Theory of the Photomagnetoelectric Effect in Semiconductors

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Results are obtained for the photomagnetoelectric (PME) effect that are more general and exact than those of earlier theory. Through an ambipolar treatment, the underlying general theory for current carrier transport with magnetic field, which can provide similarly unrestricted results for the Hall, Suhl, and magnetic rectifier effects, is first developed. The PME effect is considered in detail for the infinite slab with strongly absorbed steady radiation on one surface and parallel, steady, uniform magnetic field. Small Hall angles and constant surface recombination velocities and lifetime are assumed. Small-signal theory is given as well as nonlinear theory for arbitrary light intensity. The latter provides methods for determining lifetime that require only negligible dark-surface concentration of added carriers, as well as a method for determining surface recombination velocity; curves for these are given for germanium. Expressed in terms of conductance increase, PME current or voltage does not depend explicitly on light intensity nor on recombination velocity for the illuminated surface. Distance along the slab between equipotential probes on opposite surfaces as obtained from a nul measurement in which Dember and PME potentials cancel determines directly the sum of the magnitudes of the Hall angles, upon which the PME effect depends.

1. INTRODUCTION

THE photomagnetoelectric effect, or PME effect, may be described as a Hall effect associated with the diffusion of optically injected current carriers. It was first observed in cuprous oxide at low temperature,[1] shortly following which Frenkel provided a theoretical explanation based on the concept of the optical excitation of electron-hole pairs.[2] It was observed comparatively recently in germanium at room temperature, and, notably through the work of Aigrain and Bulliard[3,4] and Moss and Pincherle,[5] has been used to study recombination in the volume and on surfaces of germanium[3–8] and other semiconductors.[4,9]

The detailed theory of the present paper for the PME effect involves fewer restrictive assumptions and approximations than have previously been employed. In Sec. 2, the underlying general theory for the transport of current carriers with magnetic field is developed. This can provide similarly unrestricted results for the Hall,[10] Suhl,[11] and magnetic rectifier[12] effects; with added carriers (in concentrations which may be negative as well as positive), these effects present a unified aspect. Under the assumptions often made for homogeneous semiconductors, an ambipolar treatment[13] furnishes partial differential equations and other equations which, provided Boltzmann statistics remain valid, are applicable for unrestricted added carrier concentration and whatever be the n- or p-type conductivity at thermal equilibrium. These are applied, in Sec. 3, to the PME effect in an infinite slab with strongly absorbed steady radiation incident on one surface and steady parallel magnetic field. Constant surface recombination velocities and lifetime are assumed, as well as small Hall angles for which magnetoresistance is negligible.[14]

In this theory, phenomenological distinction is made between Hall and drift mobilities, scattering models[15] not being considered. An extension to the case of slow and fast holes is given which shows that relatively few fast holes can make a relatively large contribution to PME current or voltage. The same equations still apply, however, in which the Hall and drift mobilities

[1] I. K. Kikoin and M. M. Noskov, Physik. Z. Sowjetunion **5**, 586 (1934); I. K. Kikoin, Physik Z. Sowjetunion **6**, 478 (1934); G. Groetzinger, Physik. Z. **36**, 169 (1935).

[2] J. Frenkel, Physik. Z. Sowjetunion **5**, 597 (1934), **8**, 185 (1935).

[3] P. Aigrain and H. Bulliard, Compt. rend. **236**, 595, 672 (1953). H. Bulliard, Ann. phys. **15**, 52 (1954); P. Aigrain, Ann. radioélec. Compagn. Gén. de T. S. F. **9**, 219 (1954). The term "photomagnetoelectric" is in accord with the usage of these authors.

[4] H. Bulliard, Phys. Rev. **94**, 1564 (1954).

[5] Moss, Pincherle, and Woodward, Proc. Phys. Soc. (London) **66B**, 743 (1953); T. S. Moss, Physica **20**, 989 (1954); L. Pincherle, Proceedings of the Atlantic City Conference on Photoconductivity, November 4, 1954.

[6] J. J. Oberly, Phys. Rev. **93**, 911 (1954).

[7] T. M. Buck and W. H. Brattain, J. Electrochem. Soc. **102**, 636 (1955).

[8] G. Grosvalet, Ann. radioélec. Compagn. Gén. de T. S. F. **9**, 360 (1954).

[9] Kurnick, Strauss, and Zitter, Phys. Rev. **94**, 1791 (1954); T. S. Moss, reference 5; Proc. Phys. Soc. (London) **66B**, 993 (1953).

[10] R. H. Fowler, *Statistical Mechanics* (Cambridge University Press, Cambridge, 1936), p. 428; H. Welker, Z. Naturforsch. **6a**, 184 (1951); R. Landauer and J. Swanson, Phys. Rev. **91**, 555 (1953). These treatments take added carriers into account, in contrast to "classical" ones which apply to zero lifetime; see O. Madelung, Z. Naturforsch. **9a**, 667 (1954).

[11] H. Suhl and W. Shockley, Phys. Rev. **75**, 1617; **76**, 180 (1949); W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand Company, Inc., New York, 1950), pp. 71–75, 325 ff. See also H. Suhl, Bell System Tech. J. **32**, 647 (1953).

[12] H. Welker, Z. Naturforsch. **6a**, 184 (1951); E. Weisshaar and H. Welker, Z. Naturforsch. **8a**, 681 (1953); Lehovec, Marcus, and Schoeni, Phys. Rev. **98**, 229 (1955); O. Madelung, Naturwiss. **14**, 406 (1955); Madelung, Tewordt, and Welker, Z. Naturforsch. **10a**, 476 (1955); E. Weisshaar, Z. Naturforsch. **10a**, 488 (1955).

[13] W. van Roosbroeck, Phys. Rev. **91**, 282 (1953).

[14] For less than 1% magnetoresistance in germanium or silicon, Hall angles should in general not exceed 0.1 to 0.2 radian or about 5 to 10 degrees: G. L. Pearson and H. Suhl, Phys. Rev. **83**, 768 (1951); G. L. Pearson and C. Herring, Physica **20**, 975 (1954). A similar restriction applies also to InSb: G. L. Pearson and M. Tanenbaum, Phys. Rev. **90**, 153 (1953); Tanenbaum, Pearson, and Feldman, Phys. Rev. **93**, 912 (1954); H. J. Hrostowski (private communication).

[15] See O. Madelung, reference 10. Theory for the large-magnetic-field PME effect has been given by I. I. Ansel'm, Zhur. Tekh. Fiz. **24**, 2064 (1954).

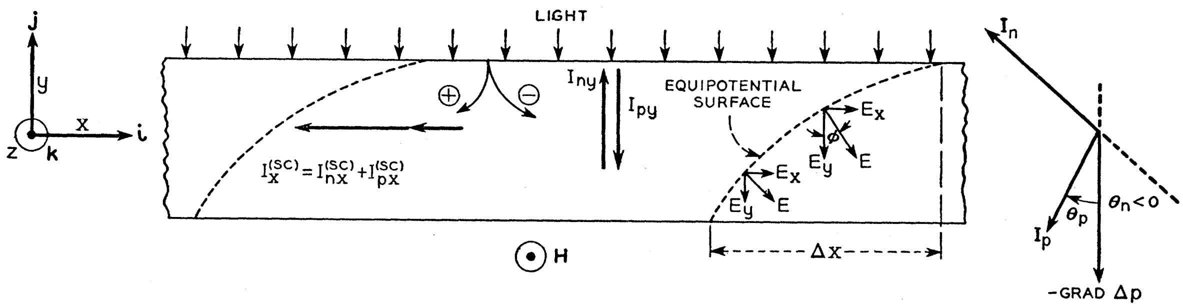

FIG. 1. The PME effect in an infinite semiconductor slab.

for holes are certain weighted averages that correspond to the apparent values determined by measurements at given temperature of Hall effect at given magnetic field and of drift velocity or conductivity. It is indicated briefly how the theory for arbitrary Hall angles can be developed in a straightforward manner for semiconductors like InSb and InAs, whose mobility ratios are large[16] and whose magnetoresistive behavior is comparatively simple if they are not degenerate.[17] Magnetoresistance in these materials, negligible only for small Hall angles,[14] can perhaps to advantage be taken into account semiempirically on the basis of measurements without added carriers.

The condition that the curl of the electrostatic field must vanish requires a constant (depth-independent) PME field along the infinite slab and a nonvanishing curl of local total current density **I**. Thus, PME open-circuit voltage is the same on both surfaces, and the associated **I** constitutes a circulating current. As previously indicated,[18] neglect of this circulating current by the otherwise plausible assumption that the open-circuit **I** is everywhere zero—valid only if there is no magnetic field—has led to some theoretical results that are inconsistent and also at variance with the experimental observation[19] of PME open-circuit voltages substantially the same on both surfaces.

In advance of the proper derivation, it may be well to enlarge upon this aspect of the theory in descriptive terms. With reference to Fig. 1, consider first the short-circuit condition with field E_x along the slab zero, as if the ends of the infinite slab were joined. Optically injected electrons and holes are deflected respectively to the right and left by the applied magnetic field **H**, and their corresponding flow densities make the Hall angles θ_n and θ_p with the negative of the concentration gradient of added carriers. As indicated in the figure, the electron and hole currents along the slab add to give a total PME short-circuit current density $I_x^{(sc)}$ to the left. Since the total current density across the slab is zero, $I_x^{(sc)}$ is proportional to the sum of the tangents of the Hall angles.

In general, and in particular for the open-circuit condition, the field along the slab is constant. The field across the slab is the field of the Dember effect[20]: Since the diffusion constant for electrons exceeds that for holes, zero-current diffusion involves a field opposite to the concentration gradient which assists the transport of holes and retards that of electrons. This field is proportional to the concentration gradient and is thus largest near the illuminated surface. The open-circuit equipotentials, orthogonal to the resultant field, are accordingly parallel curved surfaces, as indicated in the figure by the dashed lines. For Hall angles not so large that magnetoresistance occurs, the Dember effect predominates and the equipotentials are inclined mostly along the slab. It is shown in Sec. 3.5 that the distance Δx along the slab between the intersections of an equipotential with the surfaces of the slab is a measure of the sum θ of the magnitudes of the (small) Hall angles, upon which the PME effect depends. For the distance measurement, directly opposite points on the slab might first be found, say, as points for which there is no change in the Dember potential measured under the magnetic field upon reversal in direction of this field. The value of θ found by this proposed nul method, in which the PME and Dember potentials cancel, may be compared with that computed from the Hall mobilities, if the latter are known for the particular sample and magnetic field.

If $I_x^{(sc)}$ were substantially uniform across the slab, then, under open circuit, it would just be canceled by the drift current density to the right associated with the PME open-circuit field, $E_x^{(oc)}$. Because of recombination, however, $I_x^{(sc)}$ decreases with depth into the slab, being largest at the illuminated surface, where the concentration gradient of added carriers is largest. The sum of $I_x^{(sc)}$ and the drift current density is the open-circuit circulating current density, $I_x^{(oc)}$; its integral across the slab is zero. As shown qualitatively in Fig. 2

[16] H. Welker, Z. Naturforsch. **7a**, 744 (1952), **8a**, 248 (1953); Physica **20**, 893 (1954); M. Tanenbaum and J. P. Maita, Phys. Rev. **91**, 1009 (1953); O. Madelung and H. Weiss, Z. Naturforsch. **9a**, 527 (1954); H. J. Hrostowski and M. Tanenbaum, Physica **20**, 1065 (1954).

[17] E. Burstein, Phys. Rev. **93**, 632 (1954).

[18] W. van Roosbroeck, Phys. Rev. **98**, 1533 (1955).

[19] T. M. Buck (unpublished); Moss, Pincherle, and Woodward, reference 5.

[20] H. Dember, Physik. Z. **32**, 554, 856 (1931); **33**, 207 (1932). The effect was first observed in cuprous oxide and later measured by many other workers in this material and other materials as well. Its theory was given by J. Frenkel, Nature **132**, 312 (1933); Physik. Z. Sowjetunion **8**, 185 (1935).

for a slab of finite length with no electrodes, the open-circuit current is principally photomagnetoelectric and to the left near the illuminated surface over a minor fraction of the thickness, and principally a drift current to the right of smaller average density over a major fraction of the thickness.

In a slab with high-conductivity electrodes, the current flows between the electrodes in paths which are straight but otherwise similar to the ones in the figure; and the corresponding electrostatic field **E** and equipotentials are qualitatively as shown in the lower diagram. It is clear from this diagram that end effects result in principle in some difference between the PME open-circuit voltages between directly opposite pairs of probes on the illuminated and dark surfaces. This difference is minimized if the probes are located symmetrically about the center. However, the condition of constant PME open-circuit field that applies to the infinite slab is in principle not correct for a slab with perpendicular end electrodes. Also, if the electrodes are maintained at the same potential, as in measurements of PME short-circuit current, then **E** nevertheless has components along the slab. These are directed towards the electrodes; only at the center of the slab, about which **E** is symmetrical, is **E** simply the Dember field. While the general equations given in Sec. 2 are applicable to two-dimensional boundary-value problems for the slab with electrodes, specific results are obtained in Sec. 3 only for the infinite slab. The use of relatively long slabs is accordingly always a necessary experimental precaution.

Linear calculations given in Sec. 3 readily provide PME short-circuit current and open-circuit field for the limiting small-signal and large-signal cases. Included also is theory for the general nonlinear case of arbitrary light intensity which takes into account the concentration dependence of added carrier diffusivity. In conjunction with experiment, appropriate cases of this theory can serve to determine lifetime τ or surface recombination velocity with good accuracy, after having provided a critical check of the validity of the underlying assumptions. It is shown that by simultaneous measurement of photoconductance a useful simplification can be effected: Expressed in terms of conductance increase, PME current or voltage does not depend explicitly on light intensity nor on the recombination velocity s_1 for the illuminated surface. If absorbed light intensity is known, then s_1 can also be determined, in addition to τ or to the recombination velocity s_2 for the dark surface.

In particular, a method is developed for determining τ in a thick slab in which no added carriers reach the dark surface. For applying this method to germanium Fig. 6 gives theoretical curves that serve to determine the number $2Y_0$ of diffusion lengths in the slab thickness from measured quantities or quantities otherwise known with good accuracy, namely a dimensionless PME short-circuit current $\mathcal{I}$ and the relative conductance

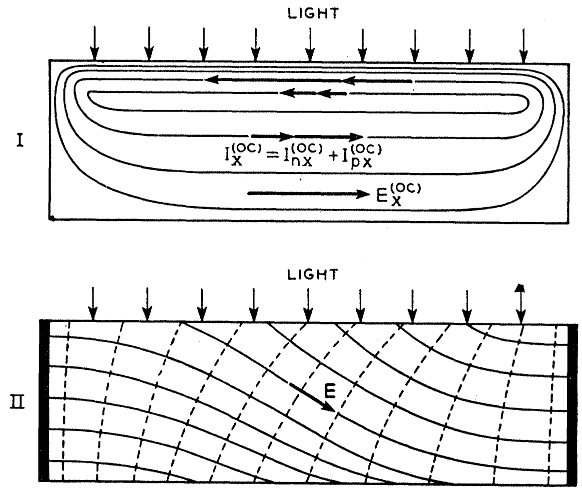

Fig. 2. The PME effect in a semiconductor slab of finite length: I—Open-circuit current, without electrodes; II—open-circuit field and equipotentials, with electrodes.

increase $\Delta G/G_0$. A more widely applicable method for τ in a slab of any thickness is also developed which depends on negligible added carrier concentration at the dark surface, as may be realized by sandblasting this surface. This method appears to be well suited for accurate determination of lifetime, even in slabs only of the order of a diffusion length in thickness; it can serve to check whether volume recombination is negligible within experimental error in a particular sample. Figure 7 gives families of curves for *n*- and *p*-type germanium that specify $2Y_0$ according to this method in terms of $\mathcal{I}/(\Delta G/G_0)$, with $\mathcal{I}$ as parameter. The small- and large-signal asymptotes for infinite $2Y_0$ shown in this figure indicate that the thick-slab approximation applies only for thicknesses of at least several diffusion lengths. For negligible volume recombination, Fig. 8 and Fig. 9 illustrate for *n*- and *p*-type germanium a method developed for the determination of surface recombination velocity which is based on curves of $\mathcal{I}/(\Delta G/G_0)$ *versus* $\Delta G/G_0$ with a dimensionless form of the recombination velocity s_2 as (unknown) parameter. These curves show that PME open-circuit voltage generally saturates slowly with increasing light intensity, the large-signal approximation applying only at large conductance increases that are not readily obtained in practice.

Time dependence is not considered in detail. The PME current or voltage following suddenly applied illumination, the steady state for which is established about as quickly as that for the photoconductance, does not at present seem well suited for quantitative studies.[21] It may, on the other hand, be desirable to extend the theory that has been given for the dependence of the relative amplitude and phase of PME and photoconductive response on the frequency of ac illumination.[8]

[21] Reference 4 and L. H. Hall, Phys. Rev. **97**, 1471 (1955).

2. GENERAL THEORY[22]

2.1 Fundamental Equations

With the hole and electron currents $\mathbf{I}_p$ and $\mathbf{I}_n$ suitably specified, the fundamental equations for steady magnetic fields are

$$p-p_0\equiv\Delta p=\Delta n\equiv n-n_0; \quad (1)$$

$$\mathrm{div}\mathbf{I}=0, \quad \mathbf{I}=\mathbf{I}_p+\mathbf{I}_n; \quad (2)$$

$$\mathrm{curl}\mathbf{E}=0, \quad \mathbf{E}=-\mathrm{grad}V; \quad (3)$$

$$\begin{aligned}\partial p/\partial t=\partial n/\partial t=\partial\Delta p/\partial t&=-e^{-1}\,\mathrm{div}\mathbf{I}_p-\Delta p/\tau\\&=e^{-1}\,\mathrm{div}\mathbf{I}_n-\Delta p/\tau. \quad (4)\end{aligned}$$

The continuity equations for holes and electrons, Eqs. (4), are the general equations simplified by use of the condition of local electrical neutrality, Eq. (1), and by introduction of the lifetime function τ for added carrier concentration.[13]

If Boltzmann statistics apply, then the hole and electron currents in a homogeneous semiconductor under steady magnetic field may be expressed by use of tensors (of the second rank) as

$$\begin{aligned}\mathbf{I}_p&=\boldsymbol{\sigma}_p\cdot\mathbf{E}-e\mathbf{D}_p\cdot\mathrm{grad}p=\boldsymbol{\sigma}_p\cdot[\mathbf{E}-(kT/e)\,\mathrm{grad}\,\ln p],\\ \mathbf{I}_n&=\boldsymbol{\sigma}_n\cdot\mathbf{E}+e\mathbf{D}_n\cdot\mathrm{grad}n=\boldsymbol{\sigma}_n\cdot[\mathbf{E}+(kT/e)\,\mathrm{grad}\,\ln n],\end{aligned} \quad (5)$$

the conductivity and diffusivity tensors being related to mobility tensors by

$$\begin{aligned}\boldsymbol{\sigma}_p&=ep\boldsymbol{\mu}_p, \quad \mathbf{D}_p=(kT/e)\boldsymbol{\mu}_p,\\ \boldsymbol{\sigma}_n&=en\boldsymbol{\mu}_n, \quad \mathbf{D}_n=(kT/e)\boldsymbol{\mu}_n.\end{aligned} \quad (6)$$

The form of the current equations and the proportionality of the diffusivity and mobility tensors in accordance with Einstein's relation obtain since steady magnetic field does not change the statistics: It affects neither the distribution in velocity, which remains Maxwellian, nor the Boltzmann density distribution.[23]

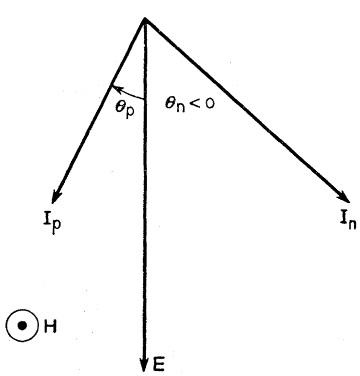

FIG. 3. The Hall angles.

The tensors may be separated into symmetric and antisymmetric parts, and the terms involving the antisymmetric tensors written as vector products, so that

$$\begin{aligned}\mathbf{I}_p&=\boldsymbol{\sigma}_p{}^{(s)}\cdot\boldsymbol{\mathcal{E}}_p+\mathbf{S}_p\times\boldsymbol{\mathcal{E}}_p,\\ \mathbf{I}_n&=\boldsymbol{\sigma}_n{}^{(s)}\cdot\boldsymbol{\mathcal{E}}_n+\mathbf{S}_n\times\boldsymbol{\mathcal{E}}_n.\end{aligned} \quad (7)$$

Here $\boldsymbol{\sigma}_p{}^{(s)}$ and $\boldsymbol{\sigma}_n{}^{(s)}$ are the symmetric conductivity tensors, and $\boldsymbol{\mathcal{E}}_p$ and $\boldsymbol{\mathcal{E}}_n$ (which may be written as negative gradients of electrochemical potentials) are the vectors in brackets in Eqs. (5). The vectors $\mathbf{S}_p$ and $\mathbf{S}_n$ may be referred to as Hall vectors.[24] It follows from the principle of microscopic reversibility that[25] the components of the symmetric tensors are even functions of $H\equiv|\mathbf{H}|$, while those of the Hall vectors are odd functions of H.

By specializing Eqs. (7), which are applicable to the general case of the normally anisotropic semiconductor with magnetoresistance, explicit dependence of the tensors on magnetic field can be exhibited. If principal axes of the tensor ellipsoids of $\boldsymbol{\sigma}_p{}^{(s)}$ and $\boldsymbol{\sigma}_n{}^{(s)}$ are collinear with the Hall vector, then "forces" $\boldsymbol{\mathcal{E}}_p$ and $\boldsymbol{\mathcal{E}}_n$ perpendicular to the Hall vector will give currents $\mathbf{I}_p$ and $\mathbf{I}_n$ which are also perpendicular. The angles between these forces and the corresponding currents may be identified with the Hall angles if the tensor ellipsoids are ellipsoids of rotation about the Hall vector, so that the angles are independent of orientation in the transverse plane and the Hall vectors are collinear with the magnetic field $\mathbf{H}$. These conditions may be realized in cubic crystals by $\mathbf{H}$ in, say, the 100 or 111 direction. The Hall angles θ_p and θ_n for holes and electrons are shown in Fig. 3 for drift (in uniform concentrations) under perpendicular fields.[26] Equations of definition,

$$\begin{aligned}\tan\theta_p&=R_p\sigma_{pt}H\doteq c^{-1}\mu_{pH}H,\\ -\tan\theta_n&=R_n\sigma_{nt}H=c^{-1}\mu_{nH}H,\end{aligned} \quad (8)$$

relate these angles to Hall coefficients R_p and R_n and, in context with theoretical considerations, to Hall mobilities μ_{pH} and μ_{nH}. Here σ_{pt} and σ_{nt} are the transverse conductivities and, with cgs units, c is the speed of light. In this paper, only Hall angles, as the most directly phenomenological quantities, will be employed; R_p and μ_{pH} may, for example, depend on H even for comparatively small H if holes of differing mobilities are actually present.[27]

Consistently with these considerations, the hole and electron currents for the case of the isotropic semiconductor with conductivities σ_p and σ_n independent

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[22] The notation employed is consistent with that of reference 13.

[23] S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, Cambridge, 1939), pp. 322 ff.; W. Shockley, reference 11, pp. 301–302.

[24] This terminology is here more convenient than that resulting from the definition of a Hall vector in terms of the antisymmetric resistivity tensor: S. R. de Groot, *Thermodynamics of Irreversible Processes* (Interscience Publishers, Inc., New York, 1951), p. 51.

[25] J. Meixner, Ann. Physik **40**, 165 (1941); S. R. de Groot, reference 24, pp. 8, 15–17, 50–51.

[26] W. Shockley, reference 11, pp. 204 ff. The angle θ_n for electrons is negative by convention.

[27] Willardson, Harman, and Beer, Phys. Rev. **96**, 1512 (1954).

of H are given by

$$\begin{aligned}\mathbf{I}_p-(\mathbf{I}_p\times\mathbf{k})\tan\theta_p&=\mathbf{I}_p^*,\\ \mathbf{I}_n-(\mathbf{I}_n\times\mathbf{k})\tan\theta_n&=\mathbf{I}_n^*,\end{aligned}\tag{9}$$

in which $\mathbf{k}$ is a unit vector in the direction of $\mathbf{H}$ and

$$\begin{aligned}\mathbf{I}_p^*&\equiv\sigma_p\mathbf{E}-eD_p\,\mathrm{grad}p=\sigma_p[\mathbf{E}-(kT/e)\,\mathrm{grad}\ln p],\\ \mathbf{I}_n^*&\equiv\sigma_n\mathbf{E}+eD_n\,\mathrm{grad}n=\sigma_n[\mathbf{E}+(kT/e)\,\mathrm{grad}\ln n]\end{aligned}\tag{10}$$

are currents respectively proportional to $\boldsymbol{\mathcal{E}}_p$ and $\boldsymbol{\mathcal{E}}_n$. Equations (9) and (10) are illustrated for transport transverse to $\mathbf{H}$ by the vector diagrams of Fig. 4. These equations and diagrams apply with $\mathbf{I}_p^*$ and $\mathbf{I}_n^*$ given by Eqs. (10) since analogous equations and diagrams with the same Hall angles apply separately for the drift and diffusion contributions to $\mathbf{I}_p^*$ and $\mathbf{I}_n^*$. The electrostatic field $\mathbf{E}$ and concentration gradients need not, of course, be collinear vectors. Solving Eqs. (9) for $\mathbf{I}_p$ and $\mathbf{I}_n$ gives

$$\begin{aligned}\mathbf{I}_p&=\cos^2\theta_p\,[\mathbf{I}_p^*+\tan\theta_p\,(\mathbf{I}_p^*\times\mathbf{k})]+\sin^2\theta_p\,(\mathbf{I}_p^*\cdot\mathbf{k})\mathbf{k},\\ \mathbf{I}_n&=\cos^2\theta_n\,[\mathbf{I}_n^*+\tan\theta_n\,(\mathbf{I}_n^*\times\mathbf{k})]+\sin^2\theta_n\,(\mathbf{I}_n^*\cdot\mathbf{k})\mathbf{k}.\end{aligned}\tag{11}$$

The generalization for tensor ellipsoids of rotation about the magnetic field is readily effected: Eqs. (9) and Eqs. (11) apply with Eqs. (10) replaced by

$$\begin{aligned}\mathbf{I}_p^*\equiv{}&\sigma_{pt}[\mathbf{E}-(kT/e)\,\mathrm{grad}\ln p]\cdot[\mathbf{ii}+\mathbf{jj}]\\ &+\sigma_{pl}[\mathbf{E}-(kT/e)\,\mathrm{grad}\ln p]\cdot\mathbf{kk},\\ \mathbf{I}_n^*\equiv{}&\sigma_{nt}[\mathbf{E}+(kT/e)\,\mathrm{grad}\ln n]\cdot[\mathbf{ii}+\mathbf{jj}]\\ &+\sigma_{nl}[\mathbf{E}+(kT/e)\,\mathrm{grad}\ln n]\cdot\mathbf{kk}.\end{aligned}\tag{12}$$

For drift in a semiconductor with no added carriers, the conductivities σ_{pt} and σ_{nt} are the reciprocals of the transverse resistances, and σ_{pl} and σ_{nl} are the reciprocals of the longitudinal resistances, as may be verified by calculating the reciprocals of the conductivity tensors[28] or the hole and electron resistances themselves. The latter are defined in terms of the powers dissipated by the hole and electron drift currents, and equal

$$[\sigma_{pt}(E_x^2+E_y^2)\cos^2\theta_p+\sigma_{pl}E_z^2]/[\sigma_{pt}^2(E_x^2+E_y^2)\cos^2\theta_p+\sigma_{pl}^2E_z^2]$$

for holes; there is a similar expression for electrons.[29] The dependence on the Hall angles of the resistance $\mathbf{E}\cdot\mathbf{I}/I^2$, which is the sum of these partial resistances multiplied respectively by $(I_p/I)^2$ and $(I_n/I)^2$, indicates that there is magnetoresistance in general even for scalar conductivities σ_p and σ_n. Theory of the PME effect for this case of large Hall angles is straightforward but rather involved. It is, however, materially simplified, at least for an n-type semiconductor, if the mobility ratio is large so that hole transport can be neglected.

2.2 Differential Equations for Small Hall Angles

For small Hall angles, Eqs. (11) reduce to

$$\begin{aligned}\mathbf{I}_p&=\mathbf{I}_p^*+\theta_p\mathbf{I}_p^*\times\mathbf{k},\\ \mathbf{I}_n&=\mathbf{I}_n^*+\theta_n\mathbf{I}_n^*\times\mathbf{k},\end{aligned}\tag{13}$$

with $\mathbf{I}_p^*$ and $\mathbf{I}_n^*$ given by Eqs. (10).

Adding these equations gives

$$\begin{aligned}\mathbf{I}=\sigma\mathbf{E}+e(D_n-E_p)\,\mathrm{grad}\Delta p+(\theta_p\sigma_p+\theta_n\sigma_n)\mathbf{E}\times\mathbf{k}\\ -e(\theta_pD_p-\theta_nD_n)\,\mathrm{grad}\Delta p\times\mathbf{k},\end{aligned}\tag{14}$$

where the terms on the right-hand side represent the drift, Dember, Hall, and PME contributions, respectively. Solving this equation for $\mathbf{E}$ results in

$$\begin{aligned}\sigma\mathbf{E}=\mathbf{I}-e(D_n-D_p)\,\mathrm{grad}\Delta p-\sigma^{-1}(\theta_p\sigma_p+\theta_n\sigma_n)\mathbf{I}\times\mathbf{k}\\ +\theta eD\,\mathrm{grad}\Delta p\times\mathbf{k},\end{aligned}\tag{15}$$

if terms quadratic in Hall angles are neglected; three terms on the right-hand side represent the Dember, Hall, and PME contributions, respectively. Here

$$\theta\equiv\theta_p-\theta_n=\theta_p+|\theta_n|,\tag{16}$$

and D is the general ambipolar diffusivity for added carrier concentration[13]:

$$\begin{aligned}D\equiv\sigma^{-1}(\sigma_nD_p+\sigma_pD_n)=kT\mu_n\mu_p(n+p)/\sigma\\ =(n+p)/(n/D_p+p/D_n).\end{aligned}\tag{17}$$

Equation (15) shows drift and Dember fields to be the only ones that can be realized separately; $\mathbf{I}/\sigma$ is the drift field if the other terms are absent. If there are no added carriers in the presence of magnetic field, then drift and Hall terms occur, while if the PME effect obtains then all four terms are in general present. The

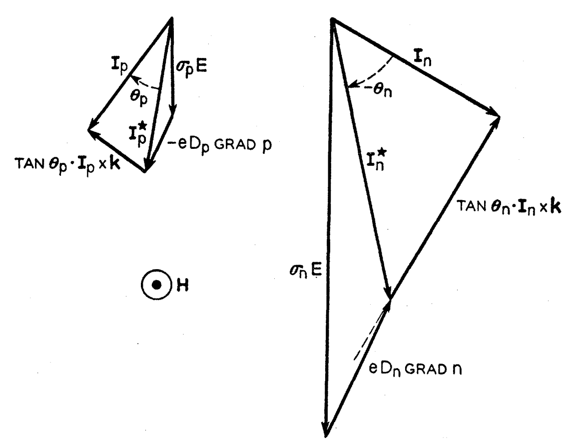

FIG. 4. The hole and electron currents.

[28] W. Shockley, reference 11, pp. 301–302.

[29] It is easily shown that the angle between the hole current $\mathbf{I}_p$ and $\mathbf{I}_p^*$ is $\sin^{-1}(\sin\theta_p\sin\Phi_p)$, where Φ_p is the angle between $\mathbf{I}_p^*$ and $\mathbf{H}$, and similarly for electrons. As may be expected, this angle equals the Hall angle θ_p for $\mathbf{I}_p^*$ and $\mathbf{H}$ perpendicular and vanishes for these vectors parallel.

coefficient of $\mathbf{I}\times\mathbf{k}$ in the Hall term of Eq. (15) is the negative of the Hall angle for total current density.[30] By substituting from this equation in Eqs. (13), the currents are obtained in ambipolar form as

$$\mathbf{I}_p=(\sigma_p/\sigma)\mathbf{I}+\mathbf{I}^{\equiv},\quad \mathbf{I}_n=(\sigma_n/\sigma)\mathbf{I}-\mathbf{I}^{\equiv}, \tag{18}$$

where

$$\mathbf{I}^{\equiv}\equiv -eD\ \mathrm{grad}\Delta p+\theta(\sigma_n\sigma_p/\sigma^2)\mathbf{I}\times\mathbf{k} - (\theta_p\sigma_n+\theta_n\sigma_p)(eD/\sigma)\ \mathrm{grad}\Delta p\times\mathbf{k}, \tag{19}$$

if terms quadratic in Hall angles are neglected.

The continuity equation for Δp is readily derived by noting that since $\mathbf{E}$ and $\mathrm{grad}\Delta p$ are lamellar, the equations

$$\begin{aligned}\mathrm{div}\mathbf{I}_p&=\mathrm{div}(\sigma_p\mathbf{E})-eD_p\ \mathrm{div\ grad}\Delta p+\theta_p[\![\mathrm{grad}\sigma_p,\mathbf{E},\mathbf{k}]\!]\\ &=-e(\partial\Delta p/\partial t+\Delta p/\tau)\\ &=-\mathrm{div}(\sigma_n\mathbf{E})-eD_n\ \mathrm{div\ grad}\Delta p\\ &\qquad -\theta_n[\![\mathrm{grad}\sigma_n,\mathbf{E},\mathbf{k}]\!]=-\mathrm{div}\mathbf{I}_n\end{aligned} \tag{20}$$

obtain, in which the heavy brackets denote scalar triple products. Multiplying respectively by σ_n and σ_p, adding, and simplifying gives

$$\partial\Delta p/\partial t=\mathrm{div}D\ \mathrm{grad}\Delta p-\mathbf{v}\cdot\mathrm{grad}\Delta p-\Delta p/\tau, \tag{21}$$

where, with $n_s\equiv n_0-p_0=n-p$,

$$\mathbf{v}\equiv(e\mu_n\mu_p n_s/\sigma)\mathbf{E}+\mathrm{grad}D + (\theta_p n-\theta_n p)(e\mu_n\mu_p/\sigma)\mathbf{E}\times\mathbf{k} \tag{22}$$

is the drift velocity for Δp. By use of Eq. (14), the equation

$$dD/d\Delta p=e^2\mu_n\mu_p(D_n-D_p)n_s/\sigma^2 \tag{23}$$

and the identity

$$\mu_n n^2+\mu_p p^2=\sigma^2/e^2(\mu_n+\mu_p)+\bar{\mu}n_s^2, \tag{24}$$

where $\bar{\mu}\equiv\mu_n\mu_p/(\mu_n+\mu_p)$, Eq. (22) may be written as[31]

$$\begin{aligned}\mathbf{v}&=(e\mu_n\mu_p n_s/\sigma^2)\mathbf{I}+\theta\bar{\mu}(1+e^2\mu_n\mu_p n_s^2/\sigma^2)\mathbf{E}\times\mathbf{k}\\ &=(e\mu_n\mu_p n_s/\sigma^2)\mathbf{I}+\theta(1+e^2\mu_n\mu_p n_s^2/\sigma^2)(\bar{\mu}/\sigma)\mathbf{I}\times\mathbf{k}.\end{aligned} \tag{25}$$

The second form applies, from Eq. (15), with the neglect of terms quadratic in Hall angles.

A complete formulation involves also Eq. (2) as a second differential equation which, from Eq. (14), may be written as

$$\mathrm{div}[\sigma\mathbf{E}+e(D_n-D_p)\ \mathrm{grad}\Delta p] +e(\theta_p\mu_p+\theta_n\mu_n)[\![\mathrm{grad}\Delta p,\mathbf{E},\mathbf{k}]\!]=0. \tag{26}$$

It is convenient to use, in addition to Δp, the potential

$$\psi\equiv V-[(b-1)/(b+1)](kT/e)\ln(\sigma/\sigma_0) \tag{27}$$

as second dependent variable, in terms of which the total current density may be written as

$$\mathbf{I}=-\sigma\ \mathrm{grad}\psi-(\theta_p\sigma_p+\theta_n\sigma_n)\ \mathrm{grad}\psi\times\mathbf{k} -\theta eD\ \mathrm{grad}\Delta p\times\mathbf{k}, \tag{28}$$

and the continuity equation and Eq. (26) as

$$\begin{aligned}&\mathrm{div}(D\ \mathrm{grad}\Delta p)+\bar{\mu}[n_s\ \mathrm{grad}\psi+(\theta_p n-\theta_n p)\ \mathrm{grad}\psi\times\mathbf{k}]\cdot\mathrm{grad}\ \ln\sigma-\Delta p/\tau=0,\\ &\mathrm{div\ grad}\psi+[\mathrm{grad}\psi+(\theta_p\mu_p+\theta_n\mu_n)\ \mathrm{grad}\psi\times\mathbf{k}/(\mu_n+\mu_p)]\cdot\mathrm{grad}\ \ln\sigma=0.\end{aligned} \tag{29}$$

Equations (29) are fundamental differential equations of the theory for small Hall angles. The condition that $\mathbf{E}$ be lamellar, Eq. (3), is implicit in the introduction of ψ. If the transport geometry is a simple one, it is often better not to solve for this potential, but to apply Eq. (3) directly; the continuity equation in the single dependent variable Δp is then the only differential equation that need be solved.

2.3 Extension for Holes of Different Mobilities

Various experiments on infrared absorption, magnetoresistance, Hall effect, and cyclotron resonance and their theoretical interpretation have indicated the presence in germanium of holes of essentially two effective masses.[32] The present treatment may easily be extended to take these into account. Denote by p_1 and p_2 the respective concentrations of slow and fast holes; by μ_{p1} and μ_{p2}, their drift mobilities; by $\sigma_{p1}=e\mu_{p1}p_1$ and $\sigma_{p2}=e\mu_{p2}p_2$, their contributions to conductivity; by D_{p1} and D_{p2}, their diffusion constants; and by θ_{p1} and θ_{p2}, their Hall angles for given H. The hole current densities for scalar conductivities and small Hall angles are then

$$\mathbf{I}_{pi}=\mathbf{I}_{pi}^*+\theta_{pi}\mathbf{I}_{pi}^*\times\mathbf{k};\quad i=1,2 \tag{30}$$

where

$$\begin{aligned}\mathbf{I}_{pi}^*&=\sigma_{pi}\mathbf{E}-eD_{pi}\ \mathrm{grad}p_i\\ &=\sigma_{pi}[\mathbf{E}-(kT/e)\ \mathrm{grad}\ \ln p_i];\quad i=1,2.\end{aligned} \tag{31}$$

The second equations of Eqs. (9) and (10) give the electron current density.

It will be assumed that the ratio of the hole concentrations is fixed, so that

$$p_2/p=r,\quad p_1/p=1-r;\quad p=p_1+p_2, \tag{32}$$

with r a constant. With this assumption and the neutrality condition, Eq. (1), the expression for $\mathbf{I}$ obtained by adding the three current-density equations may be solved for $\mathbf{E}$ in terms of $\mathbf{I}$ and Δp. If the defi-

[30] Suitably specialized, this angle is the one derived by W. Shockley, reference 11, pp. 215 ff.

[31] A term in $\mathrm{grad}\Delta p\times\mathbf{k}$ is deleted; expressions for $\mathbf{v}$ differing only by terms that do not contribute to $\mathbf{v}\cdot\mathrm{grad}\Delta p$ are considered equivalent.

[32] Much of this work has recently been summarized by C. Kittel and by A. C. Beer, Phys. Rev. 98, 1542 (1955). See also reference 27.

nitions

$$\mu_p \equiv (\sigma_{p1}+\sigma_{p2})/e(p_1+p_2) = (1-r)\mu_{p1}+r\mu_{p2}; \quad D_p \equiv kT\mu_p/e,$$

$$\theta_p \equiv (\sigma_{p1}\theta_{p1}+\sigma_{p2}\theta_{p2})/\sigma_p = [(1-r)\mu_{p1}\theta_{p1}+r\mu_{p2}\theta_{p2}]/\mu_p, \quad (33)$$

$$D_1 \equiv \sigma_{p1}D/\sigma_p = (1-r)\mu_{p1}D/\mu_p,$$

$$D_2 \equiv \sigma_{p2}D/\sigma_p = r\mu_{p2}D/\mu_p; \quad D_1+D_2=D$$

are employed, with $\sigma_{p1}+\sigma_{p2}=\sigma_p$, then, to the first order in Hall angles, $\mathbf{E}$ is given by Eq. (15) and the hole current densities in ambipolar form are

$$\mathbf{I}_{pi} = (\sigma_{pi}/\sigma)\mathbf{I} - eD_i \operatorname{grad}\Delta p + (\sigma_{pi}/\sigma^2)[(\theta_{pi}-\theta_n)\sigma_n + (\theta_{pi}-\theta_p)\sigma_p]\mathbf{I}\times\mathbf{k} - e(\sigma_{pi}/\sigma_p)(\theta_{pi}-\theta\sigma_p/\sigma)D \operatorname{grad}\Delta p\times\mathbf{k}; \quad i=1,2, \quad (34)$$

with the corresponding total hole current density and electron current density as previously given. Since it is implicit in Eqs. (32) that the lifetime τ applies to concentrations of added holes of either mobility, the definition of Eqs. (33) thus lead to Eqs. (21) and (25) for the continuity equation for added carriers; and Eqs. (29) apply as the fundamental differential equations.

In accordance with these considerations, all results of this paper apply for holes of two mobilities provided that where boundary conditions are involved the conditions hold with surface recombination velocities for these holes that are the same. By further extension of an obvious nature any discrete or continuous distributions of concentration ratio r for holes with respect to mobility can similarly be taken into account. Computation of θ shows that a small concentration of fast holes in germanium—a value of r of about 0.02, with μ_{p2}/μ_{p1} about 8, has been found to account for the dependence on H of the Hall effect and magnetoresistance[32]—contribute about 30% to PME current or voltage, the slow holes contributing only about two-thirds as much and the electrons 50%.

3. THE PME EFFECT IN AN INFINITE SLAB

3.1 Formulation

From the symmetry of the infinite slab, $\mathbf{I}$ equals $I_x\mathbf{i}$, with I_x a function of y, and is parallel to the slab surfaces; and $\operatorname{grad}\Delta p$ equals $(d\Delta p/dy)\mathbf{j}$ and is perpendicular to the surfaces; both are perpendicular to the magnetic field which is assumed parallel to the surfaces, as shown in Fig. 1. Thus, from Eqs. (15), (18), and (19), if I_{py} is the (scalar) hole current density across the slab,

$$E_x = \sigma^{-1}(I_x - \theta I_{py}) = \sigma^{-1}(I_x + \theta eDd\Delta p/dy) \quad (35)$$

holds to the first order in Hall angles, with E_x a constant, in accordance with Eq. (3); and[33,34] from Eqs. (21) and (25) the continuity equation is

$$\partial\Delta p/\partial t = d(Dd\Delta p/dy)/dy + \theta\bar{\mu}E_x(1+e^2\mu_n\mu_p n_s^2/\sigma^2)d\Delta p/dy - \Delta p/\tau. \quad (36)$$

For the theory of the Hall effect in the infinite slab and the magnetic rectifier, this equation (with the time derivative zero) applies in general without further material simplification.[35] For the PME effect, however, usually either short-circuit current or open-circuit voltage is of interest. For the former, the condition is $E_x=0$, for which the drift term does not occur in Eq. (36); for the latter, this term is of order θ^2 and may be neglected. From Eq. (35), the PME short-circuit current is

$$I^{(sc)} = \theta\int_{-y_0}^{y_0} I_{py}dy = -\theta e\int_{\Delta p_2}^{\Delta p_1} Dd\Delta p = -\theta eD_i[\Delta p_1 - \Delta p_2 - \tfrac{1}{2}(b-1)(b+1)^{-1}n_s \ln(\sigma_1/\sigma_2)] \quad (37)$$

per unit width along the magnetic field, where y_0 and $-y_0$ and subscripts 1 and 2 denote the illuminated and dark surfaces. The integrated form, in which $D_i \equiv 2D_nD_p/(D_n+D_p)$ is the diffusivity for intrinsic material,[36] is obtained by use of Eq. (17). The open-circuit condition is

$$\int_{-y_0}^{y_0} I_x dy = 0, \quad (38)$$

which, with Eqs. (35) and (37), gives E_x equal to the PME open-circuit field

$$E_x^{(oc)} = -I^{(sc)}/G, \quad G \equiv \int_{-y_0}^{y_0}\sigma dy, \quad (39)$$

and I_x equal to the open-circuit circulating current density[37]

$$I_x^{(oc)} = -\sigma I^{(sc)}/G - \theta eDd\Delta p/dy, \quad (40)$$

in which G is the conductance of the illuminated slab per unit width along the magnetic field. With the drift term of Eq. (36) for the open-circuit condition thus of

[33] In Eq. (26), the second differential equation, the scalar triple product vanishes for this geometry, and the equation merely provides (for small Hall angles) the Dember field E_y across the slab. See Sec. 3.5 and reference 34.

[34] The magnitude of the apparent Dember potential is reduced by an amount quadratic in (small) Hall angles since, from Eq. (14), E_y includes the term $\sigma^{-1}(\theta_p\sigma_p+\theta_n\sigma_n)E_x^{(oc)}$ for the open-circuit condition. This term represents a Hall field associated with the PME circulating current, or, more precisely stated, with the difference between the open-circuit and short-circuit current densities. See G. Groetzinger and J. Aron, Phys. Rev. **100**, 978 (1955).

[35] Equations (26) and (27) of Landauer and Swanson, reference 10, give the corresponding small-signal equation but without the drift term. This continuity equation is approximately valid provided the change in potential in a diffusion length associated with E_x is small compared with $kT/e\theta$.

[36] W. van Roosbroeck, Bell System Tech. J. **29**, 560 (1950); see also reference 13.

[37] The associated nonuniform magnetic field is in the direction of $\mathbf{H}$; its contribution is relatively small.

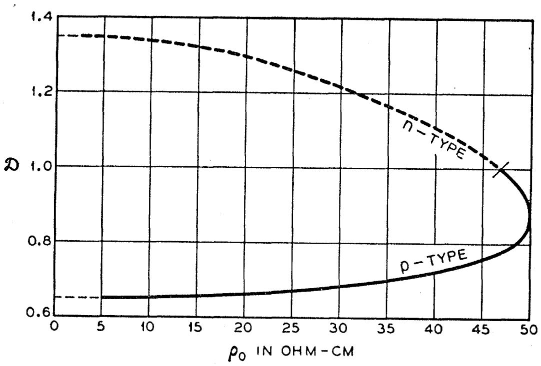

FIG. 5. The dependence of $\mathfrak{D}\equiv D_i/D_0$ on resistivity ρ_0 for germanium at 300°K.

order θ^2, the equation may be written as

$$d(Dd\Delta p/dy)/dy-\Delta p/\tau=0 \qquad (41)$$

for the steady state. With a theory based on Eq. (41), it is evidently better in principle to measure short-circuit current rather than open-circuit voltage. If both are measured, and the conductance without magnetic field as well, then consistency according to Eq. (39) is an indication of the validity of the assumption of small Hall angles.

Boundary conditions for the slab are

$$\begin{aligned} e\mathfrak{R}+I_{py}&=es_1\Delta p, \quad y=y_0,\\ I_{py}&=-es_2\Delta p, \quad y=-y_0, \end{aligned} \qquad (42)$$

where $\mathfrak{R}$ is the rate of generation of carrier pairs per unit area by strongly absorbed radiation, and s_1 and s_2 are surface recombination velocities. Note that in

$$I_{py}=-eDd\Delta p/dy-\theta(\sigma_n\sigma_p/\sigma^2)I_x, \qquad (43)$$

which follows from Eqs. (18) and (19), the second term may be neglected with I_x of order θ.

3.2 The Small-Signal Case

If the relative increase in local conductivity is everywhere small compared with unity, then D and σ may be replaced by their thermal equilibrium values D_0 and σ_0, and τ, s_1, and s_2 are also constants.[13] Equation (41), the continuity equation, and Eqs. (42), the boundary conditions, are then linear and give the solution

$$\Delta p=\frac{\mathfrak{R}L}{D_0}\frac{S_2\sinh(Y_0+Y)+\cosh(Y_0+Y)}{(1+S_1S_2)\sinh 2Y_0+(S_1+S_2)\cosh 2Y_0}, \qquad (44)$$

in which

$$\begin{aligned} &Y\equiv y/L, \quad Y_0\equiv y_0/L, \quad L\equiv(D_0\tau)^{\frac{1}{2}},\\ &S_1\equiv s_1L/D_0, \quad S_2\equiv s_2L/D_0. \end{aligned} \qquad (45)$$

The conductance increase above the thermal equilibrium value G_0 is

$$\Delta G\equiv G-G_0=e(\mu_n+\mu_p)\int_{-y_0}^{y_0}\Delta p dy=e(\mu_n+\mu_p)\tau\mathfrak{R} \times\frac{S_2(\cosh 2Y_0-1)+\sinh 2Y_0}{(1+S_1S_2)\sinh 2Y_0+(S_1+S_2)\cosh 2Y_0} \qquad (46)$$

per unit width along the magnetic field. Thus, from Eqs. (37), (39), (44), (45), and (46), the PME small-signal current and voltage are given by[38]

$$\begin{aligned} I^{(sc)}&=-G_0E_x^{(oc)}=-\theta eD_0(\Delta p_1-\Delta p_2)\\ &=-\theta e\mathfrak{R}L\frac{S_2\sinh 2Y_0+\cosh 2Y_0-1}{(1+S_1S_2)\sinh 2Y_0+(S_1+S_2)\cosh 2Y_0}\\ &=-\theta\frac{(D_0/\tau)^{\frac{1}{2}}}{\mu_n+\mu_p}\frac{S_2+\tanh Y_0}{S_2\tanh Y_0+1}\Delta G. \end{aligned} \qquad (47)$$

Note that the result in terms of ΔG does not depend explicitly on $\mathfrak{R}$ nor on s_1.

For $Y_0\gg1$, or the "thick" slab, Eqs. (47) reduce to

$$\begin{aligned} I^{(sc)}&=-G_0E_x^{(oc)}=-\theta e\mathfrak{R}L(S_1+1)^{-1}\\ &=-\theta(D_0/\tau)^{\frac{1}{2}}(\mu_n+\mu_p)^{-1}\Delta G, \end{aligned} \qquad (48)$$

a result useful for the determination of τ (and s_1 as well, if $\mathfrak{R}$ is known). For $Y_0\ll1$, or negligible volume recombination, the result

$$\begin{aligned} I^{(sc)}&=-\theta e\mathfrak{R}\cdot 2y_0S_2'(S_1'+S_2'+2S_1'S_2')^{-1}\\ &=-\theta(\mu_n+\mu_p)^{-1}s_2(1+s_2y_0/D_0)^{-1}\Delta G, \end{aligned} \qquad (49)$$

in which $S_1'\equiv s_1y_0/D_0$ and $S_2'\equiv s_2y_0/D_0$, may be used to determine s_2 (and s_1 as well, if $\mathfrak{R}$ is known). For s_2 large compared with s_1 and $D_0/\tau s_1$, lifetime τ may be found from

$$\begin{aligned} I^{(sc)}&=-\theta e\mathfrak{R}L(S_1+\coth 2Y_0)^{-1}\\ &=-\theta(D_0/\tau)^{\frac{1}{2}}(\mu_n+\mu_p)^{-1}\coth Y_0\,\Delta G. \end{aligned} \qquad (50)$$

For $s_1=s_2$, so that $S_1=S_2\equiv S$, the result in terms of $\mathfrak{R}$ assumes the form,

$$I^{(sc)}=-\theta e\mathfrak{R}L(S+\coth Y_0)^{-1}. \qquad (51)$$

3.3 The Intrinsic and Large-Signal Cases

Provided s_1, s_2, and τ are constant,[39] the continuity equation and boundary conditions that apply in general to an intrinsic semiconductor are, since D has the constant value D_i, those of the small-signal case. These equations apply also to an extrinsic semiconductor for

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[38] By specializing these results (to the case of Hall and drift mobilities of electrons and holes all equal and D_0 equal to the limiting value D_p or D_n for extrinsic material), they can be shown to agree with the PME voltage of H. Bulliard, reference 3, specialized to the small-signal, small-Hall-angle case.

[39] Other volume recombination laws may obtain in some cases, such as the mass-action law or one of the type discussed by W. Shockley and W. T. Read, Jr., Phys. Rev. 87, 835 (1952).

relative increases in local conductivity everywhere large compared with unity. Thus, with D_0 construed as D_i wherever it occurs, Δp is given by Eqs. (44) and (45), and from Eqs. (46) and

$$I^{(sc)}=-GE_x^{(oc)}=-\theta e D_i(\Delta p_1-\Delta p_2), \quad (52)$$

$I^{(sc)}$ is given by Eqs. (47) and $E_x^{(oc)}$ by

$$E_x^{(oc)}=\theta\left(\frac{e\mathcal{R}L}{G_0}\right)\times\frac{S_2\sinh 2Y_0+\cosh 2Y_0-1}{(1+S_1S_2)\sinh 2Y_0+(S_1+S_2)\cosh 2Y_0+e(\mu_n+\mu_p)\tau[S_2(\cosh 2Y_0-1)+\sinh 2Y_0]\mathcal{R}/G_0}. \quad (53)$$

The approximate form of Eq. (53) for $G\sim\Delta G\gg G_0$ is

$$E_x^{(oc)}=\theta\frac{(D_0/\tau)^{\frac{1}{2}}}{\mu_n+\mu_p}\frac{S_2+\tanh Y_0}{S_2\tanh Y_0+1}, \quad (54)$$

which shows that $E_x^{(oc)}$ "saturates" for constant τ in the large-signal cases, $I^{(sc)}$ being proportional to G. These results for $E_x^{(oc)}$ are consistent with theory previously given for the dependence on light intensity in which constant D is assumed and which is, thus, strictly speaking, theory for the intrinsic case.[38]

3.4 Cases of Arbitrary Light Intensity

The nonlinear continuity equation and boundary conditions for the slab, Eqs. (41) and (42), will be considered for τ constant. They are advantageously expressed in dimensionless form as

$$\frac{d}{dY}\frac{1+\mathfrak{D}\Delta P}{1+\Delta P}\frac{d\Delta P}{dY}-\Delta P=0, \quad (55)$$

and

$$\mathfrak{L}-\frac{1+\mathfrak{D}\Delta P}{1+\Delta P}\frac{d\Delta P}{dY}=S_1\Delta P, \quad Y=Y_0,$$
$$\frac{1+\mathfrak{D}\Delta P}{1+\Delta P}\frac{d\Delta P}{dY}=S_2\Delta P, \quad Y=-Y_0, \quad (56)$$

where[40]

$$\Delta P\equiv\Delta p/m, \quad m\equiv\sigma_0/e(\mu_n+\mu_p),$$
$$\mathfrak{L}\equiv e(\mu_n+\mu_p)L\mathcal{R}/\sigma_0 D_0=e(\mu_n+\mu_p)\mathcal{R}/\sigma_0(D_0/\tau)^{\frac{1}{2}}, \quad (57)$$
$$\mathfrak{D}\equiv D_i/D_0=2m/(n_0+p_0)=2\sigma_0/e(\mu_n+\mu_p)(n_0+p_0),$$

the other quantities that occur being defined by Eqs. (45). Note that the dimensionless added carrier concentration ΔP is the relative increase in local conductivity, $\Delta\sigma/\sigma_0$, and that

$$D/D_0=(1+\mathfrak{D}\Delta P)/(1+\Delta P). \quad (58)$$

In Fig. 5, $\mathfrak{D}$ is plotted against $\rho_0=\sigma_0^{-1}$ for[41] germanium at 300°K.

Dimensionless hole current across the slab is

$$C_{py}\equiv\left(\frac{L}{emD_0}\right)I_{py}=(\mu_n+\mu_p)\left(\frac{I_{py}}{\sigma_0}\right)\left(\frac{D_0}{\tau}\right)^{-\frac{1}{2}}=-\frac{1+\mathfrak{D}\Delta P}{1+\Delta P}\frac{d\Delta P}{dY}. \quad (59)$$

This quantity may be expressed in terms of ΔP; it is readily verified that a first integral of Eq. (55) is

$$\frac{1+\mathfrak{D}\Delta P}{1+\Delta P}\frac{d\Delta P}{dY}=[A+2(\mathfrak{D}-1)[\ln(1+\Delta P)-\Delta P]+\mathfrak{D}\Delta P^2]^{\frac{1}{2}}, \quad (60)$$

in which A is a constant of integration. Three equations that determine A and the dimensionless surface concentrations ΔP_1 and ΔP_2 in terms of $\mathfrak{L}$, $\mathfrak{D}$, Y_0, S_1, and S_2 are thus

$$\mathfrak{L}-S_1\Delta P_1=[A+2(\mathfrak{D}-1)[\ln(1+\Delta P_1)-\Delta P_1]+\mathfrak{D}\Delta P_1^2]^{\frac{1}{2}},$$
$$S_2\Delta P_2=[A+2(\mathfrak{D}-1)[\ln(1+\Delta P_2)-\Delta P_2]+\mathfrak{D}\Delta P_2^2]^{\frac{1}{2}}, \quad (61)$$

and

$$\int_{\Delta P_2}^{\Delta P_1}\frac{(1+\mathfrak{D}\Delta P)d\Delta P}{(1+\Delta P)[A+2(\mathfrak{D}-1)[\ln(1+\Delta P)-\Delta P]+\mathfrak{D}\Delta P^2]^{\frac{1}{2}}}=2Y_0, \quad (62)$$

the first two being the boundary conditions and the third following directly from Eq. (60). From Eqs. (37), (39), (57), and (58), dimensionless PME short-circuit current is

$$\mathfrak{I}\equiv-(\mu_n+\mu_p)I^{(sc)}/\theta\sigma_0 D_0=(\mu_n+\mu_p)GE_x^{(oc)}/\theta\sigma_0 D_0=\mathfrak{D}(\Delta P_1-\Delta P_2)-(\mathfrak{D}-1)\ln[(1+\Delta P_1)/(1+\Delta P_2)]. \quad (63)$$

From Eqs. (46) and (57) the relative conductance increase is given by

$$\Delta G/G_0=(2Y_0)^{-1}(\mathfrak{L}-S_1\Delta P_1-S_2\Delta P_2), \quad (64)$$

which, from Eqs. (59), (60), and (61), is proportional to the difference between the magnitudes of the hole currents at the surfaces. It is readily seen from Eqs. (61) to (64) that $\mathfrak{I}$ may be obtained (in general by numerical calculation) as a function of $\Delta G/G_0$ with $\mathfrak{D}$, $2Y_0$, and S_2 as parameters and no explicit dependence on $\mathfrak{L}$ and S_1. These various relationships can be shown to provide the results derived directly for the small-signal case.

[40] Use of the concentration unit m seems best in the present connection; with n_s, two parameters, including the mobility ratio b, occur explicitly.

[41] Values of n_0p_0 and drift mobilities are employed as in reference 13; these mobilities obtain provided σ_0 is not too large. In the notation of this reference, $\mathfrak{D}=1+[(b-1)/(b+1)]\tanh W_0$.

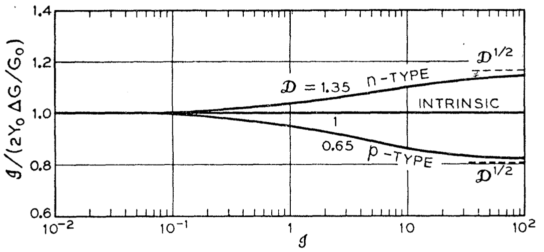

FIG. 6. Dependence of $\mathcal{J}/(2Y_0\Delta G/G_0)$ on $\mathcal{J}$ for thick germanium slabs at 300°K. The dimensionless PME current-photoconductance ratio $\mathcal{J}/(\Delta G/G_0)$ equals the number $2Y_0$ of diffusion lengths in the slab thickness for the small-signal and intrinsic cases and $\mathfrak{D}^{\frac{1}{2}}$ times this for the limiting large-signal case.

3.41 The Thick Slab

Considerable simplification results if the slab is a number of diffusion lengths thick, so that the added carrier concentration and hole flow density at the dark surface are much smaller than at the illuminated surface. The idealization is the slab of infinite thickness, for which ΔP_2 and A are zero; the thickness integral of Eq. (62) then of course diverges, and, from the first of Eqs. (61), the boundary condition for the illuminated surface may be written as

$$(\mathcal{L}-S_1\Delta P_1)^2 = 2(\mathfrak{D}-1)[\ln(1+\Delta P_1)-\Delta P_1]+\mathfrak{D}\Delta P_1^2. \quad (65)$$

Equations (63) and (64) reduce to

$$\mathcal{J}=\mathfrak{D}\Delta P_1-(\mathfrak{D}-1)\ln(1+\Delta P_1) \quad (66)$$

and

$$\Delta G/G_0=(2Y_0)^{-1}(\mathcal{L}-S_1\Delta P_1) = \tau(2y_0m)^{-1}(\mathcal{R}-s_1\Delta p_1). \quad (67)$$

From Eq. (66), $\mathcal{J}$ depends for given $\mathfrak{D}$ only on ΔP_1, and from Eqs. (65) and (67), the quantity $2Y_0\Delta G/G_0$ does likewise. In Fig. 6, the ratio $\mathcal{J}/(2Y_0\Delta G/G_0)$ is plotted against $\mathcal{J}$ for the three values of $\mathfrak{D}$ that correspond to intrinsic and (sufficiently strongly extrinsic[42]) n- and p-type germanium. For intrinsic material, for which $\mathfrak{D}=1$, the ratio is unity, as it is in the small-signal limit, D being D_i and independent of Δp. Analysis by means of these curves of measurements of $\Delta G/G_0$ and $I^{(sc)}$, the latter providing $\mathcal{J}$, determines $2Y_0$ and hence τ. As the curves show, τ may be obtained from small-signal measurements by means of the relationship $2Y_0=\mathcal{J}/(\Delta G/G_0)$ or

$$\tau=D_0[\theta\Delta G/(\mu_n+\mu_p)I^{(sc)}]^2. \quad (68)$$

Agreement between theory and experiment over an extended range in $\mathcal{J}$ would show that the assumption of constant τ is valid. If $\mathcal{R}$ is also known and Δp_1 is evaluated in terms of $\mathcal{J}$ from Eq. (66), then s_1 may be found by use of the second form of Eq. (67).

3.42 Method of the High-Recombination-Velocity Dark Surface

The method of the thick slab has the practical limitation that in general it does not apply unless $2Y_0$ is sufficiently large, while large $2Y_0$ makes for small $\Delta G/G_0$ which may be difficult to measure accurately. This limitation is largely obviated by a method employing slabs in which $2Y_0$ is not large and in which large s_2 has been produced, as by sandblasting the dark surface. The condition of negligible ΔP_2 may thus be realized substantially independently of the value of $2Y_0$. For this method, Eq. (66) applies as well as Eq. (62) with the lower limit of the integral set equal to zero. This integral converges since A, from Eqs. (59) and (60) and the second of Eqs. (61), equals $(S_2\Delta P_2)^2$ or C_{py}^2 for the dark surface and is not zero.

Equations (61) and (64) give

$$2Y_0\Delta G/G_0=[A+2(\mathfrak{D}-1)[\ln(1+\Delta P_1) -\Delta P_1]+\mathfrak{D}\Delta P_1^2]^{\frac{1}{2}}-A^{\frac{1}{2}}, \quad (69)$$

so that, with Eq. (66), $2Y_0\Delta G/G_0$ depends for given $\mathfrak{D}$ on A and $\mathcal{J}$. By numerical integration of the thickness integral, $2Y_0$ may also be obtained for given $\mathfrak{D}$ in terms of A and $\mathcal{J}$. Eliminating A furnishes a relationship between $2Y_0$, $2Y_0\Delta G/G_0$, and $\mathcal{J}$ from which a family of curves of $2Y_0$ *versus* $\mathcal{J}/(\Delta G/G_0)$ with $\mathcal{J}$ as parameter may be obtained. Figure 7 gives such families of curves for (sufficiently strongly extrinsic[42]) n- and p-type germanium. The dashed curves for zero and infinite $\mathcal{J}$ correspond to $\mathcal{J}/(\Delta G/G_0)$ equal to $2Y_0\coth Y_0$ and to $\mathfrak{D}^{\frac{1}{2}}\cdot 2Y_0\coth(\mathfrak{D}^{-\frac{1}{2}}Y_0)$ for the small- and large-signal cases, respectively.[43] The value of $2Y_0$ determines τ, and if $\mathcal{R}$ is known as well then s_1 may be found as for the thick slab.

Use of a dark surface with negligible recombination velocity s_2 would provide in principle a more sensitive measure of slight volume recombination. In practice, though, it may be difficult to secure sufficiently small s_2. The small-signal $\mathcal{J}/(\Delta G/G_0)$ for no volume recombination and $s_2\neq 0$, namely $2/(1+D_0/s_2y_0)$, would result in a fictitious $2Y_0$ in accordance with the relationship $\mathcal{J}/(\Delta G/G_0)=2Y_0\tanh Y_0$ that applies for $s_2=0$ in the small-signal and intrinsic cases; and the condition that this fictitious $2Y_0$ be small compared with the true one is $s_2\ll y_0/\tau$ for slight volume recombination. If y_0 is $10s_2\tau$, say, and s_2 is 50 cm sec^{-1}, then $2y_0$ is appreciably less than a diffusion length L if, for n-type germanium at 300°K, L and τ are appreciably less than about 0.05 cm and 50 μsec. These are conditions under which use of the small s_2 could confer some advantage; Fig. 7 indicates that if $2y_0$ is of order L, then the method of large s_2 should provide good accuracy.

[42] In the notation of reference 13, the approximate condition is $|W_0|>3$.

[43] Note that the small-signal relationship applies for all $\mathcal{J}$ to intrinsic material.

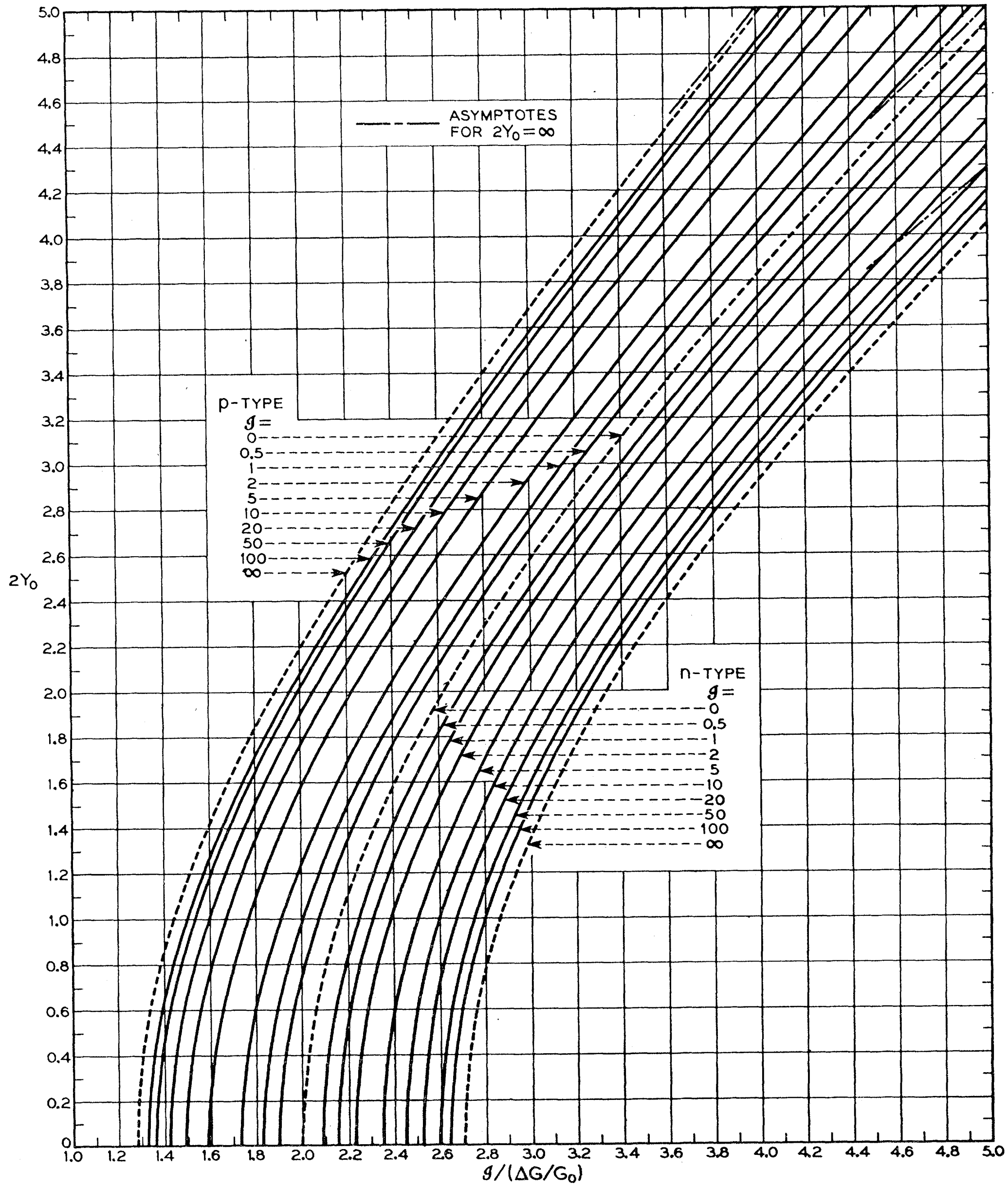

FIG. 7. Dependence of the number $2Y_0$ of diffusion lengths in the slab thickness on the dimensionless PME current-photoconductance ratio $\mathcal{G}/(\Delta G/G_0)$ for germanium slabs at 300°K with high-recombination velocity dark surfaces.

3.43 Negligible Volume Recombination

For slab thickness small compared with a diffusion length, the condition of constant I_{py} represents a first integral of the continuity equation. Thus, C_{py}' given by

$$C_{py}'\equiv(y_0/emD_0)I_{py}=(\mu_n+\mu_p)y_0I_{py}/\sigma_0D_0$$
$$=-\frac{1+\mathfrak{D}\Delta P}{1+\Delta P}\frac{d\Delta P}{dY'},\quad Y'\equiv y/y_0 \tag{70}$$

is constant; primes here distinguish dimensionless quantities based on y_0 as length unit rather than L.

The equation

$$(\mathfrak{D}-1)\ln[(1+\Delta P_1)/(1+\Delta P_2)]$$
$$-\mathfrak{D}(\Delta P_1-\Delta P_2)=2C_{py}' \tag{71}$$

is obtained by integrating Eq. (70), writing the result for the respective surfaces, $Y'=\pm1$, and subtracting. The boundary conditions, Eqs. (42), are

$$\mathcal{L}'+C_{py}'=S_1'\Delta P_1,$$
$$C_{py}'=-S_2'\Delta P_2, \tag{72}$$

where

$$\mathcal{L}'\equiv(y_0/mD_0)\mathcal{R}=e(\mu_n+\mu_p)y_0\mathcal{R}/\sigma_0D_0;$$
$$S_1'\equiv s_1y_0/D_0,\quad S_2'\equiv s_2y_0/D_0. \tag{73}$$

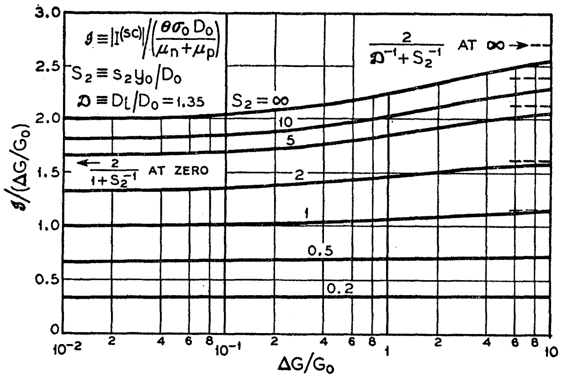

FIG. 8. Dependence of the dimensionless PME current-photoconductance ratio $\mathcal{J}/(\Delta G/G_0)$ on the relative conductance increase $\Delta G/G_0$ for n-type germanium slabs at 300°K with no volume recombination.

With constant I_{py}, the PME short-circuit current is given simply by

$$\mathcal{J}=-2C_{py}'=2S_2'\Delta P_2, \tag{74}$$

from Eqs. (37), (63), (70), and (72). Eliminating C_{py}' from Eqs. (71) and (72) results in the equations

$$(\mathfrak{D}-1)\ln[(1+\Delta P_1)/(1+\Delta P_2)]-\mathfrak{D}(\Delta P_1-\Delta P_2)=-2S_2'\Delta P_2=-\mathcal{J} \tag{75}$$

and

$$S_1'\Delta P_1+S_2'\Delta P_2=\mathcal{L}', \tag{76}$$

which determine ΔP_1 and ΔP_2. The relative conductance increase is

$$\begin{aligned}\Delta G/G_0&=\tfrac{1}{2}\int_{-1}^{1}\Delta P dY'\\&=-(2C_{py})^{-1}\int_{\Delta P_2}^{\Delta P_1}\Delta P(1+\mathfrak{D}\Delta P)(1+\Delta P)^{-1}d\Delta P\\&=\mathcal{J}^{-1}[(\mathfrak{D}-1)(\ln[(1+\Delta P_1)/(1+\Delta P_2)]\\&\quad-\Delta P_1+\Delta P_2)+\tfrac{1}{2}\mathfrak{D}(\Delta P_1^2-\Delta P_2^2)]\\&=\tfrac{1}{2}\mathfrak{D}\mathcal{J}^{-1}(\Delta P_1-\Delta P_2)(\Delta P_1+\Delta P_2+2/\mathfrak{D})-1,\end{aligned} \tag{77}$$

obtained by use of Eqs. (70), (74), and (75).

To facilitate numerical computation, Eq. (75) may be written as

$$\begin{aligned}&\ln(1+\Delta P_1)=[\mathfrak{D}/(\mathfrak{D}-1)]\Delta P_1-K,\\&K\equiv(\mathfrak{D}+2S_2')(\mathfrak{D}-1)^{-1}\Delta P_2-\ln(1+\Delta P_2).\end{aligned} \tag{78}$$

Then, with $\mathfrak{D}$ known, values may be assigned to S_2' as parameter and to ΔP_2 which determine, besides $\mathcal{J}$, also K and thus ΔP_1 as root of Eq. (78). To compute $\Delta G/G_0$ from Eq. (77) with reasonable accuracy, this root must be evaluated to a number of significant figures. Figures 8 and 9 give $\mathcal{J}/(\Delta G/G_0)$ *versus* $\Delta G/G_0$ so obtained for values of S_2' as parameter for (sufficiently strongly extrinsic[42]) n- and p-type germanium.

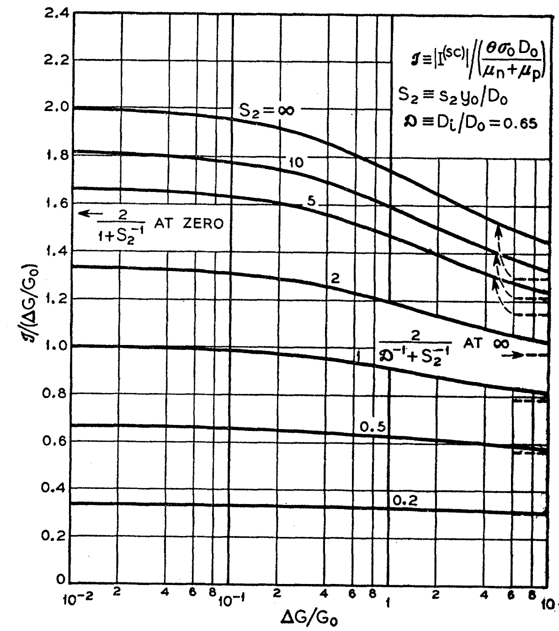

FIG. 9. Dependence of the dimensionless PME current-photoconductance ratio $\mathcal{J}/(\Delta G/G_0)$ on the relative conductance increase $\Delta G/G_0$ for p-type germanium slabs at 300°K with no volume recombination.

With this method, volume recombination, which increases $\mathcal{J}/(\Delta G/G_0)$, would give an apparent s_2 larger than the true one. It can be shown that the correction $\delta\mathcal{J}/(\Delta G/G_0)$ at fixed $\Delta G/G_0$ for slight volume recombination is $[2-4(1+3/2S_2')/3(1+1/S_2')^2]Y_0^2$ for the small-signal and intrinsic cases.

3.5 The Open-Circuit Equipotentials

The equipotential surfaces shown in Fig. 1 for the infinite slab are specified by

$$\Delta x=\int_{-y_0}^{y_0}\cot\varphi dy,\quad \cot\varphi=-E_y/E_x, \tag{79}$$

which relates to the y coordinate of a point on an equipotential its x distance from the intersection with the dark surface. For the open-circuit condition, the constant field E_x is $E_x^{(oc)}$, which is of the first order in Hall angles. It follows then from Eq. (14) that, to the same approximation, E_y is the Dember field,[34]

$$E_y=-(e/\sigma)(D_n-D_p)d\Delta p/dy. \tag{80}$$

The integral obtained by substituting for E_y in Eq. (79) is easily evaluated. For open circuit, it gives

$$\begin{aligned}\Delta x&=(kT/e)(b-1)(b+1)^{-1}[\ln(\sigma/\sigma_2)]/E_x^{(oc)}\\&=\tfrac{1}{2}[(\mu_p^{-1}-\mu_n^{-1})G\ln(\sigma/\sigma_2)]/\theta e[\Delta p_1-\Delta p_2\\&\quad-\tfrac{1}{2}(b-1)(b+1)^{-1}n_s\ln(\sigma_1/\sigma_2)],\end{aligned} \tag{81}$$

obtained by use of Einstein's relation and Eqs. (37) and (39).

The equipotentials for the small-signal case are given by

$$\Delta x = \frac{(D_n - D_p)\cdot 2y_0}{\theta D_0}\frac{\Delta p - \Delta p_2}{\Delta p_1 - \Delta p_2}, \tag{82}$$

as obtained by expanding the logarithms. This result may be written explicitly in terms of y by use of the appropriate expressions for Δp and $E_x^{(oc)}$. From it, the distance along the slab between intersections of an equipotential with the surfaces is given by

$$\Delta x = (D_n - D_p)\cdot 2y_0/\theta D_0$$

for the small-signal case. This Δx is a direct measure of θ. It is large compared with the thickness $2y_0$ of the slab, since θ has been assumed small.

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Hyperfine Structure and Nuclear Moments of Gadolinium*

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Study of the optical hyperfine structure of several Gd I lines using enriched isotopes shows that the spins of the odd isotopes Gd^{155} and Gd^{157} each are $\frac{3}{2}$. The ratio of the magnetic moments is $\mu_I(Gd^{155})/\mu_I(Gd^{157}) = 0.80 \pm 0.02$. The magnetic moment for Gd^{157} obtained from two lines is -0.37 ± 0.04 nm. Deviations from the interval rule in these two lines can be accounted for with a quadrupole moment of approximately 1.0×10^{-24} cm² for Gd^{157} and 1.1×10^{-24} cm² for Gd^{155}. The known anomolous isotope shift between neutron numbers 88 and 90 (Gd^{152} and Gd^{154}) is accurately measured for several lines.

INTRODUCTION

SPECTROSCOPIC measurements of the nuclear moments of the heavier odd-neutron nuclei have been hampered by the presence in the corresponding elements of several even-even isotopes. With the availability of enriched samples, however, it has become possible to add to the rather scanty data on such nuclides for comparison with predictions of the unified shell model developed by Bohr[1] and by Bohr and Mottelson.[2] The recent work of Mottelson and Nilsson[3] shows that marked deviations are to be expected from the moments derived from the single-particle model in those regions of the periodic table where the nuclear deformations are large. Thus whereas a spin $I=7/2$ was assigned by Klinkenberg[4] to the odd isotopes $_{64}Gd^{155}$ and $_{64}Gd^{157}$ on the basis of the shell model, it appears that if the nuclei are sufficiently deformed the lowest level should be either $3/2^-$ or $5/2^+$.[5]

Previous studies[6–9] with natural gadolinium have been chiefly concerned with the isotope shifts, and Murakawa[8] has confirmed the anomalously large shift between isotopes 152 and 154 that was expected by analogy with Nd and Sm. This author, and somewhat earlier Suwa,[9] attempted to draw conclusions about the spins and magnetic moments of the odd isotopes of Gd from the unresolved structure underlying the strong components due to the even isotopes. Both investigators concluded that the spins were probably greater than 3/2, and Murakawa assumed the value 7/2 in estimating the magnetic moments. It is therefore apparent that further study of the hyperfine structure of Gd with separated isotopes was needed. The very complete classification of the lines of both Gd I and Gd II by Russell[10] makes it possible to derive the nuclear moments from the observed splittings.

The composition of the four samples of enriched isotopes used in the present work, as compared with that of natural Gd, is shown in Table I.

* Work supported by the National Science Foundation.

[1] A. Bohr, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **26**, No. 14 (1952).
[2] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab Mat.-fys. Medd. **27**, No. 16 (1953).
[3] B. R. Mottelson and S. G. Nilsson, Phys. Rev. **99**, 1615 (1955).
[4] P. F. A. Klinkenberg, Revs. Modern Phys. **24**, 63 (1952).
[5] Reference 3, Fig. 2, p. 1616.
[6] P. F. A. Klinkenberg, Physica **12**, 33 (1946).
[7] P. Brix and H. D. Engler, Z. Physik **133**, 362 (1925).
[8] K. Murakawa, Phys. Rev. **96**, 1543 (1954).
[9] S. Suwa, J. Phys. Soc. (Japan) **8**, 377 (1953) and Phys. Rev. **86**, 247 (1952).
[10] H. N. Russell, J. Opt. Soc. Am. **40**, 550 (1950).