In (3) G is the conductance function[3] defined in A by

$$G=\pi\omega\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\int_0^\infty \rho(E_R+\hbar\omega) \times|\langle E_R|Q|E_R+\hbar\omega\rangle|^2\rho(E_R)f(E_R)dE_R. \quad (4)$$

Expressions (3) and (4) are exact for the Hamiltonian (1), and describe the approach to equilibrium of an oscillator which is coupled to a resistance in equilibrium with a heat bath at temperature T, and which has energy U_0 at time $t=0$.

It remains to be shown that expression (4) is the same as the classical frequency dependent conductance measured by the response of the conductance to harmonic driving forces. For an ensemble of oscillators consisting of an inductance L, capacity C, and conductance $G_c(\omega)$, the classical expression for the energy U as a function of time is

$$U=U_0\exp\left[-\frac{2\,\mathrm{Re}[G_c(\omega_1)]}{C(1+\omega^2/|\omega_1|^2)}t\right]. \quad (5)$$

In (5), ω_1 is the complex natural frequency of the damped oscillator, ω is the natural frequency of the undamped oscillator, and $\mathrm{Re}G_c(\omega_1)$ is the real part of the conductance G_c evaluated for the complex ω_1. We wish to show that $G_c(\omega)=G(\omega)$, where $G(\omega)$ is defined by (4). Comparing (3) with (5), we see that

$$\frac{2\,\mathrm{Re}G_c(\omega_1)}{1+\omega^2/|\omega_1|^2}=G(\omega). \quad (6)$$

Now in (5) imagine C to become very large and L to become very small such that the undamped frequency ω remains the same. Then it is clear that $\omega_1\to\omega$. Since $G_c(\omega)$ is real, it then follows from (6) that

$$G_c(\omega)=G(\omega). \quad (7)$$

[3] The conductance is the reciprocal of the resistance if there is no series reactance associated with the resistance.

PHYSICAL REVIEW VOLUME 101, NUMBER 6 MARCH 15, 1956

Fluctuation Dissipation Theorem*

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(Received September 29, 1955)

The fluctuation dissipation theorem (Nyquist formula) is shown to be exact and a number of generalizations of it are given, including a four-dimensional formulation which is useful in the quantum theory of fields. The more general theorem can be used to calculate vacuum expectation values of field operators, and to deduce covariant commutation relations for the fields. For a four-potential field, the vacuum expectation values for operators at two space-time points $\mathbf{x}$ and $\mathbf{x}'$ are

$$\langle A_\mu(\mathbf{x})A_\alpha(\mathbf{x}')\rangle_0=\frac{\hbar}{\pi}\int_0^\infty\frac{d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega)}{\omega}d\omega,$$

where $d_{\mu\alpha}$ is a dissipation tensor.

The covariant commutation relations are

$$[A_\mu(\mathbf{x}),A_\alpha(\mathbf{x}')]=\frac{\hbar}{\pi}\int_0^\infty\frac{[d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega)-d_{\alpha\mu}([\mathbf{x}'-\mathbf{x}],\omega)]}{\omega}d\omega.$$

A well-defined cut-off procedure is given for calculating observable fluctuations in cases where the theorem gives infinite results.

For measurements with a linear device which has energy $E=\hbar\omega_c$, the observable fluctuations of the vacuum electromagnetic fields are given by the exact expression

$$\langle V^2\rangle=\frac{\hbar}{\pi}\int_0^{\omega_c}R(\omega)\omega d\omega.$$

INTRODUCTION

NYQUIST[1] first deduced the fluctuation dissipation theorem, using classical statistics. His work was prompted by Johnson's experiments on electrical noise. The Nyquist formula, applicable to linear electrical networks, states that the mean squared fluctuation voltage in an angular frequency interval $d\omega$ is given by

$$\langle V^2\rangle=2kTR(\omega)d\omega/\pi. \quad (1)$$

In (1), $R(\omega)$ is the real part of the impedance function.

Later Callen and Welton[2] gave a quantum-theoretical deduction of (1) and showed that it was applicable to a

* This work was started at the University of Maryland under Office of Naval Research sponsorship.
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[1] H. Nyquist, Phys. Rev. **32**, 110 (1928).
[2] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).

wider class of phenomena than electrical noise alone. They applied it to Brownian motion and pressure fluctuations in a gas. Their expression for the mean squared force fluctuations is

$$\langle V^2\rangle=\frac{2}{\pi}\int_0^\infty R(\omega)\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]d\omega. \quad (2)$$

They deduced (2) using first-order perturbation theory. In this paper we will show that (2) is exact for systems which meet a certain definition of linearity, and we will discuss certain useful generalizations and other forms of this equation. These generalizations enable the theorem to be used for calculating expectation values in quantum theory, including field theory. Covariant commutation laws for the fields are also deduced from these generalizations.

EXACTNESS OF THE FLUCTUATION DISSIPATION THEOREM

In order to show that the more general expression (2) is exact, we employ a number of relations which have already been deduced by Callen and Welton.[2] The following expressions (3), (4), (5), (6), (7), and (9) are a partial summary of their paper.

We assume that a system which may be driven by external forces has a Hamiltonian of the form

$$\mathcal{H}=H_0+V_eQ. \quad (3)$$

H_0 is the Hamiltonian in the absence of driving forces, V_e represents an external driving "force," and Q is a function of the coordinates and momenta of the system. The impedance function $Z(\omega)$ is defined to be the ratio of the Fourier transforms of V_e and $\dot{Q}$ and the resistance function $R(\omega)$ is the real part. The admittance function $Y(\omega)$ is the reciprocal of the impedance function and the conductance function $G(\omega)$ is the real part of the admittance function.

First we calculate the ensemble average of the quantity $\dot{Q}^2$, in the absence of external driving forces. The Hamiltonian is H_0 and for the nth eigenstate we write $\langle\dot{Q}^2\rangle_{nn}=\sum_m\langle\dot{Q}\rangle_{nm}\langle\dot{Q}\rangle_{mn}$. We replace the summation over m by an integral and take the ensemble average. Let $\rho(E)$ be the density in energy of quantum states in the vicinity of E and let $F(E)$ be a statistical weighting factor. Then the ensemble average of $\dot{Q}^2$ is given by

$$\langle\dot{Q}^2\rangle=\int_0^\infty\hbar\omega^2\left[\int_0^\infty\rho(E)F(E)\times\{|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)+|\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)\}dE\right]d\omega. \quad (4)$$

We can make use of the definition of the impedance function Z, to deduce, from (4) an expression for the ensemble average of the force fluctuations V^2, again in the absence of external driving forces. The result is

$$\langle V^2\rangle=\int_0^\infty|Z|^2\hbar\omega^2\left[\int_0^\infty\rho(E)F(E)\times\{|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)+|\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)\}dE\right]d\omega. \quad (5)$$

We make use of the relation $F(E+\hbar\omega)/F(E)=\exp(-\hbar\omega/kT)$, and make a change of variable in the second integral of expressions (4) and (5). We can then write for $\langle\dot{Q}^2\rangle$

$$\langle\dot{Q}^2\rangle=\int_0^\infty\hbar\omega^2\left[1+\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\times\int_0^\infty|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)\times\rho(E)F(E)dEd\omega, \quad (6)$$

and for $\langle V^2\rangle$

$$\langle V^2\rangle=\int_0^\infty|Z|^2\hbar\omega^2\left[1+\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\times\int_0^\infty|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)\times\rho(E)F(E)dEd\omega. \quad (7)$$

We can calculate the ensemble average of Q^2, using the same procedure which was employed to obtain (6). The result is

$$\langle Q^2\rangle=\int_0^\infty\hbar\left[1+\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\int_0^\infty|\langle E+\hbar\omega|Q|E\rangle|^2\times\rho(E+\hbar\omega)\rho(E)F(E)dEd\omega. \quad (8)$$

Expressions (6), (7), and (8) are ensemble averages, for the unperturbed system with Hamiltonian H_0. Now we imagine the external driving forces to be applied so that the Hamiltonian is given by (3), with V_e varying sinusoidally with time with angular frequency ω. Using first-order perturbation theory we calculate the transition probability, assuming first that the system is in an energy eigenstate. From this the exchange of energy and power can be calculated. The exchange of power is then averaged over an ensemble and the result is proportional to the square of the amplitude of the external driving force. The conductance function can then be shown to be

$$G(\omega)=\frac{R(\omega)}{|Z(\omega)|^2}=\pi\omega\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\times\int_0^\infty|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)\rho(E)F(E)dE. \quad (9)$$

We now imagine the external forces to be removed. Callen and Welton substitute the perturbation theory result (9) into (7) to obtain expression (2) for the mean squared (spontaneously) fluctuating forces, in terms of the system response to external forces. In an accompanying note,[3] it has been shown that expression (9) is exact for systems having a dense distribution of eigenstates and which can be coupled to a harmonic oscillator by a term proportional to pQ. p is either a coordinate or momentum of the harmonic oscillator and Q is a function of the coordinates and momenta of the system, independent of the harmonic oscillator coordinates. Such systems will have power dissipation quadratic in the amplitude because for large amplitudes (where the notion of amplitude has a meaning) the damped linear oscillator has power dissipation quadratic in the amplitude.

On the other hand, if a system is known *a priori* to have power dissipation quadratic in the driving forces, then it follows without any other criteria that expression (9) must be exact even though it was obtained using first-order perturbation theory. This is true because all higher order terms in the perturbation expansion lead to dissipation higher than quadratic in the driving forces, and the contribution of all higher order terms must therefore be zero.

We adopt the definition that dissipative systems are linear if they have power dissipation always quadratic in the driving force amplitude, for large harmonic driving forces. This is consistent with the usual definition that linear systems are those which are described by linear differential equations. From the foregoing remarks it is clear that for linear systems the first-order term in perturbation theory gives exact results.[4] If a dissipative system is not known *a priori* to be linear, then a sufficient criterion for linearity is that it couples to a harmonic oscillator by a term linear in either the oscillator coordinate or momentum.

It can also be shown that a dissipative system which has a Lagrangian[5] quadratic in the coordinates and momenta, and coupling terms to external forces which are linear in the coordinates and momenta will satisfy our definition of linearity.

For linear systems, then, we can substitute the exact expression (9) into (6), (7), and (8) to obtain the following exact forms of the fluctuation dissipation theorem:

$$\langle \dot{Q}^2 \rangle = \frac{2}{\pi} \int_0^\infty G(\omega) \left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \right] d\omega, \quad (10)$$

$$\langle V^2 \rangle = \frac{2}{\pi} \int_0^\infty R(\omega) \left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \right] d\omega, \quad (11)$$

$$\langle Q^2 \rangle = \frac{2}{\pi} \int_0^\infty \frac{\hbar G(\omega)}{\omega} \left[\frac{1}{2} + \frac{1}{\exp(\hbar\omega/kT)-1} \right] d\omega. \quad (12)$$

[3] J. Weber, previous paper [Phys. Rev. **101**, 1619 (1956)].

[4] It follows from these remarks that the results given by Julius Jackson, Phys. Rev. **87**, 471 (1952), are exact for linear systems.

[5] The author wishes to acknowledge an enlightening discussion of this point with Professor John A. Wheeler.

It should be noted that the definition of conductance (9) is the same as the classical one.[6]

Expressions (10), (11), and (12) have the following interpretation, for systems which meet our definition of linearity. Imagine the system to be driven by external forces such that the conductance and resistance functions are $G(\omega)$ and $R(\omega)$, respectively. Then, in the absence of driving forces, there are fluctuations given by expressions (10), (11), and (12). For $T=0$ these expressions give the zero-point fluctuations.

Callen and Welton employed (11) to calculate expectation values of the square of the electric field intensity at a point. Their divergent result contained a term having the form of the Planck distribution law. In the following sections we generalize the theorem so that (convergent) expectation values for field operators at different space-time points can be calculated.

MODIFICATIONS OF THE THEOREM TO INCLUDE TIME AVERAGED QUANTITIES

In many cases measurements are not instantaneous values, but are time averages over a finite time interval. We can generalize (10), (11), and (12) to cover such problems, in the following way. Suppose we have an operator O, and we want to calculate $[O_{Av}]^2$ where O_{Av} is the time average of O over an interval τ.

$$[O_{Av}]^2 = \frac{1}{\tau^2} \int_0^\tau \int_0^\tau OO' dt dt', \quad (13)$$

$$\langle (O_{Av})^2 \rangle_{nn} = \frac{1}{\tau^2} \int_0^\tau \int_0^\tau \sum_m \langle E_n | O | E_m \rangle \exp(i\omega_{nm} t) \times \langle E_m | O | E_n \rangle \exp(-i\omega_{nm} t') dt dt'. \quad (13A)$$

Instead of starting with $\langle O^2 \rangle_{nn}$, we can start with (13A) to deduce expressions corresponding to (10), (11), and (12). The results are then

$$\langle ((\dot{Q})_{Av})^2 \rangle = \frac{2}{\pi\tau^2} \int_0^\infty \int_0^\tau \int_0^\tau G(\omega) \exp[i\omega(t-t')] \times \left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \right] d\omega dt dt', \quad (14)$$

$$\langle ((\dot{Q})_{Av})^2 \rangle = \frac{2}{\pi} \int_0^\infty G(\omega) \left[\frac{\sin(\omega\tau/2)}{\omega\tau/2} \right]^2 \times \left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \right] d\omega, \quad (14A)$$

$$\langle (V_{Av})^2 \rangle = \frac{2}{\pi\tau^2} \int_0^\infty \int_0^\tau \int_0^\tau R(\omega) \exp[i\omega(t-t')] \times \left[\frac{\hbar\omega}{2} + \frac{\hbar\omega}{\exp(\hbar\omega/kT)-1} \right] d\omega dt dt', \quad (15)$$

[6] J. Weber, Phys. Rev. **90**, 977 (1953).

$$\langle (V_{Av})^2\rangle = \frac{2}{\pi}\int_0^\infty R(\omega)\left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2 \times\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]d\omega, \quad (15A)$$

$$\langle (Q_{Av})^2\rangle = \frac{2}{\pi\tau^2}\int_0^\infty\int_0^\tau\int_0^\tau \frac{\hbar G(\omega)}{\omega}\exp[i\omega(t-t')] \times\left[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right]d\omega dt dt', \quad (16)$$

$$\langle (Q_{Av})^2\rangle = \frac{2}{\pi}\int_0^\infty \frac{\hbar G(\omega)}{\omega}\left[\frac{\sin(\omega\tau/2)}{\omega\tau/2}\right]^2 \times\left[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right]d\omega. \quad (16A)$$

Expression (15) is appropriate for situations where we have driving voltages, while (16) is appropriate for situations such as the quantum theory of the Maxwell field, where driving charges and currents will be employed.

FOUR-DIMENSIONAL FORMULATION OF THE FLUCTUATION DISSIPATION THEOREM

We imagine that a system of "charges" and currents interacts with a field. The field is regarded as the dissipative system, and it is assumed to be linear. We calculate the dissipation function for the field which we assume to be driven by the charges and currents. Then expression (16) allows us to calculate the expectation values of the field variables, in the absence of driving charges and currents.

We follow existing convention and use Latin indices which can have the values 1, 2, and 3, and Greek indices which can have the values 1, 2, 3, and 4. A repeated index is to be summed over.

The interaction Hamiltonian is assumed to be of the form

$$H = I\int f_\mu A_\mu d_3x. \quad (17)$$

f_μ in (17) is a "charge" and "current" distribution function. Charges and currents everywhere are assumed to be proportional to $I(t)$. f_μ is a function of the space coordinates and all four components of f_μ can be independently specified. A_μ represents a four-potential field with which the charges and currents interact. The function Q is defined by

$$Q = \int f_\mu A_\mu d_3x. \quad (18)$$

We denote the Fourier transform of I by $I_f(\omega)$, and the Fourier transform of $\dot{Q}$ by $\dot{Q}_f(\omega)$. Then the admittance is

$$Y(\omega) = \dot{Q}_f(\omega)/I_f(\omega). \quad (19)$$

The conductance is

$$G(\omega) = \mathrm{Re}(i\omega Q_f(\omega)/I_f(\omega)). \quad (20)$$

We assume that the conductance can be written in the form of an integral over the charges and currents and we define a dissipation tensor $d_{\mu\alpha}$ by letting

$$G(\omega)\exp[i\omega(t-t')] = \int d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega) f_\mu(\mathbf{r}) f_\alpha(\mathbf{r}') d_3x d_3x'. \quad (21)$$

In (21), $\mathbf{x}$ and $\mathbf{x}'$ are position four-vectors. We now employ (21) and (16) to write

$$\int_V \langle A_\mu(\mathbf{x}) A_\alpha(\mathbf{x}')\rangle f_\mu(\mathbf{r}) f_\alpha(\mathbf{r}') d_4x d_4x' = \frac{2\hbar}{\pi}\int_0^\infty\int_V \frac{d_{\mu\alpha} f_\mu f_\alpha}{\omega}\left[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right] \times d\omega d_4x d_4x'. \quad (22)$$

Now expression (22) is an identity, valid for all regions $d_4x d_4x'$, in V; also the different components of f_μ can be arbitrarily specified. This allows us to write

$$\langle A_\mu(\mathbf{x}) A_\alpha(\mathbf{x}')\rangle_T = \frac{2\hbar}{\pi}\int_0^\infty \frac{d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega)}{\omega} \times\left[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\right]d\omega. \quad (23)$$

In most cases the temperature-independent part of (23) is all that is needed, since this gives us (corresponding to $T=0$) the vacuum expectation values:

$$\langle A_\mu(\mathbf{x}) A_\alpha(\mathbf{x}')\rangle_0 = \frac{\hbar}{\pi}\int_0^\infty \frac{d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega)d\omega}{\omega}. \quad (24)$$

We can employ expression (24) to write the covariant commutation rules:

$$[A_\mu(\mathbf{x}), A_\alpha(\mathbf{x}')] = \frac{\hbar}{\pi}\int_0^\infty \frac{d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega)-d_{\alpha\mu}([\mathbf{x}'-\mathbf{x}],\omega)}{\omega}d\omega. \quad (24A)$$

Equations (23) and (24) are four-dimensional formulations of the fluctuation dissipation theorem, valid either for a scalar field (by setting $\mu=\alpha$) or a four-potential field which can be imagined to interact with a system of "charges" and "currents" according to (17).

We could have discussed a tensor field by starting with the interaction Hamiltonian

$$H = I\int f_{\mu\alpha} A_{\mu\alpha} d_3x. \quad (25)$$

We would then have

$$G(\omega)\exp[i\omega(t-t')]=\int d_{\alpha\beta\gamma\delta}([\mathbf{x}-\mathbf{x}'],\omega)f_{\alpha\beta}f_{\gamma\delta}d_3xd_3x', \quad (26)$$

and corresponding to (24) we would have

$$\langle A_{\alpha\beta}(\mathbf{x})A_{\gamma\delta}(\mathbf{x}')\rangle_0=\frac{\hbar}{\pi}\int\frac{d_{\alpha\beta\gamma\delta}([\mathbf{x}-\mathbf{x}'],\omega)d\omega}{\omega}. \quad (27)$$

We illustrate the usefulness of (24) by employing it to calculate expectation values for the Maxwell field.[7]

For the Maxwell field we have a four-potential which interacts with the four-current, and the interaction Hamiltonian is

$$H=-\frac{1}{c}\int J_\mu(\mathbf{x})A_\mu(\mathbf{x})d_3x=-\frac{J(t)}{c}\int f_\mu(\mathbf{r})A_\mu(\mathbf{x})d_3x. \quad (28)$$

We let

$$I=-J(t)/c, \quad Q=\int f_\mu(\mathbf{r})A_\mu(\mathbf{x})d_3x. \quad (29)$$

To calculate the conductance we assume all quantities to be harmonically varying with time; then the ratio of Fourier transforms of $\dot{Q}$ and I will be simply the ratio of $\dot{Q}$ to I and the conductance

$$G(\omega)=\mathrm{Re}\left(\frac{i\omega Q_f(\omega)}{I_f(\omega)}\right)=\mathrm{Re}\left[\frac{-i\omega\int f_\mu A_\mu d_3x}{J/c}\right]. \quad (30)$$

We can evaluate (30) by noting that A_μ is given in terms of the four-current density Jf_μ by the retarded potentials (in Lorentz gauge)

$$A_\mu=\frac{1}{c}\int\frac{Jf_\mu\exp[-i(\omega/c)|\mathbf{r}-\mathbf{r}'|]d_3x}{|\mathbf{r}-\mathbf{r}'|}. \quad (31)$$

Inserting (31) in (30) gives us

$$G(\omega)=-\omega\delta_{\mu\alpha}\times\int\frac{f_\mu(\mathbf{r})f_\alpha(\mathbf{r}')\sin[(\omega/c)|\mathbf{r}-\mathbf{r}'|]d_3xd_3x'}{|\mathbf{r}-\mathbf{r}'|}. \quad (32)$$

Comparing (32) with (21) we see that the dissipation tensor is

$$d_{\mu\alpha}=-\frac{\delta_{\mu\alpha}\omega\sin[(\omega/c)|\mathbf{r}-\mathbf{r}'|]\exp[i\omega(t-t')]}{|\mathbf{r}-\mathbf{r}'|}. \quad (33)$$

[7] The electromagnetic field is linear, the vacuum is not.

Inserting (33) into (24) gives

$$\langle A_\mu(\mathbf{x})A_\alpha(\mathbf{x}')\rangle_0=-\frac{\hbar}{\pi}\delta_{\mu\alpha}\int_0^\infty\frac{\sin[(\omega/c)|\mathbf{r}-\mathbf{r}'|]\exp[i\omega(t-t')]d\omega}{|\mathbf{r}-\mathbf{r}'|}. \quad (34)$$

Equation (34) may be written in a more familiar form by noting that

$$\frac{\sin[(\omega/c)|\mathbf{r}-\mathbf{r}'|]}{(\omega/c)|\mathbf{r}-\mathbf{r}'|}=-\frac{1}{2}\int_0^\pi\exp[-i(\omega/c)|\mathbf{r}-\mathbf{r}'|\cos\theta]\sin\theta d\theta,$$

and introducing a four-vector $\mathbf{K}$ such that $K_1^2+K_2^2+K_3^2=\omega^2/c^2$. (34) then can be written

$$\langle A_\mu(\mathbf{x})A_\alpha(\mathbf{x}')\rangle_0=\frac{\hbar c\delta_{\mu\alpha}}{2\pi^2}\int\exp[i\mathbf{K}\cdot(\mathbf{x}-\mathbf{x}')]\delta(K^2)\theta(K)d_4K. \quad (35)$$

In (35) we have taken $d_4K=d_3KdK_0$, and

$$K_1^2+K_2^2+K_3^2+K_4^2=K^2=K_1^2+K_2^2+K_3^2-K_0^2,$$

$$\theta(K)=1, \quad K_0>0,$$

$$\theta(K)=0, \quad K_0<0.$$

From (35), the well-known covariant commutation law for the Maxwell field can be immediately written as

$$[A_\mu(\mathbf{x}),A_\alpha(\mathbf{x}')]=\frac{\hbar c\delta_{\mu\alpha}}{2\pi^2}\int\exp[i\mathbf{K}\cdot(\mathbf{x}-\mathbf{x}')]\delta(K^2)\epsilon(K)d_4K, \quad (35A)$$

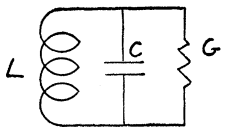
where $\epsilon(K)=+1$ for $K_0>0$ and $\epsilon(K)=-1$ for $K_0<0$.

At temperature T, (35) becomes

$$\langle A_\mu(\mathbf{x})A_\alpha(\mathbf{x}')\rangle_T=\frac{\hbar c\delta_{\mu\alpha}}{\pi^2}\int\left[\frac{1}{2}+\frac{1}{\exp(\hbar K_0c/kT)-1}\right]\times\exp[i\mathbf{K}\cdot(\mathbf{x}-\mathbf{x}')]\delta(K^2)\theta(K)d_4K. \quad (35B)$$

Expression (35) is the well-known result of the quantum theory of the Maxwell field. In making this calculation we have worked with two fields, the electron field and the Maxwell field which was regarded as the dissipative system. It is apparent that the relation (16) can be used at least in field problems where we have an interaction Hamiltonian of the form (17) or (25), and we are calculating expectation values for a classically describable field. Extension of this formalism to spinor fields may be possible and is being investigated. The connection between dissipation and vacuum expectation values appears to be quite fundamental. For example a

FIG. 1. Electric circuit in equilibrium with a heat bath.



connection is implied between gravitational radiation and the quantization of the gravitational field.

CUT-OFF PROCEDURE FOR CALCULATING OBSERVABLE FLUCTUATIONS

Let us consider two electric circuits illustrated by Figs. 1 and 2.

If the circuits are in equilibrium with a heat bath at temperature T, then from classical statistical mechanics, we shall expect that for both Fig. 1, and Fig. 2, the capacitor voltage fluctuations $\langle V^2\rangle$ is obtained from

$$\begin{aligned}\tfrac{1}{2}C\langle V^2\rangle&=\tfrac{1}{2}kT,\\ \langle V^2\rangle&=kT/C.\end{aligned}\tag{36}$$

If expression (1) is integrated over all frequencies, it gives (36) for both Fig. 1 and Fig. 2.

For the circuit of Fig. 1, expression (2) gives (G is constant)

$$\langle V^2\rangle=\frac{2}{\pi}\int_0^\infty \frac{\omega^2L^2G}{\omega^2L^2G^2+(1-\omega^2LC)^2}\times\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]d\omega.\tag{37}$$

Equation (37) diverges logarithmically, because of the temperature independent part. For the circuit of Fig. 2 expression (2) gives (R is constant):

$$\langle V^2\rangle=\frac{2}{\pi}\int_0^\infty \frac{R}{(1-\omega^2LC)^2+\omega^2C^2R^2}\times\left[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]d\omega.\tag{38}$$

Equation (38) does not diverge. If R is small it reduces to the simple form

$$\langle V^2\rangle=\frac{1}{C}\left[\frac{\hbar\omega_0}{2}+\frac{\hbar\omega_0}{\exp(\hbar\omega_0/kT)-1}\right],\tag{39}$$

where ω_0 is the natural frequency.

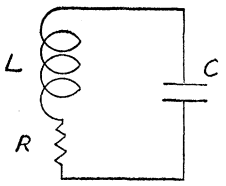
We see then that expression (2) gives different results for two situations, and one result is divergent. Classically we should expect to obtain the same result for both cases.[8]

Actually, the circuits of Figs. 1 and 2 are not good models of physically realizable circuits. At low frequencies a circuit can be described as a set of conducting boundaries having many modes corresponding to solutions of the field equations which satisfy the boundary conditions. At frequencies above the x-ray region, the circuit is better described as an ensemble of particles. For high energy, the cross section of the circuit particles will be small, and the high-frequency contribution to (2) or (15) will be mainly from the vacuum fields. For the vacuum,

$$\langle V^2\rangle=\frac{1}{\tau^2}\int_0^\tau\int_0^\tau\int_0^l\int_0^l\langle E_iE_j'\rangle dl_i dl_j' dt dt'.\tag{40}$$

It is well known that (40) diverges for the vacuum.[9] We can conclude from this that (2) and (15) diverge, as they stand, for all physically realizable networks. However the experiments which are actually done to measure electrical noise do not measure the quantity given by (2) and (15). We seek a modification of (2) and (15) to cover actual experiments. This can be done in the following way.

FIG. 2. Electric circuit in equilibrium with a heat bath.



Measurements are usually carried out by interacting with a system at certain times, to measure an eigenvalue of some operator, or its time average over a short time interval. The system returns to equilibrium with the heat bath between interactions. The interaction will induce transitions, with exchange of energy between the measuring apparatus and the system. If energy $\hbar\omega_i$ is exchanged, the transition will involve[10] matrix elements $\langle E\pm\hbar\omega_i|O|E\rangle$ of the operator O. The measuring apparatus is usually not a thermodynamic system and its energy is well defined. Ordinarily, for example, low-frequency noise measurements are carried out using electrons, with well-defined energy.

We denote the wave functions and energy eigenvalues of the measuring apparatus by ψ_a and E_a, and the wave functions and energy eigenvalues of the thermodynamic system by ψ_s and E_s. If the measuring[11] apparatus has energy $E_a=\hbar\omega_c$ then it is clear that all matrix elements $\langle E_a-\hbar\omega|O|E_a\rangle$ will contribute zero for $\omega>\omega_c$. Now energy is conserved in these transitions and therefore we can say that for transitions in which the system gains energy, eigenfunctions having eigenvalues exceeding $E_s+\hbar\omega_c$ will not contribute to the observed fluctuations. Eigenfunctions having eigenvalues $<E_s-\hbar\omega_c$ will contribute because the system energy is not well defined,

[8] In this section it is assumed that apparatus energy is low so that pair creation is of no consequence.

[9] Expression (40) diverges because it is a line integral, while expression (22) does not diverge because it is a volume integral.

[10] Expression (3.5) of I suggests this procedure for carrying out measurements of Q^2, by measuring $|\langle E_n|Q|E_n\pm\hbar\omega\rangle|^2$.

[11] We assume that the measurements are carried out with a linear device.

since only its temperature is known. Accordingly we should rewrite (4), as

$$\langle\dot{Q}^2\rangle=\int_0^{\omega_c}\hbar\omega^2\Big[\int_0^{\infty}\rho(E)F(E)\times|\langle E+\hbar\omega|Q|E\rangle|^2\rho(E+\hbar\omega)dE\Big]d\omega+\int_0^{\infty}\hbar\omega^2\Big[\int_0^{\infty}\rho(E)F(E)\times|\langle E-\hbar\omega|Q|E\rangle|^2\rho(E-\hbar\omega)dE\Big]d\omega. \quad (41)$$

A similar cutoff can be applied to (5). With this modification we obtain for the observable fluctuations of current, voltage, and function Q the general forms corresponding to (10), (11), and (12) as

$$\langle\dot{Q}^2\rangle=\frac{2}{\pi}\int_0^{\omega_c}G(\omega)\Big[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\Big]d\omega+\frac{2}{\pi}\int_{\omega_c}^{\infty}\frac{G(\omega)\hbar\omega d\omega}{\exp(\hbar\omega/kT)-1}, \quad (42)$$

$$\langle V^2\rangle=\frac{2}{\pi}\int_0^{\omega_c}R(\omega)\Big[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\Big]d\omega+\frac{2}{\pi}\int_{\omega_c}^{\infty}\frac{R(\omega)\hbar\omega d\omega}{\exp(\hbar\omega/kT)-1}, \quad (43)$$

$$\langle Q^2\rangle=\frac{2}{\pi}\int_0^{\omega_c}\frac{\hbar G(\omega)}{\omega}\Big[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\Big]d\omega+\frac{2}{\pi}\int_{\omega_c}^{\infty}\frac{\hbar G(\omega)d\omega}{\omega[\exp(\hbar\omega/kT)-1]}, \quad (44)$$

and for the time averaged quantities (14), (15), and (16) we have

$$\langle((\dot{Q})_{Av})^2\rangle=\frac{2}{\pi\tau^2}\int_0^{\omega_c}\int_0^{\tau}\int_0^{\tau}G(\omega)\exp[i\omega(t-t')]\times\Big[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\Big]d\omega dtdt'+\frac{2}{\pi\tau^2}\int_{\omega_c}^{\infty}\int_0^{\tau}\int_0^{\tau}\times\frac{G(\omega)\exp[i\omega(t-t')]\hbar\omega d\omega dtdt'}{\exp(\hbar\omega/kT)-1}, \quad (45)$$

$$\langle(V_{Av})^2\rangle=\frac{2}{\pi\tau^2}\int_0^{\omega_c}\int_0^{\tau}\int_0^{\tau}R(\omega)\exp[i\omega(t-t')]\times\Big[\frac{\hbar\omega}{2}+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\Big]d\omega dtdt'+\frac{2}{\pi\tau^2}\int_{\omega_c}^{\infty}\int_0^{\tau}\int_0^{\tau}\times\frac{R(\omega)\exp[i\omega(t-t')]\hbar\omega d\omega dtdt'}{\exp(\hbar\omega/kT)-1}, \quad (46)$$

$$\langle(Q_{Av})^2\rangle=\frac{2}{\pi\tau^2}\int_0^{\omega_c}\int_0^{\tau}\int_0^{\tau}\frac{\hbar G(\omega)}{\omega}\exp[i\omega(t-t')]\times\Big[\frac{1}{2}+\frac{1}{\exp(\hbar\omega/kT)-1}\Big]d\omega dtdt'+\frac{2}{\pi\tau^2}\int_{\omega_c}^{\infty}\int_0^{\tau}\int_0^{\tau}\times\frac{\hbar G(\omega)\exp[i\omega(t-t')]d\omega dtdt'}{\omega[\exp(\hbar\omega/kT)-1]}. \quad (47)$$

These expressions do not diverge, and give the results of the kinds of measurements which can be carried out. ω_c is defined by the measuring apparatus energy $E=\hbar\omega_c$.

In earlier papers,[12,13] experiments were discussed for observing directly the fluctuations of the vacuum electromagnetic fields in an enclosure. It follows from the preceding discussion that the observable voltage fluctuations are given by

$$\langle V^2\rangle=\frac{\hbar}{\pi}\int_0^{\omega_c}R(\omega)\omega d\omega, \quad (48)$$

and that this result is exact for fluctuations measured by a linear device.

CONCLUSION

We have shown that the fluctuation dissipation theorem is exact and have given the generalizations (14), (15), (16), (23), (24), and a cut-off procedure. The theorem appears to be a very fundamental relation of quantum theory and is of wide applicability.

[12] J. Weber, Phys. Rev. **94**, 215–217 (1954).
[13] J. Weber, Phys. Rev. **96**, 556–559 (1954).