}{2}}}\sum_{i=1}^{A-1}\times\begin{vmatrix} w_1(\mathbf{x}_1) & \cdots & \Delta_A\omega_1{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A) & \cdots & w_1(\mathbf{x}_{A-1})\\ \vdots & & \vdots & & \vdots\\ w_{A-1}(\mathbf{x}_1)\cdots & & \Delta_A\omega_{A-1}{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A) & & \cdots w_{A-1}(\mathbf{x}_{A-1})\end{vmatrix}. \tag{39}$$

The coordinates of the Ath nucleon occur in $\omega_n{}^{(1)}(\mathbf{x}_i,\mathbf{x}_A)$ only in the matrix element $v_{nn'}$. $\Delta_A v_{nn'}$ is

$$\Delta_A v_{nn'}=\kappa^2 v_{nn'}+2\pi Ce^2 w_n(\mathbf{x}_A)w_{n'}(\mathbf{x}_A). \tag{40}$$

The correlation energy, E_{corr}, is given approximately by

$$E_{\text{corr}}=(1-\beta)(\Psi_0{}^{(0)},\ (V-\bar{V})\Psi_0{}^{(1)})-(\hbar^2/2m)(\Psi_0{}^{(1)},\ \Delta_A\Psi_0{}^{(1)}). \tag{41}$$

This expression will be evaluated below. As will be shown there, $E_{\text{corr}}{}^2$ contributes a negligible amount to the second moment.

Keeping only highest order terms in M_2, we may approximate (31) by

$$M_2=\int|(1-\beta)(V-\bar{V})\Psi_0{}^{(0)}-(\hbar^2/2m)\Delta_A\Psi_0{}^{(1)}|^2d\tau$$

$$=\int|(1-\beta)V\Psi_0{}^{(0)}-(\hbar^2/2m)\Delta_A\Psi_0{}^{(1)}|^2d\tau-(1-\beta)^2\bar{V}^2. \tag{42}$$

Using (32) in (6) we obtain

$$\bar{V}=\sum_{i=1}^{A-1}v_{ii}. \tag{43}$$

Using the expressions (32) and (38) for $\Psi_0{}^{(0)}$ and $\Psi_0{}^{(1)}$ respectively, the second moment (42) can be written in terms of one- and two-nucleon integrals in the following way:

$$M_2=-(1-\beta)^2\Big(\sum_{i=1}^{A-1}v_{ii}\Big)^2+\sum_{i=1}^{A-1}\int|(1-\beta)v(|\mathbf{x}_A-\mathbf{x}|)w_i(\mathbf{x})-(\hbar^2/2m)\Delta_A\omega_i{}^{(1)}(\mathbf{x}_A,\mathbf{x})|^2d\mathbf{x}$$

$$-\sum_{i=1}^{A-1}\sum_{\substack{j=1\\ i\neq j}}^{A-1}\Big\{(1-\beta)^2(v_{ij})^2-2(\hbar^2/2m)(1-\beta)v_{ij}\times\int w_i\Delta_A\omega_j{}^{(1)}d\mathbf{x}+(\hbar^2/2m)^2\Big(\int w_i\Delta_A\omega_j{}^{(1)}d\mathbf{x}\Big)^2$$

$$-(1-\beta)^2v_{ii}v_{jj}+2(\hbar^2/2m)(1-\beta)v_{ii}\times\int w_j\Delta_A\omega_j{}^{(1)}d\mathbf{x}-(\hbar^2/2m)^2\times\Big(\int w_i\Delta_A\omega_i{}^{(1)}d\mathbf{x}\Big)\Big(\int w_j\Delta_A\omega_j{}^{(1)}d\mathbf{x}\Big)\Big\}. \tag{44}$$

The first and the last three terms in the curly bracket are, respectively, the exchange and the direct integrals. The last two direct integrals vanish identically. All of the remaining terms on the right-hand side of (44) combine in the following way:

$$M_2=\sum_{i=1}^{A-1}\sum_{j=A}^{\infty}\left|(1-\beta)v_{ij}-\beta(\hbar^2/2m)\frac{\Delta_A v_{ij}}{\mathcal{E}_i-\mathcal{E}_j}\right|^2. \tag{45}$$

We note that, thus far, the only way in which we have used the assumption that the extra nucleon is at the center of the core is in the neglect of the terms in-

[10] This potential was also used by Lane, Thomas, and Wigner in their calculation of the second moment. It is taken from the work of H. Feshbach and J. Schwinger, Phys. Rev. **84**, 194 (1951) and H. H. Hall and J. L. Powell, Phys. Rev. **90**, 912 (1953).

volving $\nabla_A u_{cp}$. We now set $\mathbf{x}_A=0$ in v_{ij} and $\Delta_A v_{ij}$. If the potential $v(|\mathbf{x}-\mathbf{o}|)$ is large only for distances which are small compared to the radius a of the core, then v_{ij} and $\Delta_A v_{ij}$ are large only if i and j both refer to s-wave states. We can therefore write

$$M_2=4\sum_s\sum_{s'}\left|(1-\beta)v_{ss'}-\beta(\hbar^2/2m)\frac{\Delta_A v_{ss'}}{\mathcal{E}_s-\mathcal{E}_{s'}}\right|^2, \quad (46)$$

where s refers to an occupied s-wave state and s' to an unoccupied s-wave state. The factor 4 arises from the fact that there are four nucleons in each occupied s-state.

From (33), $v_{ss'}$ is found to be

$$v_{ss'}=-(1/4)(Ce^2/a)\ln\left[\frac{(\kappa a/\pi)^2+(s+s')^2}{(\kappa a/\pi)^2+(s-s')^2}\right], \quad (47)$$

which is obviously always negative. Similarly, in terms of (40) and (33)

$$-(\hbar^2/2m)\frac{\Delta_A v_{ss'}}{\mathcal{E}_s-\mathcal{E}_{s'}}=-\frac{Ce^2/a}{s^2-s'^2}$$
$$\times\left\{ss'-(\kappa a/2\pi)^2\ln\left[\frac{(\kappa a/\pi)^2+(s+s')^2}{(\kappa a/\pi)^2+(s-s')^2}\right]\right\}. \quad (48)$$

Since the first term of the curly bracket is always larger than the second, (48) is obviously always positive.

The quantities $v_{ss'}$ and $(\hbar^2/2m)\Delta_A v_{ss'}/(\mathcal{E}_s-\mathcal{E}_{s'})$, for a given s, may be regarded, respectively, as the components of two vectors in the Hilbert space whose basis is given by the s-wave states. If these vectors are parallel it is possible to choose β so that the contribution of the occupied core state s to the second moment vanishes. The condition that each term in the sum over occupied states, in (46), must be larger than or equal to zero, for all values of β, is equivalent to Schwarz's inequality.

The second moment (46) was calculated assuming a radius of 9.0×10^{-13} cm for the core. This radius ($a=1.45A^{\frac{1}{3}}\times10^{-13}$ cm) corresponds to a very heavy nucleus ($A\sim240$). For such a nucleus the first three s-states are occupied.[11]

The result obtained for the square root of the second moment is shown on Fig. 1 as a function of the amount of core polarization. The square roots of the separate contributions of the $1s$, $2s$, and $3s$ states to M_2 are also shown. The value of $M_2^{\frac{1}{2}}$ for $\beta=0$, ($M_2^{\frac{1}{2}}=22.3$ Mev) agrees very well with the result obtained by Lane, Thomas, and Wigner ($M_2^{\frac{1}{2}}=22.5$ Mev) although these latter authors used a different approximation for the core functions in evaluating M_2 and a smaller radius. If the larger radius of the present article is used in the second moment calculated by Lane, Thomas, and

[11] This is consistent with the fact that the $4s$ state lies at an energy of 40.5 Mev, that is, at an energy which is higher than the depth of the potential $\bar{V}$ as calculated by Lane, Thomas, and Wigner for this choice of the radius.

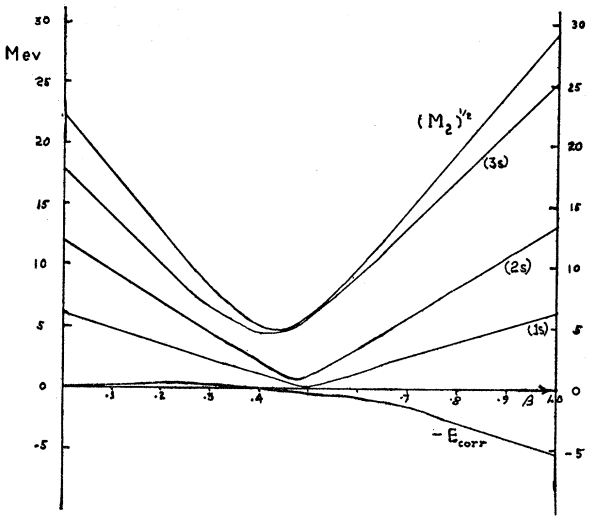

FIG. 1. The second moment of the strength function in terms of the amount of core polarization. The curve labeled $(M_2)^{\frac{1}{2}}$ is the square root of the second moment. The curves labeled $1s$, $2s$, and $3s$ are, respectively, the square roots of the contributions of the $1s$, $2s$, and $3s$ states to the second moment. The curve labeled $-E_{\text{corr}}$ is the negative of the correlation energy given by (50).

Wigner their result for $M_2^{\frac{1}{2}}$ is reduced to 18.8 Mev. The minimum value[12] of $(M_2)^{\frac{1}{2}}$[$(M_2)^{\frac{1}{2}}=4.75$ Mev] differs only by a factor of two or three from the result obtained with complex square wells[4] for the half-width of the strength function. This relatively small difference may well lie in the crudeness of the present calculation or in the relation between the width of the strength function and the square root of the second moment.

The correlation energy E_{corr}, (41), can be calculated in a similar way to be

$$E_{\text{corr}}=\sum_s\sum_{s'}\left\{\beta(1-\beta)\frac{(v_{ss'})^2}{\mathcal{E}_s-\mathcal{E}_{s'}}\right.$$
$$\left.-\beta^2(\hbar^2/2m)\frac{v_{ss'}\Delta_A v_{ss'}}{(\mathcal{E}_s-\mathcal{E}_{s'})^2}\right\}, \quad (50)$$

where s and s' again refer to occupied and unoccupied states, respectively. The term $\sum_s\sum_{s'}(v_{ss'})^2/(\mathcal{E}_s-\mathcal{E}_{s'})$ of (50) corresponds to a quantity calculated by Wigner,[9] that is, the second approximation to the energy obtained by treating the potential $V-\bar{V}$ as a perturbation to the wave functions of the shell model. The result obtained by Wigner for this quantity, (2.3 Mev),[13]

[12] The maximum amount by which the result obtained by Lane, Thomas, and Wigner for M_2 can be reduced is a factor of 23. Previously, A. M. Lane and L. Verlet as well as the present author reported a reduction of only a factor of 3 [Phys. Rev. **100**, 956(A) (1955)]. The former authors obtained this smaller reduction in an entirely different calculation of the core polarization. The result reported at that time by the present author was due to a mistake in the present calculation.

[13] In order to correspond with the present calculation equation (12) in the appendix of reference 9 must be divided by 4 because of the difference between the two-nucleon potentials used in reference 9 and the present article. As prescribed in reference 9,

agrees very well with the (3.1 Mev) obtained in our calculation. E_{corr} is also shown on Fig. 1. It has a zero near the minimum of the second moment. Since E_{corr}^2 is never greater than three percent of M_2 the neglect of E_{corr}^2 in the calculation of the second moment was justified. This small value of E_{corr} ensures, further, that the maxima of our partial strength functions occur near the energies, ϵ_p, of the single-particle levels in the shell model.

The approximation methods used in the above calculation of the second moment are crude and can certainly be improved. The perturbation calculation for the core functions should be fairly accurate since the square integral, N_1^2, of the first-order wave functions, $\Psi_0^{(1)}$, is small. N_1^2 is

$$N_1^2=\beta^2\sum_s\sum_{s'}\frac{v_{ss'}^2}{(\mathcal{E}_s-\mathcal{E}_{s'})^2}, \tag{51}$$

where s and s' refer to occupied and unoccupied states respectively. N_1^2 is found to be 0.013 β^2 so that $\Psi_0^{(1)}$ is at least an order of magnitude smaller than $\Psi_0^{(0)}$.

the reduced mass $m/2$ instead of the total mass of the nucleon, m, should be used in this calculation. Because of this Eq. (12) in the appendix of reference 9 must be divided by a further factor of 2.

The approximation involved in placing the extra nucleon, A, at the center of the core could be justified if the contributions of the various core nucleons to the second moment were proportional to their densities at the center of the core. The argument for this justification would be similar to that which was used in the preceding sections to show that $\epsilon_c(\mathbf{x}_A)$ was constant over most of the volume of the core. At the center of the core the relative densities of the $1s$, $2s$, and $3s$ nucleons are, respectively, 1, 4, and 9. As seen on Fig. 1, the contributions of these nucleons to the second moment have these relative values for $\beta=0$. For other values of β the $3s$ nucleons contribute a greater portion of the second moment. The extent to which this effect impairs the validity of the approximation used in the above calculation has not been investigated.

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Angular Distributions of the D+D Neutrons*

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The angular distributions of the neutrons produced in the D(d,n)He^3 reaction have been investigated experimentally in the deuteron energy range of 0.25 to 0.825 Mev. A two-crystal neutron spectrometer providing discrimination against gamma radiation was used as the detector. It is noted that the experimental data can be fitted by the deuteron stripping theory.

INTRODUCTION

ALTHOUGH the angular distributions of the neutrons from the reaction D+D→He^3+n are of the greatest importance from both a theoretical and experimental standpoint, there is still some uncertainty in the experimental data. Konopinski *et al.*[1] have shown that the energy-dependence of the angular distribution coefficients can be accounted for by the differences in centrifugal barriers corresponding to the different components of the incident deuteron waves, provided that spin-orbit coupling is introduced. Conversely, the experimentally observed energy dependence of the coefficients can be used to determine the amount of spin orbit coupling. In Konopinski's work no distinction is made between the D(d,n)He^3 and D(d,p)H^3 reactions. Fairbairn[2] has shown that the angular distributions of the neutrons from the reaction D(d,n)He^3 can be fitted by deuteron stripping at energies above 4 Mev. The stripping calculation utilized an interaction radius of about $4(10)^{-13}$ cm.

Recent experiments[3–10] on the reaction

D+D→n+He^3+3.25 Mev

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