THE
PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 101, NO. 6 MARCH 15, 1956

Exact Quantum Theory Solution for the Damped Harmonic Oscillator*

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(Received September 29, 1955)

It is shown that the perturbation theory solution previously given for the damped harmonic oscillator is exact. The expression given for the conductance is also exact.

IN an earlier paper,[1] hereinafter to be denoted by A, the Hamiltonian of a harmonic oscillator coupled to a resistance was taken as

$$\mathcal{H}=\tfrac{1}{2}(p^2+\omega^2q^2)+H_R+(pQ/\sqrt{C}). \quad (1)$$

In (1), H_R is the unperturbed Hamiltonian of the resistance, q and p are the coordinate and momentum of the oscillator, and Q is a function of the coordinates and momenta of the resistance. C is a constant which is the capacity of the oscillator if it is an electrical oscillator.

If the oscillator is in an eigenstate of its unperturbed Hamiltonian with energy E_F, then using first-order perturbation theory, the transition probability can be written

$$W_r=\frac{2\pi}{\hbar}\Big[\rho(E_R+\hbar\omega)|\langle E_R|Q|E_R+\hbar\omega\rangle|^2 \times\left|\left\langle E_F\left|\frac{p}{\sqrt{C}}\right|E_F-\hbar\omega\right\rangle\right|^2 +\rho(E_R-\hbar\omega)|\langle E_R|Q|E_R-\hbar\omega\rangle|^2 \times\left|\left\langle E_F\left|\frac{p}{\sqrt{C}}\right|E_F+\hbar\omega\right\rangle\right|^2\Big]. \quad (2)$$

In (2), $\rho(E_R+\hbar\omega)$ is the density in energy of the quantum states of the resistance in the vicinity of $E_R+\hbar\omega$. ω is the natural frequency of the harmonic oscillator.

Let us consider the higher order perturbation theory approximations to (2). Suppose the original eigenstate of the harmonic oscillator is ψ_m and the original eigenstate of the resistance is ϕ_m.

Because of the linear nature of the interaction term $pQ/\sqrt{C}$, the *only* harmonic oscillator matrix elements which are not zero are those to states $\psi_{m\pm1}$. This means that if the oscillator is in the state ψ_m, the *only* two states to which it can go are ψ_{m+1}, and ψ_{m-1}. The higher order approximations all involve the two adjacent states as intermediate states. If the oscillator is in the state ψ_m, it must pass through either the state ψ_{m+1} or the state ψ_{m-1} before going to any other state. The calculation of the statistical exchange of energy includes all higher approximations by considering only the two adjacent states, and therefore (2) is exact for this purpose.[2]

The remainder of A was concerned with statistical averaging of expressions obtained from (2) for a resistance in equilibrium with a heat bath at a temperature T. The result for the energy of a harmonic oscillator which is coupled to a resistance at temperature T, at time $t=0$, is

$$U=\left[\tfrac{1}{2}\hbar\omega+\frac{\hbar\omega}{\exp(\hbar\omega/kT)-1}\right]\left[1-\exp\left(-\frac{Gt}{C}\right)\right] +U_0\exp\left(-\frac{Gt}{C}\right). \quad (3)$$

* Supported by the Office of Naval Research.
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[1] J. Weber, Phys. Rev. **90**, 977 (1953). In the earlier paper all squared matrix elements are squares of absolute values.
[2] It is assumed that the dissipative element has a dense distribution of eigenstates, that the perturbation expansion converges, and that the expression (2) does not vanish.

In (3) G is the conductance function[3] defined in A by

$$G=\pi\omega\left[1-\exp\left(-\frac{\hbar\omega}{kT}\right)\right]\int_0^\infty \rho(E_R+\hbar\omega) \times|\langle E_R|Q|E_R+\hbar\omega\rangle|^2\rho(E_R)f(E_R)dE_R. \quad (4)$$

Expressions (3) and (4) are exact for the Hamiltonian (1), and describe the approach to equilibrium of an oscillator which is coupled to a resistance in equilibrium with a heat bath at temperature T, and which has energy U_0 at time $t=0$.

It remains to be shown that expression (4) is the same as the classical frequency dependent conductance measured by the response of the conductance to harmonic driving forces. For an ensemble of oscillators consisting of an inductance L, capacity C, and conductance $G_c(\omega)$, the classical expression for the energy U as a function of time is

$$U=U_0\exp\left[-\frac{2\,\mathrm{Re}[G_c(\omega_1)]}{C(1+\omega^2/|\omega_1|^2)}t\right]. \quad (5)$$

In (5), ω_1 is the complex natural frequency of the damped oscillator, ω is the natural frequency of the undamped oscillator, and $\mathrm{Re}G_c(\omega_1)$ is the real part of the conductance G_c evaluated for the complex ω_1. We wish to show that $G_c(\omega)=G(\omega)$, where $G(\omega)$ is defined by (4). Comparing (3) with (5), we see that

$$\frac{2\,\mathrm{Re}G_c(\omega_1)}{1+\omega^2/|\omega_1|^2}=G(\omega). \quad (6)$$

Now in (5) imagine C to become very large and L to become very small such that the undamped frequency ω remains the same. Then it is clear that $\omega_1\to\omega$. Since $G_c(\omega)$ is real, it then follows from (6) that

$$G_c(\omega)=G(\omega). \quad (7)$$

[3] The conductance is the reciprocal of the resistance if there is no series reactance associated with the resistance.

PHYSICAL REVIEW VOLUME 101, NUMBER 6 MARCH 15, 1956

Fluctuation Dissipation Theorem*

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(Received September 29, 1955)

The fluctuation dissipation theorem (Nyquist formula) is shown to be exact and a number of generalizations of it are given, including a four-dimensional formulation which is useful in the quantum theory of fields. The more general theorem can be used to calculate vacuum expectation values of field operators, and to deduce covariant commutation relations for the fields. For a four-potential field, the vacuum expectation values for operators at two space-time points $\mathbf{x}$ and $\mathbf{x}'$ are

$$\langle A_\mu(\mathbf{x})A_\alpha(\mathbf{x}')\rangle_0=\frac{\hbar}{\pi}\int_0^\infty\frac{d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega)}{\omega}d\omega,$$

where $d_{\mu\alpha}$ is a dissipation tensor.

The covariant commutation relations are

$$[A_\mu(\mathbf{x}),A_\alpha(\mathbf{x}')]=\frac{\hbar}{\pi}\int_0^\infty\frac{[d_{\mu\alpha}([\mathbf{x}-\mathbf{x}'],\omega)-d_{\alpha\mu}([\mathbf{x}'-\mathbf{x}],\omega)]}{\omega}d\omega.$$

A well-defined cut-off procedure is given for calculating observable fluctuations in cases where the theorem gives infinite results.

For measurements with a linear device which has energy $E=\hbar\omega_c$, the observable fluctuations of the vacuum electromagnetic fields are given by the exact expression

$$\langle V^2\rangle=\frac{\hbar}{\pi}\int_0^{\omega_c}R(\omega)\omega d\omega.$$

INTRODUCTION

NYQUIST[1] first deduced the fluctuation dissipation theorem, using classical statistics. His work was prompted by Johnson's experiments on electrical noise. The Nyquist formula, applicable to linear electrical networks, states that the mean squared fluctuation voltage in an angular frequency interval $d\omega$ is given by

$$\langle V^2\rangle=2kTR(\omega)d\omega/\pi. \quad (1)$$

In (1), $R(\omega)$ is the real part of the impedance function.

Later Callen and Welton[2] gave a quantum-theoretical deduction of (1) and showed that it was applicable to a

* This work was started at the University of Maryland under Office of Naval Research sponsorship.
† Guggenheim and Merck (National Research Council) Fellow.
[1] H. Nyquist, Phys. Rev. **32**, 110 (1928).
[2] H. B. Callen and T. A. Welton, Phys. Rev. **83**, 34 (1951).